Control and Cybernetics

vol. **31** (2002) No. 3

Hőlder-like properties of minimal points in vector optimization

by

Ewa M. Bednarczuk

Systems Research Institute, Polish Academy of Sciences
01-447 Warsaw, Newelska 6

Abstract: We derive conditions for Hőlder calmness of minimal points of a given set, as a function of a parameter appearing in the description of the set. Different criteria are proved depending on whether the ordering cone has a nonempty interior or not.
Keywords: vector optimization, stability, Hőlder-like continuity

1. Introduction.

We investigate Hőlder-like properties of minimal points of a set depending upon a parameter. The goal is to provide a general framework for stability analysis of parametric vector optimization problems. From the results obtained one can easily derive the conditions for stability of minimal points in parametric vector optimization problems. These conditions, in turn, can be viewed as vector counterparts of conditions for stability of the optimal value function in scalar parametric optimization problems. Our results depend essentially on the behaviour of the containment and the weak containment rate functions, introduced in the present paper. These functions are specific for stability analysis in vector case. Their appearance is caused by the fact that in vector optimization we work with partial orders only.

Lipschitz-like properties of multifunctions were investigated by many authors, e.g. by Robinson (1981, 1976), Aubin (1984, 1985), Clarke (1983). They play an important role in stability of nonlinear programming problems, see e.g. Henrion and Outrata (2001), Klatte and Kummer (2001). We define Hőlder counterparts of these notions with orders other than 1 (and not necessarily smaller than 1). This allows us to investigate the influence of the order of change of a given multifunction, and of the speed of growth of the containment and the weak containment rate functions, upon the order of change of minimal point multifunction.

In Theorem 3.1 we give conditions for Hőlder calmness of minimal points. It is worth noticing that, as a consequence of assumptions, we obtain that $\operatorname{int}\mathcal{K} \neq \emptyset$.

separately and requires other techniques. We propose dual approach. We exploit the quasi-interior of a cone and the description of a cone by its dual. The main result related to the case $\operatorname{int}\mathcal{K} = \emptyset$ is given in Theorem 5.1.

Let Y and U be normed spaces and let B_Y denote the open unit ball in Y. We say that a multivalued mapping $\Gamma : U \rightrightarrows Y$, is

(H1) *upper pseudo-Hőlder or Hőlder calm* at (u_0, y_0), $y_0 \in \Gamma(u_0)$, if, for a neighbourhood V of y_0 and a neighbourhood U_0 of u_0, there are positive L and q such that
$$\Gamma(u) \cap V \subset \Gamma(u_0) + L\|u - u_0\|^q B_Y, \quad u \in U_0.$$

(H2) *lower pseudo-Hőlder* at (u_0, y_0), $y_0 \in \Gamma(u_0)$, if, for a neighbourhood V of y_0 and a neighbourhood U_0 of u_0, there are positive L and q such that
$$\Gamma(u_0) \cap V \subset \Gamma(u) + L\|u - u_0\|^q B_Y, \quad u \in U_0.$$

For $q = 1$, $(H1)$ reduces to calmness (see Henrion, Outrata, 2001, Klatte, Kummer, 2002). Criteria for calmness of different multifunctions can be found eg., in Henrion and Outrata (2001). For instance, if $S(y) = [-s(y), s(y)]$, where $s(y) = 1 + \sqrt{|y|}$, $y \in R$, then S is not calm at $(0, 1)$ (see Klatte and Kummer, 2001), but it is Hőlder calm at $(0, 1)$ with order 1/2.

Let $A \subset Y$ be a subset of Y and let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y, $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$. We say that $y_0 \in A$ is

(M1) *minimal point*, $y_0 \in Min_A$, if $A \cap (y_0 - \mathcal{K}) = \{y_0\}$,

(M2) *local minimal point*, if $y_0 \in Min_{A\cap V}$, where V is a neighbourhood of y_0.

When $A \subset Y$ is a convex subset of Y,
$$Min_{A\cap V} \subset Min_A. \tag{1}$$
To see this, suppose that $y_0 \notin Min_A$, i.e., there exists $y_1 \in A$ such that $y_1 - y_0 \in -\mathcal{K}$. By convexity, $\lambda y_0 + (1 - \lambda)y_1 \subset A \cap (y_0 - \mathcal{K})$, $0 \le \lambda \le 1$, and $\lambda y_0 + (1 - \lambda)y_1 \in V$, for $0 \le \lambda \le \overline{\lambda} \le 1$. Hence, $y_0 \notin Min_{A\cap V}$.

2. Containment property and its characterizing functions

Let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y. For any subset $C \subset Y$ the point to set distance $d(x, C)$ is given as $d(x, C) = \inf\{\|x - c\| \mid c \in C\}$, and the ε neighbourhood of the set C is given as $B(C, \varepsilon) = \{y \in Y \mid d(y, C) < \varepsilon\}$. Denote $C(\varepsilon) = \{c \in C \mid d(c, Min_C) \ge \varepsilon\}$.

We say that the *containment property* (CP) (Bednarczuk, 2002) holds for a subset $C \subset Y$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that
$$C(\varepsilon) + \delta B_Y \subset Min_C + \mathcal{K}.$$
We define *the cone containment function*, *cont* : $\mathcal{K} \to R^+$, as follows
$$cont(k) = \sup\{r \ge 0 \mid k + rB_Y \subset \mathcal{K}\}.$$
If $\operatorname{int}\mathcal{K} = \emptyset$, then *cont* $\equiv 0$. Since $\mathcal{K}$ is closed, the supremum is always attained, i.e., $k + cont(k)B_Y \subset \mathcal{K}$. The cone containment function is posi-

$\lambda \cdot cont(k)$. If there were $cont(\lambda \cdot k) = \beta > \lambda \cdot cont(k)$ for some $\lambda > 0$, it would be $k + \frac{\beta}{\lambda} B_Y \subset \mathcal{K}$, contradictory to the definition of $cont(k)$. Moreover, $cont(k_1+k_2) \geq cont(k_1)+cont(k_2)$. In consequence, $cont(\cdot)$ is a concave function and

$$\operatorname{dom} cont = \{k \in \mathcal{K} \mid cont(k) > -\infty\} = \mathcal{K}.$$

For $k \in \mathcal{K}$, $cont(k) = -S_{\mathcal{K}}(k)$, where $S_{\mathcal{K}}(k) = \inf_{y \in \mathcal{K}} \|k - y\| - \inf_{y \in Y \setminus \mathcal{K}} \|k - y\|$. The function $S_{\mathcal{K}}$ was introduced by Hiriart-Urruty (1979a,b), see also Gorokhovik (1990). The function $\mu_C : Min_C + \mathcal{K} \to R_+$ defined as

$$\mu_C(y) = \sup\{cont(y - \eta_y) \mid \eta_y \in Min_C \cap (y - \mathcal{K})\},$$

is the *rate of containment of an element* $y \in Y$ (Bednarczuk, 2002) with respect to C and $\mathcal{K}$. The function $\delta_C : R_+ \to \overline{R} = R \cup \{\pm\infty\}$, given as

$$\delta_C(\varepsilon) = \inf\{\mu_C(y) \mid y \in C(\varepsilon)\}$$

is the *rate of containment of a set* C (Bednarczuk, 2002) with respect to $\mathcal{K}$.

REMARK 2.1 *If* $\operatorname{int} \mathcal{K} = \emptyset$, *then* $\mu_C \equiv 0$, *and* $\delta_C \equiv 0$. *On the other hand,* $\mu_C(y) = 0$, *implies that* y *lies on the boundary of* $Min_C + \mathcal{K}$, $y \in \partial(Min_C + \mathcal{K})$.

REMARK 2.2 *The containment property* (CP) *can be characterized by the containment rate function* δ *as follows.* (CP) *holds for a subset* $C \subset Y$ *if and only if* $\delta_C(\varepsilon) > 0$ *for any* $\varepsilon > 0$. *We say that the domination property* (DP) *holds for* C *if* $C \subset Min_C + \mathcal{K}$. (DP) *holds for* C *if and only if* $\delta_C(\varepsilon) \geq 0$ *for any* $\varepsilon > 0$.

Below, we give conditions under which the supremum in the definition of the function μ is attained. Recall that a convex subset Θ of a cone $\mathcal{K}$ is a base of $\mathcal{K}$ if $0 \notin \operatorname{cl}\Theta$, $\mathcal{K} = \bigcup\{\lambda\Theta \mid \lambda \geq 0\}$. Following Borwein and Zhuang (1993) we say that $R_\sigma(C)$ is the generalized weak recession cone of a set C if

$$R_\sigma(C) = \{v \in Y \mid \text{there exist } \lambda_n > 0 \ \lambda_n \to 0 \ c_n \in C \text{ such that } \lambda_n c_n \text{ tends weakly to } v\}.$$

A set $C \subset Y$ is $\mathcal{K}$−lower bounded if there is a constant $M > 0$ such that

$$C \subset MB_Y + \mathcal{K}.$$

If C is $\mathcal{K}$−lower bounded, then $R_\sigma(C) \subset \mathcal{K}$, see Borwein and Zhuang (1993).

PROPOSITION 2.1 *Let* $Y = (Y, \|\cdot\|)$ *be a normed space. Let* $\mathcal{K} \subset Y$ *be a closed convex pointed cone in* Y *and let* $C \subset Y$ *be a subset of* Y. *Let* $V \subset Y$ *be an open subset of* Y *and let* $y \in Min_{C \cap V} + \mathcal{K}$. *If either of the conditions holds:*

(ii) *$Min_{C\cap V}$ is $\mathcal{K}-$ lower bounded and weakly closed and $\mathcal{K}$ has a weakly compact base,*

then $y = \eta_y + k_y$, with $\eta_y \in Min_{C\cap V}$, and $k_y + \mu_{C\cap V}(y)B_Y \subset \mathcal{K}$.

Proof. Let $y \in Min_{C\cap V} + \mathcal{K}$. For each $n > 0$, there exists a representation $y = \eta_n + k_n$, $\eta_n \in Min_{C\cap V} \cap (y - \mathcal{K})$, $k_n + cont(k_n)B_Y \subset \mathcal{K}$, and

$$cont(k_n) \leq \mu_{C\cap V}(y) \text{ and } cont(k_n) > \mu_{C\cap V}(y) - \frac{1}{n}.$$

We claim that under either (i) or (ii) the sequences $\{\eta_n\}$ and $\{k_n\}$ converge to η_0, and k_0, respectively, and

$$y = \eta_0 + k_0. \tag{2}$$

If (i) holds, the sequence $\{\eta_n\}$ contains a weakly convergent subsequence. Without loss of generality we can assume that $\{\eta_n\}$ weakly converges to an $\eta_0 \in Min_{C\cap V} \cap (y - \mathcal{K})$. By this, the sequence $\{k_n\}$ weakly converges to $k_0 \in \mathcal{K}$ and we get (2).

If (ii) holds, and $\Theta \subset \mathcal{K}$ is a weakly compact base of $\mathcal{K}$, then $k_n = \lambda_n\theta_n$, $\lambda_n \geq 0$, and $\{\theta_n\} \subset \Theta$ contains a weakly convergent subsequence. Again, we can assume that $\{\theta_n\}$ weakly converges to $\theta_0 \in \Theta$. If there were $\lambda_n \to +\infty$, then

$$\frac{1}{\lambda_n}(\eta_n - y) \xrightarrow{w} -\theta_0$$

and $-\theta_0 \in R_\sigma(C) \cap (-\mathcal{K})$, contradictory to $\mathcal{K}$-lower boundedness of C. Hence, $\lambda_n \to \lambda_0 < +\infty$. Consequently, $\{k_n\}$ weakly converges to $k_0 \in \mathcal{K}$, and $\{\eta_n\}$ weakly converges to $\eta_0 = y - k_0$. Since $Min_{C\cap V}$ is weakly closed, $\eta_0 \in Min_{C\cap V}$, and (2) follows.

To complete the proof we need to show that $k_0 + \mu_{C\cap V}(y)B_Y \subset \mathcal{K}$. On the contrary, if we have $k_0 + \mu_{C\cap V}(y)b_0 \notin \mathcal{K}$, $b_0 \in B_Y$, then, by separation arguments, there exists a linear continuous functional f such that

$$f(k_0 + \mu_{C\cap V}(y)b_0) < 0 \leq f(k) \text{ for } k \in \mathcal{K}.$$

Consequently, there would be

$$\begin{aligned} &f(k_\alpha + cont(k_\alpha)b_0) \\ &= f(k_0 + \mu_{C\cap V}(y)b_0) + f(k_\alpha - k_0) + f([cont(k_\alpha) - \mu_{C\cap V}(y)]b_0) < 0, \end{aligned}$$

contradictory to the fact that $k_\alpha + cont(k_\alpha)B_Y \subset \mathcal{K}$. ∎

Based on Proposition 2.1 we prove continuity of the rate of containment μ_C.

THEOREM 2.1 *Let $(Y, \|\cdot\|)$ be a normed space. Let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y and let $C \subset Y$ be a subset of Y. Let $y_0 \in \operatorname{int}(Min_C + \mathcal{K})$.*

(i) Min_C *is weakly compact,*

(ii) Min_C *is* $\mathcal{K}-$*lower bounded and weakly closed and* $\mathcal{K}$ *has a weakly compact base,*

the function μ_C *is continuous at* y_0.

Proof. Let $y_0 \in \text{int}(Min_C + \mathcal{K})$. We start by proving the lower semicontinuity of μ_C at y_0. By Proposition 2.1,

$$y_0 = \eta_0 + k_0, \quad \eta_0 \in Min_C, \quad k_0 + \mu_C(y_0)B_Y \subset \mathcal{K}. \tag{3}$$

Take any $\varepsilon > 0$ and $v \in \delta B_Y$, where $\delta = \min\{\varepsilon, \mu_C(y_0)/2\}$. By (3),

$$y_0 + v = \eta_0 + k_0 + v, \quad \eta_0 \in Min_C, \quad k_0 + v \in \mathcal{K}.$$

Moreover, since $v + (\mu_C(y_0) - \|v\|)B_Y \subset \mu_C(y_0)B_Y$, we get

$$\mu_C(y_0 + v) \geq \mu_C(y_0) - \|v\| > \mu_C(y_0) - \varepsilon,$$

which proves the lower semicontinuity of μ_C at y_0.

To show the upper semicontinuity of μ_C at y_0 suppose, on the contrary, that for a certain $\overline{\varepsilon} > 0$ and each $\delta > 0$ there would be $v_\delta \in \min\{\delta, \mu_C(y_0)\}B_Y$ such that $\mu_C(y_0 + v_\delta) \geq \mu_C(y_0) + \overline{\varepsilon}$. This would mean that for each v_δ there would be a representation

$$y_0 + v_\delta = \overline{k} + \overline{\eta}, \quad \overline{k} \in \mathcal{K}, \quad \overline{\eta} \in Min_C, \tag{4}$$

where $y_0 = \overline{k} + \overline{\eta}$, $\overline{k} + \mu_C(y_0)B_Y \subset \mathcal{K}$, such that

$$cont(\overline{k}) > \mu_C(y_0) + \overline{\varepsilon}, \quad \text{i.e.} \quad \overline{k} + [\mu_C(y_0) + \overline{\varepsilon}]B_Y \subset \mathcal{K}.$$

By (4), $y_0 = \overline{\eta} + k_1$, where $k_1 = \overline{k} - v_\delta$. Since $v_\delta + [\mu_C(y_0) + 1/2\overline{\varepsilon}]B_Y \subset [\mu_C(y_0) + \overline{\varepsilon}]B_Y$ for $\delta < 1/2\overline{\varepsilon}$, we would get

$$k_1 + [\mu_C(y_0) + 1/2\overline{\varepsilon}]B_Y \subset \overline{k} + [\mu_C(y_0) + \overline{\varepsilon}]B_Y \subset \mathcal{K},$$

contradictory to the definition of $\mu_C(y_0)$. ■

PROPOSITION 2.2 *Let* $Y = (Y, \|\cdot\|)$ *be a normed space. Let* $\mathcal{K} \subset Y$ *be a closed convex pointed cone in* Y *and let* $C \subset Y$ *be a subset of* Y. *Let* $V \subset Y$ *be an open subset of* Y. *Suppose that* (CP) *holds for* $C \cap V$. *If, for any* $y \in Min_{C\cap V} + \mathcal{K}$, *either of the conditions holds:*

(i) $Min_{C\cap V}$ *is weakly compact,*

(ii) $Min_{C\cap V}$ *is* $\mathcal{K}-$*lower bounded and weakly closed and* $\mathcal{K}$ *has a weakly compact base,*

then, for any $\varepsilon > 0$

$$(C \cap V)(\varepsilon) + \delta_{C\cap V}(\varepsilon)B_Y \subset Min_{C\cap V} + \mathcal{K}.$$

Proof. Let $\varepsilon > 0$. By (CP), $(C \cap V)(\varepsilon) \subset Min_{C\cap V} + \mathcal{K}$. By Proposition 2.1, for any $y \in (C\cap V)(\varepsilon)$ we have $y = \eta_y + k_y$, where $\eta_y \in Min_{C\cap V}$, $k_y + \mu_{C\cap V}(y)B_Y \subset \mathcal{K}$. Consequently, $y + \delta_{C\cap V}(\varepsilon)B_Y \subset Min_{C\cap V} + \mathcal{K}$. ■

3. Hőlderian calmness of minimal points for cones with nonempty interior

Let $\Gamma : U \rightrightarrows Y$ be a set-valued mapping. The set-valued mapping $M : U \rightrightarrows Y$, defined as

$$M(u) = Min_{\Gamma(u)}$$

is called *the minimal point multifunction*. Here we formulate conditions for Hölder calmness of M.

THEOREM 3.1 *Let $Y = (Y, \|\cdot\|)$ and $U = (U, \|\cdot\|)$ be normed spaces. Let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y. Let $\Gamma : U \rightrightarrows Y$ be a set-valued mapping with $\Gamma(u_0)$ convex. Let, for a neighbourhood V of $y_0 \in M(u_0)$, Γ be upper pseudo-Hőlder at (u_0, y_0) with order q_1 and constant L_1, and lower pseudo-Hőlder at (u_0, y_0) with order q_2 and constant L_2. Suppose that one of the following conditions holds:*

(i) $Min_{\Gamma(u_0)\cap V}$ *is weakly compact,*

(ii) $Min_{\Gamma(u_0)\cap V}$ *is $\mathcal{K}$−lower bounded and weakly closed and $\mathcal{K}$ has a weakly compact base.*

If

(A1) $\delta_{\Gamma(u_0)\cap V}(\varepsilon) \geq c \cdot \varepsilon^p$, $c > 0$, *for* $\varepsilon < \varepsilon_0$, $\varepsilon_0 > 0$,

then M is upper pseudo-Hőlder at u_0, i.e.,

$$M(u) \cap V \subset M(u_0) + \left(L_1 + \left(\frac{L_1 + L_2}{c}\right)^{\frac{1}{p}}\right)\|u - u_0\|^{\min\{q_1, \frac{\min\{q_1, q_2\}}{p}\}} \cdot B_Y$$

for all u in some neighbourhood of u_0.

Proof. By assumptions, there is a positive κ such that

$\Gamma(u_0) \cap V \subset \Gamma(u) + L_2\|u - u_0\|^{q_2} B_Y$, and

$\Gamma(u) \cap V \subset \Gamma(u_0) + L_1\|u - u_0\|^{q_1} \cdot B_Y$

$$\begin{aligned}
&\subset \Bigg[Min_{\Gamma(u_0)\cap V} + L_1 \cdot \|u - u_0\|^{q_1} \cdot B_Y \\
&+ \left(\frac{L_1 + L_2}{c}\right)^{\frac{1}{p}} \cdot \|u - u_0\|^{\frac{\min\{q_1, q_2\}}{p}} \cdot B_Y\Bigg] \\
&\cup \Bigg[\left(\Gamma(u_0) \setminus \left(Min_{\Gamma(u_0)\cap V} + \left(\frac{L_1 + L_2}{c}\right)^{\frac{1}{p}} \|u - u_0\|^{\frac{\min\{q_1, q_2\}}{p}} \cdot B_Y\right)\right) \\
&+ L_1 \cdot \|u - u_0\|^{q_1} \cdot B_Y\Bigg], [-2pt]
\end{aligned}$$

whenever $\|u - u_0\| < \kappa$. We claim that

$$\begin{aligned}
&M(u) \cap V \cap \Bigg[\Bigg(\Gamma(u_0) \setminus \Bigg(Min_{\Gamma(u_0)\cap V} \\
&+ \left(\frac{L_1 + L_2}{c}\right)^{\frac{1}{p}} \|u - u_0\|^{\frac{\min\{q_1, q_2\}}{p}} \cdot B_Y\Bigg)\Bigg) + L_1 \|u - u_0\|^{q_1} \cdot B_Y\Bigg] = \emptyset \quad (*)
\end{aligned}$$

for $\|u-u_0\|<\kappa$, or equivalently

$$M(u)\cap V\cap\Bigg[\Bigg(\Gamma(u_0)\cap V\setminus\Bigg(Min_{\Gamma(u_0)\cap V}$$
$$+\left(\frac{L_1+L_2}{c}\right)^{\frac{1}{p}}\|u-u_0\|^{\frac{\min\{q_1,q_2\}}{p}}\cdot B_Y\Bigg)\Bigg)+L_1\|u-u_0\|^{q_1}\cdot B_Y\Bigg]=\emptyset.$$

Let us take any $y\in\Gamma(u)\cap V$, such that $y=\gamma+b_1$, where

$$\gamma\in\Gamma(u_0)\cap V\setminus\left(Min_{\Gamma(u_0)\cap V}+\left(\frac{L_1+L_2}{c}\right)^{\frac{1}{p}}\|u-u_0\|^{\frac{\min\{q_1,q_2\}}{p}}\cdot B_Y\right),$$

and $b_1\in L_1\|u-u_0\|^{q_1}\cdot B_Y$. By Proposition 2.2,

$$\gamma=\eta_\gamma+k_\gamma,\quad \eta_\gamma\in Min_{\Gamma(u_0)\cap V},$$
$$k_\gamma+\delta_{\Gamma(u_0)\cap V}\left(\left(\frac{L_1+L_2}{c}\right)^{\frac{1}{p}}\|u-u_0\|^{\frac{\min\{q_1,q_2\}}{p}}\right)\cdot B_Y\subset\mathcal{K}.$$

By the lower pseudo-Hölder property of Γ, $\eta_\gamma=\gamma_1+b_2$, $\gamma_1\in\Gamma(u)$, $b_2\in L_2\|u-u_0\|^{q_2}\cdot B_Y$. In view of the assumptions, by chosing κ small enough, we obtain $\delta_{\Gamma(u_0\cap V}((\frac{L_1+L_2}{c})^{\frac{1}{p}}\|u-u_0\|^{\frac{\min\{q_1,q_2\}}{p}})\geq(L_1+L_2)\|u-u_0\|^{\min\{q_1,q_2\}}$, and consequently

$$y-\gamma_1=\gamma+b_1-\eta_\gamma+b_2=\eta_\gamma+k_\gamma+b_1-\eta_\gamma+b_2$$
$$\subset k_\gamma+(L_1+L_2)\|u-u_0\|^{\min\{q_1,q_2\}}\cdot B_Y$$
$$\subset k_\gamma+\delta_{\Gamma(u_0)\cap V}\left(\left(\frac{L_1+L_2}{c}\right)^{\frac{1}{p}}\|u-u_0\|^{\frac{\min\{q_1,q_2\}}{p}}\right)\cdot B_Y\subset\mathcal{K}.$$

By this, $y\notin M(u)\cap V$, and (*) follows. Hence, by (1)

$$M(u)\cap V\subset M_{\Gamma(u_0)\cap V}+L_1\cdot\|u-u_0\|^{q_1}\cdot B_Y$$
$$+\left(\frac{L_1+L_2}{c}\right)^{\frac{1}{p}}\cdot\|u-u_0\|^{\frac{\min\{q_1,q_2\}}{p}}\cdot B_Y$$
$$\subset M(u_0)+\left(L_1+\left(\frac{L_1+L_2}{c}\right)^{\frac{1}{p}}\right)\|u-u_0\|^{\min\{q_1,\frac{\min\{q_1,q_2\}}{p}\}}\cdot B_Y,$$

for $\|u-u_0\|<\kappa$, which completes the proof. ■

By Remark 2.1, the assumption $\delta(\varepsilon)\geq c\varepsilon^p$ implies that $\operatorname{int}\mathcal{K}\neq\emptyset$. The convexity of $\Gamma(u_0)$ allows us to make use of (1) and can be replaced by any other condition ensuring (1); in the case $V=Y$ the convexity of $\Gamma(u_0)$ is not necessary and can be omitted (see Bednarczuk, 2002).

4. Weak containment property and its characterizing function

As we have noted the assumption $(A1)$ of Theorem 3.1 might hold true only when $\operatorname{int} \mathcal{K} \neq \emptyset$. However, in some important spaces, standard cones of nonnegative elements have empty interiors. We propose to treat such cases via dual cones.

Let Y be a Hausdorff topological vector space with topological dual Y^*. Let $\mathcal{K} \subset Y$ be a closed convex cone in Y. The cone $\mathcal{K}^* \subset Y^*$,

$$\mathcal{K}^* = \{f \in Y^* \mid f(k) \geq 0 \quad \text{for all} \quad k \in \mathcal{K}\}$$

is the dual to $\mathcal{K}$. The quasi-interior of $\mathcal{K}^*$ (see Jahn, 1986) is given as

$$\mathcal{K}^{*i} = \{f \in Y^* \mid f(y) > 0 \text{ for all } y \in \mathcal{K} \setminus \{0\}\}.$$

Clearly, $\mathcal{K}$ is based if and only if $\mathcal{K}^{*i} \neq \emptyset$. Necessary and sufficient conditions ensuring $\mathcal{K}^{*i} \neq \emptyset$ are given in Gallagher (1995), Lemma 2.1 and Dauer, Gallagher (1990), Proposition 2.1.

If $\operatorname{int} \mathcal{K}$ is nonempty and $e \in \operatorname{int} \mathcal{K}$, then $\Theta = \{f \in \mathcal{K}^* \mid f(e) = 1\}$ is a base of $\mathcal{K}^*$. On the other hand, $\mathcal{K}^{*i}$ is always based, and for any $y_0 \in \mathcal{K} \setminus \{0\}$, the set $\Theta^{*i} = \{f \in \mathcal{K}^{*i} \mid f(y_0) = 1\}$ is a base of $\mathcal{K}^{*i}$.

The bidual cone $\mathcal{K}^{**}$,

$$\mathcal{K}^{**} = \{y \in Y \mid f(y) \geq 0 \text{ for } f \in \mathcal{K}^*\},$$

is convex and weakly closed and in locally convex spaces $\mathcal{K} = \mathcal{K}^{**}$ if and only if $\mathcal{K}$ is convex and weakly closed (see Theorem 12.C of Holmes, 1975). The quasi-interior of $\mathcal{K}$ (see Peressini, 1967, Schaefer, 1971, Krasnoselskii, Lifschitz, Sobolev, 1985, Bakhtin, 1985) is given as

$$\mathcal{K}^i = \{k \in \mathcal{K} \mid f(k) > 0 \text{ for } f \in \mathcal{K}^* \setminus \{0\}\}.$$

In locally convex space, if $\operatorname{int} \mathcal{K} \neq \emptyset$, then $\operatorname{int} \mathcal{K} = \mathcal{K}^i$. $\mathcal{K}^i$ is nonempty if and only if $\mathcal{K}^*$ is based (see Lemma 2.1 of Gallagher, 1995). We refer to any base Θ^* of $\mathcal{K}^*$ of the form

$$\Theta^* = \{f \in \mathcal{K}^* \mid f(y_0) = 1\}, \quad y_0 \in \mathcal{K}^i \tag{5}$$

as a standard base.

Example 4.1 (see Gallagher, 1995, Krasnoselskii, Lifschitz, Sobolev, 1985, Peressini, 1967, Schaefer, 1971)

1. *Let $Y = R^m$, $\mathcal{K} \subset Y$ be a closed convex pointed cone. For any convex subset A, $cor(A)$ coincides with the topological interior of A. Hence, eg., for the cone $\mathcal{K} = \{(y_1, y_2) \mid y_1 \geq 0 \ y_1 = y_2\}$ we get $\mathcal{K}^* = \{(f_1, f_2) \mid f_2 \geq$*

2. *For any* $p \in [1, +\infty)$ *consider the sequence space* ℓ^p, *of sequences* $s = \{s_i\}$ *with real terms,*
$$\ell^p = \Big\{ s = \{s_i\} \,\Big|\, \sum_{i=1}^{\infty} |s_i|^p < +\infty \Big\},$$
with the natural ordering cone
$$\ell^p_+ = \{s = \{s_i\} \in \ell^p \mid s_i \geq 0\}.$$
The ordering cone ℓ^p_+ *has empty topological interior and empty algebraic interior,* $cor(\ell^p_+) = \emptyset$. *But*
$$(\ell^p_+)^i = \{s = \{s_i\} \in \ell^p \mid s_i > 0\}.$$
3. *For any* $p \in [1, +\infty)$, *consider the space of all p-th Lebesgue integrable functions* $f : \Omega \to R$ *with the natural ordering cone*
$$L^p_+ = \{f \in L^p \mid f(x) \geq 0 \text{ almost everywhere on } \Omega\}.$$
The topological interior $\mathrm{int}(L^p_+)$ *and* $cor(L^p)$ *are both empty but* $\mathcal{K}^i \neq \emptyset$. *To see this recall that*
$$(L^p_+)^i = \Big\{ f \in L^p \,\Big|\, \int_\Omega f g \, d\mu > 0 \text{ for all } g \in L^q_+ \setminus \{0\} \Big\},$$
$\frac{1}{p} + \frac{1}{q} = 1$, *and*
$$(L^p_+)^i = \{f \in L^p \mid f(x) > 0 \text{ almost everywhere on } \Omega\}.$$

We have the following Proposition.

Proposition 4.1 *Let* Y *be a locally convex topological vector space and let* $\mathcal{K} \subset Y$ *be a closed convex cone with* $\mathcal{K}^{*i} \neq \emptyset$. *Then*

(i) $\mathcal{K}^i \subset \mathcal{K} \setminus \{0\}$,

(ii) $w - * - cl\mathcal{K}^{*i} \subset \mathcal{K}^*$.

(iii) $\mathcal{K} = \{y \in Y \mid f(y) \geq 0 \text{ for all } f \in \mathcal{K}^{*i}\}$,

(iv) $w - cl\{y \in Y \mid f(y) > 0 \text{ for all } f \in \mathcal{K}^* \setminus \{0\}\} \subset \mathcal{K}$.

Proof. (i) follows from the fact that in a locally convex space $\mathcal{K} = \{y \in Y \mid f(y) \geq 0 \text{ for all } f \in \mathcal{K}^*\}$.

(ii) Since $\mathcal{K}^{*i} \subset \mathcal{K}^*$ and $\mathcal{K}^*$ is weakly $-*-$ closed, we get $w - * - cl\mathcal{K}^{*i} \subset \mathcal{K}^*$.

(iii) If $k \in \mathcal{K} \setminus \{0\}$, then $f(k) > 0$ for any $f \in \mathcal{K}^{*i}$, which proves that $\mathcal{K} \subset \{y \in Y \mid f(y) \geq 0 \text{for all } f \in \mathcal{K}^{*i}\}$. The inclusion $\{y \in Y \mid f(y) \geq 0 \text{ for all } f \in \mathcal{K}^{*i}\} \subset \mathcal{K}$ is proved in Dauer, Gallagher (1990), Lemma 5.5.

(iv) Since $\mathcal{K}$ is weakly closed, $w - cl\mathcal{K}^i \subset \mathcal{K}$. ■

Let $C \subset Y$ be a subset of a normed space $(Y, \|\cdot\|)$ and let $\mathcal{K}^*$ has a base Θ^*.

Definition 4.1 *The* weak containment property (WCP) *holds for* C *with respect to* Θ^* *if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for each* $y \in C(\varepsilon)$ *one can find* $\eta_y \in Min_C$ *satisfying*

$$\theta^*(y - \eta_y) \geq \delta \quad \text{for each } \theta^* \in \Theta^*$$

Note that if $y - \eta_y$ satisfies (6), then $y - \eta_y \in \mathcal{K}^i$. It has been shown by Peressini (1967, sec. 4.4) that in the spaces ℓ^∞, ℓ^p, $L^p(\Omega)$, $p \geq 1$, the quasi-interior $\mathcal{K}_+^i$ of the positive cone $\mathcal{K}_+$ coincides with the set of weak order units (see Peressini, 1967, p. 184), i.e., for any $y_0 \in \mathcal{K}_+^i$ and any $y \in \mathcal{K}_+$, $y \neq 0$, there exists $z \in \mathcal{K}_+$, $z \neq 0$, such that $z \leq y_0$ and $z \leq y$. For the general result in order complete vector lattices see Schaefer (1971), Th. 7.7.

In general, (WCP) depends upon base. In the sequel we give a characterization of bases for which (WCP) holds.

Now we define functions characterizing weak containment property. The *dual cone containment function* $dcont_{\Theta^*} : \mathcal{K} \to R_+$ is defined as

$$dcont_{\Theta^*}(k) = \inf\{\theta^*(k) \mid \theta^* \in \Theta^*\}.$$

Let $C \subset Y$ be a subset of Y. The function $\nu_C : Min_C + \mathcal{K} \to R_+$ given as

$$\nu_C(y) = \sup\{dcont_{\Theta^*}(y - \eta_y) \mid \eta_y \in Min_C \cap (y - \mathcal{K})\}$$

is the *rate of weak containment of an element $y \in Y$ with respect to C and $\mathcal{K}$*. The function $d_C : R_+ \to \overline{R} = R \cup \{\pm\infty\}$, given as

$$d_C(\varepsilon) = \inf\{\nu_C(y) \mid y \in C(\varepsilon)\}$$

is the *rate of weak containment* of a set C with respect to $\mathcal{K}$ and Θ^*.

Let $y_0 \in \mathcal{K}^i$. Consider the standard base

$$\Theta^* = \{\theta^* \in \mathcal{K}^* \mid \theta^*(y_0) = 1\}.$$

For any $k \in \mathcal{K}$,

$$dcont_{\Theta^*}(k) = \inf\{\theta^*(k) \mid \theta^*(y_0) = 1,\ \theta^* \in \mathcal{K}^*\}, \tag{7}$$

is an *infinite-dimensional linear programming problem*. By duality theory (see e.g. Barbu, Precupanu 1986, Ch. 3, par. 3, p. 233), it is the dual to the problem

$$\sup\{r \mid k - r \cdot y_0 \in \mathcal{K}\}, \tag{8}$$

where r is a real number, $r \in R$ (compare also Barbu, Precupanu, 1986, Ch. 3, Th. 3.4., p. 235). Since $r_0 = 0$ is feasible for (8), by Proposition 2.1, Ch. 3, p. 197 of Barbu, Precupanu (1986), we have

$$0 \leq \sup\{r \mid k - r \cdot y_0 \in \mathcal{K}\} \leq \inf\{\theta^*(k) \mid \theta^*(y_0) = 1,\ \theta^* \in \mathcal{K}^*\}. \tag{9}$$

Suppose now that for a given $k \in \mathcal{K}$

$$\inf\{\theta^*(k) \mid \theta^*(y_0) = 1,\ \theta^* \in \mathcal{K}^*\} = \overline{r} \geq 0.$$

Hence, for any $\theta^*(y_0) = 1$, $\theta^* \in \mathcal{K}^*$, we have $\theta^*(k) \geq \overline{r}$, which entails that $k - \overline{r}y_0 \in \mathcal{K}$ and which proves that

$$\sup\{r \mid k - r \cdot y_0 \in \mathcal{K}\} = \inf\{\theta^*(k) \mid \theta^*(y_0) = 1,\ \theta^* \in \mathcal{K}^*\}. \tag{10}$$

The function

$$q(k) = \sup\{r > 0 \mid r^{-1}k \in y_0 + \mathcal{K}\},$$

has been also considered in other context (see Namioka, 1957). It is superlinear, and the graph of q,

$$Graph(q) = \{(k, r) \mid q(k) \geq r\}$$

is a cone in $Y \times R$.

Now the question arises when the optimal value $\overline{r}$ is nonzero. Clearly, if, for any $y_0 \in \mathcal{K}^i$ and any $k \in \mathcal{K}^i$, it would be $r > 0$ such that $k - ry_0 \in \mathcal{K}^i$, then $\mathcal{K}^i \subset \mathrm{cor}_{\mathcal{K}^i \cup (-\mathcal{K}^i)}(\mathcal{K}^i)$, i.e., each $k \in \mathcal{K}^i$ belongs to the *core of* $\mathcal{K}^i$ *relative to* $\mathcal{K}^i \cup (-\mathcal{K}^i)$. It is easy to point out examples when $\overline{r} = 0$.

EXAMPLE 4.2 *Let $p > 1$, $Y = \ell^p$, $\mathcal{K} = \ell^p_+$. As we observed before*

$$(\ell^p_+)^i = \{(s_i) \in \ell^p \mid s_i > 0 \textit{ for each } i \in N\}.$$

By taking $y_0 = (\frac{1}{i^2})$, and $k_0 = (\frac{1}{i^3})$, we see that for any $r > 0$ there exists an index I such that

$$\frac{1}{i^3} - r\frac{1}{i^2} < 0 \quad \textit{for} \quad i > I,$$

and hence $\overline{r} = 0$.

Now we can rewrite (WCP) property for a set C as follows: for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $y \in C(\varepsilon)$ there exists $\eta_y \in Min_C$ such that

$$y - \eta_y \in \delta \cdot y_0 + \mathcal{K}. \tag{11}$$

PROPOSITION 4.2 *Let $(Y, \|\cdot\|)$ be a normed space and let $A \subset Y$ be a subset of Y. Let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y and let $\mathcal{K}^*$ be its dual with a base Θ^*. The following conditions are equivalent:*

(i) *(WCP) holds for A,*
(ii) *$d(\varepsilon) > 0$ for each $\varepsilon > 0$.*

Proof. $(i) \rightarrow (ii)$. Take any $\varepsilon > 0$ and $y \in A(\varepsilon)$. By (WCP), there exist $\delta > 0$ and $\eta_y \in Min_A$ such that

$$dcont_{\Theta^*}(y - \eta_y) \geq \delta.$$

Hence, $\nu_A(y) \geq \delta$, and $d(\varepsilon) \geq \delta > 0$.

$(ii) \rightarrow (i)$. Let $d(\varepsilon) = \alpha > 0$. For each $y \in A(\varepsilon)$

and consequently,

$$\inf_{\theta^* \in \Theta^*} \theta^*(y - \eta_y) > \alpha/2,$$

for some $\eta_y \in Min_A \cap (y - \mathcal{K})$, i.e., (WCP) holds. ∎

PROPOSITION 4.3 *Let $(Y, \|\cdot\|)$ be a normed space and let $C \subset Y$ be a subset of Y. Let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y and let $\mathcal{K}^* \subset Y^*$ be its dual cone with a base Θ^*.*

For any $y \in Min_C + \mathcal{K}$, if $Min_C \cap (y - \mathcal{K})$ is weakly compact, then there exists $\eta_y \in Min_C$ such that

$$\nu_C(y) = dcont_{\Theta^*}(y - \eta_y).$$

Proof. Let $y \in Min_C + \mathcal{K}$. By definition, $dcont_{\Theta^*}(y - \eta) \le \nu_C(y)$, for each $\eta \in Min_C \cap (y - \mathcal{K})$, and for any $\rho > 0$, there exists $\eta_\rho \in Min_C \cap (y - \mathcal{K})$ such that for any $\theta^* \in \Theta^*$

$$\theta^*(y - \eta_\rho) \ge dcont_{\Theta^*}(y - \eta_\rho) > \nu_C(y) - \rho.$$

Since $Min_C \cap (y - \mathcal{K})$ is weakly compact, the net $\{\eta_\rho\}$ contains a weakly convergent subnet and without loss of generality we can assume that the net $\{\eta_\rho\}$ converges weakly to $\eta_y \in Min_C \cap (y - \mathcal{K})$. Since $\mathcal{K}$ is weakly closed, the net $\{k_\rho = y - \eta_\rho\}$ tends to some $k_y \in \mathcal{K}$, and $y = \eta_y + k_y$. Thus,

$$\inf_{\theta^* \in \Theta^*} \theta^*(y - \eta_y) \ge \nu_C(y),$$

which completes the proof. ∎

PROPOSITION 4.4 *Let $\mathcal{K} \subset Y$ be a closed convex pointed cone in a topological vector space Y with $\mathcal{K}^i \neq \emptyset$. If Θ_1^* and Θ_2^* are any two standard bases, with $y_1, y_2 \in \mathcal{K}^i$ such that $y_2 \in (ry_1 + \mathcal{K})$, $r > 0$, then there exists a positive real number β with*

$$dcont_{\Theta_1^*}(k) \ge \beta \cdot dcont_{\Theta_2^*}(k).$$

Proof. Let Θ_1^*, Θ_2^* be any two standard bases, i.e., for $y_1, y_2 \in \mathcal{K}^i$ we have

$$\Theta_1^* = \{\theta_1^* \in \mathcal{K}^* \mid \theta_1^*(y_1) = 1\}$$
$$\Theta_2^* = \{\theta_2^* \in \mathcal{K}^* \mid \theta_2^*(y_2) = 1\}.$$

For any $k \in \mathcal{K}$, and $\theta_1^* \in \Theta_1^*$, there exists $\overline{\theta}_2^* \in \Theta_2^*$ such that

$$\theta_1^*(k) = \theta_1^*(y_2)\overline{\theta}_2^*(k),$$

where $\theta_1^*(y_2) > 0$. Hence,

$$\theta_1^*(k) \ge \theta_1^*(y_2) \inf \overline{\theta}_2^*(k) \ge \theta_1^*(y_2) \inf \theta_2^*(k),$$

and

$$\inf_{\theta_1^*\in\Theta_1^*}\theta_1^*(k)\geq\inf_{\theta_1^*\in\Theta_1^*}\theta_1^*(y_2)\inf_{\theta_2^*\in\Theta_2^*}\theta_2^*(k), \tag{12}$$

Since $y_2\in r\cdot y_1+\mathcal{K}$, by (10), $\beta=\inf_{\theta_1^*\in\Theta_1^*}\theta_1^*(y_2)>0$, and by (12),

$$dcont_{\Theta_1^*}\geq\beta\cdot dcont_{\Theta_2^*}. \quad ■$$

5. Hőlder calmness of minimal points for cones with possibly empty interiors

In the present section we use the weak containment rate function to derive conditions for Hőlder calmness of M.

A subset F of Y^* is *equicontinuous* (Holmes, 1975, 12.D) if for any $\varepsilon>0$ there exists a $0-$neighbourhood W such that $|f(W)|<\varepsilon$ for any $f\in F$. Equivalently, there exists a balanced $0-$neighbourhood W such that $f(W)\leq 1$ for each $f\in F$, i.e., $F\subset(W)^o$. By Banach-Alaoglu theorem, W^o is weakly$-*-$compact. When Y is a normed linear space, $F\subset Y^*$ is equicontinuous if and only if it is bounded in the norm topology of Y^*.

PROPOSITION 5.1 *Let $\mathcal{K}\subset Y$ be a closed convex pointed cone in a normed space Y, $\operatorname{int}\mathcal{K}\neq\emptyset$. Then, for any subset $A\subset Y$, (CP) holds for A if and only if (WCP) holds for A.*

Proof. It follows from Lemma 2.2 of Gallagher (1995) that $\mathcal{K}^*$ has a $w-*-$compact, and hence an equicontinuous base Θ^*. By Proposition 4.4, if (WCP) holds, then it holds for any equicontinuous base. Thus, (WCP) holds for Θ^*. Take any $\varepsilon>0$. By (WCP), there exists $\delta>0$ such that for any $y\in A(\varepsilon)$ there exists $\eta_y\in Min_A$ satisfying

$$\theta^*(y-\eta_y)\geq\delta,\quad\text{for }\theta^*\in\Theta^*.$$

Since Θ^* is equicontinuous, there exists a $0-$neighbourhood O such that $|\theta^*(q)|<\delta/2$ for $q\in O$, $\theta^*\in\Theta^*$. Hence,

$$\theta^*(y-\eta_y)\geq\delta>\theta^*(q),$$

and finally

$$\theta^*(y-\eta_y)+\theta^*(q)\geq\delta/2.$$

Suppose now that (CP) holds A. There exists $\delta>0$ such that for $y\in A(\varepsilon)$ we have

$$y-\eta_y+\delta B\subset\mathcal{K}\quad\text{for some }\eta_y\in Min_A.$$

By taking any $y_0\in\mathcal{K}^i=\operatorname{int}\mathcal{K}$, we get $\overline{\delta}y_0\in\delta B$, $\overline{\delta}>0$, and

$$y-\eta_y-\overline{\delta}y_0\subset\mathcal{K},$$

which means that (WCP) holds. ■

THEOREM 5.1 *Let $Y=(Y,\|\cdot\|)$ and $U=(U,\|\cdot\|)$ be normed spaces. Let $\mathcal{K}\subset Y$ be a closed convex pointed cone in Y, and let $\mathcal{K}^*$ be its dual with an equicontinuous base Θ^*. Let $\Gamma: U\rightrightarrows Y$, be a set-valued mapping, with $\Gamma(u_0)$ convex, which is upper pseudo-Hölder of order ℓ_1 with constant L_1 and lower pseudo-Hölder of order ℓ_2 with constant L_2 at $(y_0,u_0)\in graph(\Gamma)$ for a neighbourhood V of y_0. If*

(i) *$d_{\Gamma(u_0)\cap V}(\varepsilon)\geq c\cdot\varepsilon^p$, with $c>0$, for $\varepsilon<\varepsilon_0$, $\varepsilon_0>0$,*

(ii) *$Min_{\Gamma(u_0)\cap V}$ is weakly compact,*

then M is upper pseudo-Hölder at u_0, i.e.,

$$M(u)\cap V\subset M(u_0)+\left(L_1+\left(2\frac{L_1+L_2}{c}\right)^{\frac{1}{p}}\right)\|u-u_0\|^{\min\{\ell_1,\frac{\min\{\ell_1,\ell_2\}}{p}\}}\cdot B_Y.$$

for all u in some neighbourhood of u_0.

Proof. In this proof we follow the same reasoning as in the proof of Theorem 3.1. Using the same notation we only need to show that under our assumptions, for $\|u-u_0\|<\kappa$

$$M(u)\cap V\cap\Bigg[\Bigg(\Gamma(u_0)\cap V\setminus\Bigg(Min_{\Gamma(u_0)\cap V}+\left(2\frac{L_1+L_2}{c}\right)^{\frac{1}{p}}\|u-u_0\|^{\frac{\min\{\ell_1,\ell_2\}}{p}}\cdot B_Y\Bigg)\Bigg)+L_1\|u-u_0\|^{\ell_1}\cdot B_Y\Bigg]=\emptyset,\ (*)$$

To this aim take any

$$y\in\Gamma(u)\cap V\cap\Bigg[\Bigg(\Gamma(u_0)\cap V\setminus\Bigg(Min_{\Gamma(u_0)\cap V}+\left(2\frac{L_1+L_2}{c}\right)^{\frac{1}{p}}\|u-u_0\|^{\frac{\min\{\ell_1,\ell_2\}}{p}}\cdot B_Y\Bigg)\Bigg)+L_1\|u-u_0\|^{\ell_1}\cdot B_Y\Bigg],$$

for $\|u-u_0\|<\kappa$. We have $y=\gamma+b_1$, where $\gamma\in\Gamma(u_0)\cap V\setminus(Min_{\Gamma(u_0)\cap V}+(2\frac{L_1+L_2}{c})^{\frac{1}{p}}\|u-u_0\|^{\frac{\min\{\ell_1,\ell_2\}}{p}}\cdot B_Y)$, $b_1\in L_1\|u-u_0\|^{\ell_1}\cdot B_Y$.

Since Θ^* is equicontinuous we can assume that $\theta^*(b)\leq 1$, for each $\theta^*\in\Theta^*$, and $b\in B_Y$. Hence, for each $b\in L_1\|u-u_0\|^{\ell_1}\cdot B_Y$ we have

$$-L_1\|u-u_0\|^{\ell_1}\leq\theta^*(b)\leq L_1\|u-u_0\|^{\ell_1}.$$

By Proposition 4.3, there exists $\eta_\gamma\in Min_{\Gamma(u_0)\cap V}$ satisfying

$$\theta^*(\gamma-\eta_\gamma)\geq\nu(\gamma)=\inf_{\theta^*\in\Theta^*}\theta^*(\gamma-\eta_\gamma)\geq d_{\Gamma(u_0)\cap V}(\varepsilon)\geq c\cdot\varepsilon^p\quad\text{for }\varepsilon<\varepsilon_0,$$

for each $\theta^*\in\Theta^*$. By the lower pseudo-Hölder continuity of Γ, $\eta_\gamma=\gamma_1+b_2$,

$$\theta^*(y-\gamma_1) = \theta^*(y-\gamma) + \theta^*(\gamma - \eta_\gamma) + \theta^*(\eta_\gamma - \gamma_1)$$
$$\geq -L_1\|u-u_0\|^{\ell_1} - L_2\|u-u_0\|^{\ell_2}$$
$$+d_{\Gamma(u_0)\cap V}\left(\left(2\frac{L_1+L_2}{c}\right)^{\frac{1}{p}}\|u-u_0\|^{\frac{\min(\ell_1,\ell_2)}{p}}\right)$$
$$\geq -(L_1+L_2)\|u-u_0\|^{\min\{\ell_1,\ell_2\}} + 2(L_1+L_2)\|u-u_0\|^{\min\{\ell_1,\ell_2\}} > 0.$$

Consequently, $f(y-\gamma_1) \geq 0$ for any $f \in \mathcal{K}^*$, and $y - \gamma_1 \in \mathcal{K}$, which proves $(*)$ and completes the proof. ■

As in Theorem 3.1, the inequality $d_{\Gamma(u_0)\cap V} \geq c\varepsilon^p$ is assumed to hold only for ε close to zero, and the convexity of $\Gamma(u_0)$ is needed only to ensure the inclusion $Min_{\Gamma(u_0)\cap V} \subset Min_{\Gamma(u_0)}$.

EXAMPLE 5.1 *Let $K \subset R^n$ be a convex closed cone in R^n with empty interior. Then $K^* \subset R^n$ has no base since the set $K^T = \{y \in K^* \mid y \cdot x = 0$ for each $x \in K\}$ is a nontrivial linear subspace contained in K^*. This shows that the above Theorem cannot be applied to finite-dimensional case.*

EXAMPLE 5.2 *Let $Y = c_0$ be the space of all real sequences that converge to zero with the usual positive cone $\mathcal{K} = (c_0)_+$. Then $(c_0)_+$ has no interior point, and $\mathcal{K}^*$ is the usual positive cone in the space ℓ^1, $\mathcal{K}^* = (\ell^1)_+$. The set of sequences $\{\xi_n\} \subset \ell^1$ such that $\sum \xi_n = 1$ is a base for $\mathcal{K}^*$ that is bounded and closed in the norm topology.*

The above example shows that in some spaces, for standard cones $\mathcal{K}$ of nonnegative elements there is $\operatorname{int}\mathcal{K} = \emptyset$, and $\mathcal{K}^*$ has a bounded base. This, however, is not the case for the space $L^p(\Omega)$ where the nonnegative cone has empty interior and does not possess a bounded base.

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