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Critical points for vector-valued functions

by

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Abstract: This paper contains a mountain pass theorem for continuous mappings, defined on a complete metric space and taking values in a real Banach space, ordered by a closed convex cone. We use the concept of critical point introduced by Degiovanni, Lucchetti and Ribarska, and we furnish a variant of their result, allowing for a localization both of the critical point and of the critical value.

Keywords: weak slope, critical point, Palais-Smale condition, mountain pass theorem, Pareto optimum.

1. Introduction

For a long time the search for minima/maxima, and more generally for critical points, has been a central issue in mathematics and applications, like in physics, for instance. Much more recently, motivated mainly by applications to economics, the case of functionals taking values in a Banach space ordered by a cone, and the associated vector minima/maxima problems have been considered. Instead, in this setting, critical point theory has been paid less attention, with the exceptions of some papers by Smale (1973–1976b), proposing an extension of the Morse theory, a paper by Malivert (1982), providing an extension of the Ljusternik-Schnirelman category, and the recent one by Degiovanni, Lucchetti and Ribarska (2002), where a deformation lemma, a mountain pass theorem

Here we work in the same setting as in Degiovanni, Lucchetti and Ribarska, that we describe now. We are given a continuous function $f : X \to Y$, where X is a complete metric space and Y a Banach space, ordered by a closed convex cone P. It is also given a subset P_0 of P, not containing zero. The notion of critical point for f depends on the set P_0, which has essentially the meaning of providing the effective descent directions for the function f. In the scalar case, usually one takes P as the positive reals, and P_0 the half line $[1, \infty)$.

In this paper we prove another mountain pass theorem, under the assumption that P_0 is of the form $P_0 = e + P$, for some $0 \neq e \in P$. This allows us to locate the value of the critical point. Furthermore, we provide also a localization of the critical point in the domain space X, in the spirit of the analogous result for the scalar case, given by Ghoussoub and Preiss (1989). Our proof relies on applying a scalar mountain pass theorem to a suitable real-valued function g associated with the given function f. We finally present some consideration on the connections between the critical points of f and g.

2. Some preliminary definitions and notions

Throughout this paper (X, d) will stand for a metric space and $(Y, \|\cdot\|)$ will denote a real Banach space. We denote by P a proper nonempty closed convex cone in Y, and by P_0 a nonempty closed convex subset of P not containing the origin. As a typical example, one may think of Y as the Euclidean space $\mathbb{R}^n$, with P and P_0 the positive cone in $\mathbb{R}^n$ and the set $\{y = (y_1, \dots, y_n) : y_i \geq 1\}$. Similarly, in the more general case when Y is a function space over some set T, the above choices are $P := \{f \in Y : f(t) \geq 0 \text{ for every } t \in T\}$ and $P_0 := \{f \in Y : f \in h + P\}$, for some given nonzero function $h \in P$.

In this setting, Degiovanni, Lucchetti and Ribarska (2002) propose the following definition of *weak slope* for a vector-valued function. Let $f : X \to Y$ be a continuous function, and denote by $B(x, \delta)$, as usual, the open ball in X centered at $x \in X$ with radius $\delta > 0$.

DEFINITION 2.1 *The* weak slope *of f at $x \in X$ with respect to P_0 is the supremum of $\sigma \in [0, \infty)$ so that there exist $\delta > 0$ and a continuous mapping $\eta : B(x, \delta) \times [0, \delta] \to X$ such that for every pair $(x', t) \in B(x, \delta) \times [0, \delta]$ the following are true:*

(a) $d(\eta(x', t), x') \leq t$,

(b) $f(\eta(x', t)) \in f(x') - \sigma t P_0 - P$.

We denote this supremum by $|d_{P_0} f|(x)$.

It is easily seen that the weak slope $|d_{P_0} f|(\cdot)$ is a lower semicontinuous function in X.

In the sequel the set P_0 is assumed to be of the form $P_0 = e + P$, for some nonzero element $e \in P$. In this case, as one could see, an equivalent formulation (b′) $f(\eta(x', t)) \in f(x') - \sigma te - P$.

In the scalar case $Y = \mathbb{R}$, when we take $P = [0, \infty)$ and $P_0 := [1, \infty)$, this definition agrees with the definition of weak slope given by Degiovanni and Marzocchi (1994), see also Corvellec, Degiovanni and Marzocchi (1993) and Ioffe and Schwartzman (1996), and is simply denoted by $|df|(x)$ (for more details see also Canino and Degiovanni, 1995). It is well-known that in this case the weak slope is a generalization (for functions which are not differentiable) of the norm of the derivative. Therefore, in our more general setting the following definition again given in Degiovanni, Lucchetti and Ribarska (2002) is natural:

DEFINITION 2.2 *A point $x \in X$ is said to be* critical *for f with respect to P_0 if* $|d_{P_0} f|(x) = 0$.

We continue by giving two more definitions: a subset C of Y is called *invariant* if $C = C - P$. Along the paper we will often use, without special remark, the following elementary monotonic property of a given invariant set C: if e is an element of the cone P then for any $\alpha, \beta \in \mathbb{R}$, with $\alpha < \beta$, we have $C + \alpha e \subset C + \beta e$.

Further, a set $F \subset Y$ is said to be *reachable from $A \subset Y$ with respect to P_0* if there exists $t \geq 0$ so that $F - tP_0 \subset A$. An equivalent assertion of the last definition is that for some $t \geq 0$ we have:

$$F \subset \bigcap_{y \in P_0} (A + ty).$$

If the set P_0 is of the special form $e + P$ for some non zero $e \in P$, then, if A is invariant we have:

$$\bigcap_{y \in P_0} (A + ty) = A + te.$$

We end this preliminary section by introducing some more notation as well as an appropriate scalar mountain pass theorem which will be used later. (T, S) will be a fixed pair of compact sets: by this we mean that T is a non-empty compact metric space and S is a non-empty closed subset of T. Given a complete metric space (Z, d) and a continuous mapping $\psi : S \to Z$ we denote by $\Gamma(Z, \psi)$ the following set:

$$\Gamma(Z, \psi) := \{p : T \to Z : p \text{ is continuous }, \quad p|S = \psi\}.$$

We will consider on $\Gamma(Z, \psi)$ the usual uniform metric ρ: for $p_1, p_2 \in \Gamma(Z, \psi)$, $\rho(p_1, p_2) := \sup\{d(p_1(t), p_2(t)) : t \in T\}$. Then, $(\Gamma(Z, \psi), \rho)$ is a complete metric space.

Finally, we recall a version of a scalar mountain pass theorem, that one can find in Conti (1994). Below, as usual, $d(\cdot, A)$ denotes the distance function in

THEOREM 2.1 *Let (Z, d) be a complete metric space and $\psi : S \to Z$ be continuous. Suppose that $g : Z \to \mathbb{R}$ is a continuous function and set*

$$c := \inf_{p \in \Gamma(Z,\psi)} \max_{t \in T} g(p(t)).$$

Let L be a non-empty closed subset of the set $\{z \in Z : g(z) \geq c\}$, such that, for all $p \in \Gamma(Z, \psi)$ we have:

$$p(T) \cap L \neq \emptyset \quad \wedge \quad \psi(S) \cap L = \emptyset.$$

Then, for any $\varepsilon > 0$ there exists $z_\varepsilon \in Z$ with the following properties:

1. $d(z_\varepsilon, L) \leq 2\varepsilon$;
2. $c - (1/2)\varepsilon^2 \leq g(z_\varepsilon) \leq c + 2\varepsilon^2$;
3. $|dg|(z_\varepsilon) \leq 3\varepsilon$.

We mention that the classical mountain pass theorem takes as the set T the interval $[0, 1]$ and as S its endpoints 0 and 1. In this case $\Gamma(Z, \psi)$ is the set of all continuous paths in Z connecting $z_0 := \psi(0)$ with $z_1 := \psi(1)$. On the other hand, considering the above more general setting allows getting theorems as the saddle point theorem, see Rabinowitz (1986). This is the reason why we shall make use here of a more general formulation, though we do not insist in furnishing a vector version of the saddle point theorem.

3. A mountain pass theorem

In this section we will establish a mountain pass theorem for vector-valued functions. Here we make a less general assumption on the set P_0 than in Degiovanni, Lucchetti and Ribarska (2002), but on the other hand we get a more precise localization both of the critical point x and of its value. Moreover, we do not use a deformation lemma as it was done in Degiovanni, Lucchetti and Ribarska (2002), but we appeal to a scalar mountain pass theorem, based on the use of the Ekeland variational principle.

Before establishing our theorem, we need to introduce some more notions. Let us, as above, be given a complete metric space (X, d), a real Banach space Y, a closed convex cone P in Y, and the set $P_0 = e + P$, for some nonzero element $e \in P$. Furthermore, let us be given a continuous function $f : X \to Y$. We provide now a Palais-Smale condition suitable for our vector-valued setting. To introduce such a condition, let C be an invariant closed subset of Y, L be a non-empty closed subset of X and c be a real number.

DEFINITION 3.1 *A sequence $\{x_n\} \subset X$ is said to be a* Palais-Smale sequence *for f (with respect to L, C and c), denoted by $(PS)_{LC,c}$-sequence, if*

1. $|d_{P_0} f|(x_n) \to 0$;
2. $d(x_n, L) \to 0$;

The function f satisfies the $(PS)_{LC,c}$-condition if every $(PS)_{LC,c}$-sequence has a cluster point.

From condition 1, and the fact that the weak slope is lower semicontinuous, every cluster point of a $(\mathrm{PS})_{LC,c}$-sequence is a critical point for f. Moreover, since C is closed and invariant we have

$$\bigcap_{\varepsilon>0}(C+(c+\varepsilon)e)\setminus(\operatorname{Int}C+(c-\varepsilon)e)=C\setminus\operatorname{Int}C+ce,$$

and consequently every cluster point x of a $(\mathrm{PS})_{LC,c}$-sequence, is also such that $f(x)\in(C\setminus\operatorname{Int}C)+ce$.

Finally, let the pair of compact sets (T,S) be given, let F be a closed invariant subset of Y such that $f(X)\cap F\neq\emptyset$, let $\psi:S\to f^{-1}(F)$ be a continuous function and set $\Gamma:=\Gamma(f^{-1}(F),\psi)$.

Now, we are ready to formulate and prove our mountain pass theorem:

THEOREM 3.1 *Let (X,d) be a complete metric space, Y a real Banach space with ordering closed convex cone P and let $P_0=e+P$, for some nonzero element $e\in P$. Let $f:X\to Y$ be a continuous function. Suppose F is a closed invariant set and the pair of compact sets (T,S) and the mapping ψ are as above. Let C be an invariant closed convex set in Y with nonempty interior* $\operatorname{Int}C$. *Suppose, moreover, the following conditions are true:*

1. *$F\cap f(X)$ is reachable from $\operatorname{Int}C$ and $\Gamma\neq\emptyset$;*
2. *There is $a\in\mathbb{R}$ such that $\forall p\in\Gamma\;\exists t\in T: f(p(t))\notin\operatorname{Int}C+ae$;*
3. *Set $c:=\sup\{a\in\mathbb{R}: a$ fulfills condition 2$\}$ and suppose there is a closed subset L of X, such that, for all $p\in\Gamma$,*
$$f(L)\cap(\operatorname{Int}C+ce)=\emptyset\quad\wedge\quad p(T)\cap L\neq\emptyset\quad\wedge\quad\psi(S)\cap L=\emptyset;$$
4. *f satisfies the $(PS)_{LC,c}$–condition.*

Then, f possesses a critical point $x\in L$ so that $f(x)\in((C\setminus\operatorname{Int}C)+ce)\cap F$.

Proof. We start with some remarks. First, the set $\operatorname{Int}C$ is invariant. This, together with condition 1 of the theorem, shows that the number c in 3 above is well-defined and finite. Next, the following monotonic property holds:

$$(C+\alpha e)\cap f(X)\cap F\subset\operatorname{Int}C+\beta e,$$

if $\beta>\alpha$. Let us prove it. Let $\beta>\alpha$ and take any point $y\in(C+\alpha e)\cap f(X)\cap F$. Then $y-\alpha e\in C$ and since $f(X)\cap F$ is reachable from $\operatorname{Int}C$ we have $y-te\in\operatorname{Int}C$ for some $t\geq0$. We may think that $t>\beta$. Then the point $y-\beta e$ is on the segment $[y-te,y-\alpha e[$. Since C is convex, this entails $y-\beta e\in\operatorname{Int}C$.

We divide now the proof into several steps.

Step 1. Define the function $g:X\to[-\infty,\infty]$:

$$g(x):=\inf\{\alpha\in\mathbb{R}: f(x)\in(C+\alpha e)\},\ x\in X,$$

We shall consider the restriction of the function g to the set $f^{-1}(F)$. Since the set $F \cap f(X)$ is reachable from $\operatorname{Int} C$, for every $x \in f^{-1}(F)$, there exists at least one α satisfying the property in the brackets. Now, we claim that, for no $y \in Y$, C can contain the line $y + \lambda e$, $\lambda \in \mathbb{R}$. Suppose the contrary, i.e. the existence of $y \in Y$ such that $y + \lambda e \in C$, for all $\lambda \in \mathbb{R}$. Take an element $x \in f^{-1}(F)$ such that $f(x) \notin \operatorname{Int} C + ae$, for some $a \in \mathbb{R}$: the existence of such an element is guaranteed by assumption 2. As $F \cap f(X)$ is reachable from $\operatorname{Int} C$, there is $t \geq 0$ such that $f(x) - te \in \operatorname{Int} C$. By convexity of $\operatorname{Int} C$ and the fact that $y + \lambda e \in C$ for all $\lambda \in \mathbb{R}$, it follows that $f(x) - te + \lambda e \in \operatorname{Int} C$, for all $\lambda \in \mathbb{R}$. But this implies $f(x) \in \operatorname{Int} C + ae$: a contradiction. Thus, $g(x) > -\infty$ and finally g is real valued on $f^{-1}(F)$. Observe also that, for $x \in f^{-1}(F)$, clearly the infimum in the definition of g is attained.

We shall get the existence of a critical point for f by finding a critical point of g, restricted to the set $f^{-1}(F)$, with the help of the scalar mountain pass theorem (Theorem 2.1).

Step 2. The function $g : f^{-1}(F) \to \mathbb{R}$ is continuous in the induced topology on $f^{-1}(F)$.

Let $x_0 \in f^{-1}(F)$ and $\varepsilon > 0$ be arbitrary. As $f(x_0) \in (C + g(x_0)e) \cap F$, then

$$f(x_0) \in (\operatorname{Int} C + (g(x_0) + \varepsilon)e) \cap F.$$

By the continuity of f there is some open subset U_1 of X containing x_0 so that

$$f(x) \subset (\operatorname{Int} C + (g(x_0) + \varepsilon)e) \cap F \quad \forall x \in U_1 \cap f^{-1}(F).$$

Then

$$g(x) \leq g(x_0) + \varepsilon \ \ \forall x \in U_1 \cap f^{-1}(F). \tag{1}$$

Further, by the definition of g and the monotonicity properties of the sets $C + \alpha e$ we have that $f(x_0) \notin (C + (g(x_0) - \varepsilon)e) \cap F$. Since the latter set is closed in Y and f is continuous we have the existence of an open set U_2 of X containing x_0 and so that $f(x) \notin (C + (g(x_0) - \varepsilon)e) \cap F$ for any $x \in U_2 \cap f^{-1}(F)$. Again by the monotonic properties of the sets $C + \alpha e$ and the definition of g we have that

$$g(x) \geq g(x_0) - \varepsilon \tag{2}$$

for each $x \in U_2 \cap f^{-1}(F)$. Hence, upon putting $U := U_1 \cap U_2$, by (1) and (2) we see that

$$|g(x) - g(x_0)| \leq \varepsilon \ \ \forall x \in U \cap f^{-1}(F).$$

Therefore, g is continuous on $f^{-1}(F)$. The proof of step 2 is completed.

There is nothing to prove if $|d_{P_0}f|(x) = 0$. Then, take some $0 < \sigma < |d_{P_0}f|(x)$ for which there exist $\delta > 0$ and a continuous mapping $\eta : B(x,\delta) \times [0,\delta] \to X$ so that for every $(x',t) \in B(x,\delta) \times [0,\delta]$ we have:

(a) $d(\eta(x',t),x') \le t$,
(b) $f(\eta(x',t)) \in f(x') - \sigma t e - P$.

Let $x' \in B(x,\delta) \cap f^{-1}(F)$. Then $f(x') \in F$ and since F is invariant ($F - P = F$) we conclude that

$$f(x') - \sigma t P_0 \subset F \quad \text{for every } t \ge 0.$$

Thus by (b) we get $f(\eta(x',t)) \in F$ for any $x' \in B(x,\delta) \cap f^{-1}(F)$ and $t \in [0,\delta]$. Therefore, the restriction of η to $B(x,\delta) \cap f^{-1}(F) \times [0,\delta]$ takes its values in $f^{-1}(F)$.

Moreover, take any $(x',t) \in B(x,\delta) \cap f^{-1}(F) \times [0,\delta]$. Then

$$f(x') \in (C + g(x')e) \cap F$$

whence, because C is invariant,

$$f(\eta(x',t)) \in (C + (g(x') - \sigma t)e) \cap F.$$

It follows that

(b′) $g(\eta(x',t)) \le g(x') - \sigma t$.

This, together with (a), shows that $|dg|(x) \ge \sigma$, and since $\sigma < |d_{P_0}f|(x)$ was arbitrary, we can therefore conclude that $|dg|(x) \ge |d_{P_0}f|(x)$. The proof of the third step is completed.

Step 4. Let us now see that we can apply Theorem 2.1 to the function g, to $Z = f^{-1}(F)$ and to the set $L \cap f^{-1}(F)$.

From step 2, we know that g is continuous. Further, we show that $c = \widehat{c} := \inf_{p\in\Gamma} \max_{t\in T} g(p(t))$. Let a be so that for any $p \in \Gamma$ there is $\bar{t} \in T$ with $f(p(\bar{t})) \notin \operatorname{Int} C + ae$. Then, $g(p(\bar{t})) \ge a$, otherwise there would be $\bar{a} < a$ such that $f(p(\bar{t})) \in C + \bar{a}e$ and by the monotonic properties listed in the beginning of the proof we would have $f(p(\bar{t})) \in \operatorname{Int} C + ae$. The contradiction shows that $g(p(\bar{t})) \ge a$ for any $p \in \Gamma$, and therefore, $c \le \widehat{c}$.

Suppose that $\widehat{c} > c$ and take $a \in \mathbb{R}$ with $\widehat{c} > a > c$. By the definition of c there is some $\bar{p} \in \Gamma$ such that $f(\bar{p}(t)) \in C + ae$ for any $t \in T$. But this implies $g(\bar{p}(t)) \le a$ for every $t \in T$. By the definition of $\widehat{c}$ this means $\widehat{c} \le a$, a contradiction. Thus, $c = \widehat{c}$.

Finally, as we assumed that $f(L) \cap (\operatorname{Int} C + ce) = \emptyset$, we see that $L \cap f^{-1}(F) \subset \{x : g(x) \ge c\}$. Hence, we can apply Theorem 2.1, to conclude that, for every $\varepsilon > 0$ there exists $x_\varepsilon \in f^{-1}(F)$ so that:

1. $d(x_\varepsilon, L) \le 2\varepsilon$;
2. $c - (1/2)\varepsilon^2 \le g(x_\varepsilon) \le c + 2\varepsilon^2$;

From step 3, we can conclude that x_ε fulfills also $|d_{P_0}f|(x_\varepsilon) \le 3\varepsilon$. Moreover, condition 2 amounts to saying that

$$f(x_\varepsilon) \in (C + (c + 2\varepsilon^2)e) \setminus (\mathrm{Int}\,(C + (c - (1/2)\varepsilon^2)e)).$$

Now we use this to construct a $(\mathrm{PS})_{LC,c}$-sequence which by the Palais-Smale condition has a cluster point $x \in L$, which is critical for f. Moreover x satisfies the property $f(x) \in (C \setminus \mathrm{Int}\, C) + ce$. Finally, since $x_\varepsilon \in f^{-1}(F)$, then $f(x) \in F$, and the proof of the theorem is completed. ∎

REMARK. When e is an interior point of the cone P, it is easy to see that $c - ae \in \mathrm{Int}\, C$ for every $c \in C$ and $a > 0$. This implies, in particular, that C is reachable from $\mathrm{Int}\, C$, thus a natural choice for the set F could be $F = C + ke$ (k large). However, here it is not assumed that C is reachable from $\mathrm{Int}\, C$. Observe also that, when the cone P has interior points and condition 2 of the theorem is fulfilled, the function g is everywhere finite and continuous. For, if $e \in \mathrm{Int}\, P$, for every $x \in X$ and $c \in C$ there is $a \in \mathbb{R}$ such that $e - \frac{f(x)+c}{a} \in P$, so that $f(x) \in c - P + ae \subset C + ae$, showing that g is finite at x. Continuity follows from the argument used in step 2 of Theorem 3.1.

4. Pareto optima and critical points

We end the paper with some considerations and results related to the mountain pass theorem proved in the previous section. First, let us recall the notion of Pareto optimum, see Luc (1989): let Y be a real Banach space with a positive closed convex cone P which is pointed (the latter means that $p, -p \in P$ is possible only when $p = 0$). Let $f : X \to Y$ be a mapping from X into Y. The point $x_0 \in X$ is said to be a *local Pareto minimum* (resp. maximum) with respect to P if there is a neigbourhood U of x_0 so that $(f(x_0) - P) \cap f(U) = \{f(x_0)\}$ (resp. $(f(x_0) + P) \cap f(U) = \{f(x_0)\}$).

In the scalar case, it is well-known that if a point is a local minimum for a continuous function f, then the weak slope of f at this point is zero. In other words, local minima are critical points with respect to the weak slope. It is the same when the range space is a vector space as it can be easily seen by the second condition (b) in the definition of the weak slope. The situation with the local maximum, however, is different; in a general metric one may have a function with a local maximum, without having the weak slope zero at this point. A simple example is: $X := [0,1]$, $f(x) := x$; when $|df|(1) = 1$. But when X is a finite dimensional Banach space, the answer is positive, as mentioned in Conti and Lucchetti (1995) (in infinite dimensional X the question is still open). Here we see that the same result holds when the range space is a Banach space.

PROPOSITION 4.1 *Let Y be a Banach space ordered by a closed convex pointed cone P, let P_0 be a closed subset of P, not containing zero, and let $f : \mathbb{R}^n \to Y$ be a continuous function. Suppose x_0 is a local Pareto maximum for f with*

Proof. The proof goes as the analogous result in the scalar case. We include it here for convenience of the reader. Suppose, by contradiction, that there are $\sigma > 0$, $\delta > 0$ and η as in the definition of the weak slope. We may think that δ is so small that $(f(x_0) + P) \cap f(B(x_0, \delta)) = \{f(x_0)\}$. Let $\theta : B(x_0, \delta) \to [0, \frac{\delta}{2}]$ be a continuous function such that:

1. $\theta(x_0) > 0$;
2. $\theta(x) = 0$ if $\|x - x_0\| \geq \delta/2$.

Let $h : B(x_0, \delta) \to B(x_0, \delta)$ be defined as $h(x) = \eta(x, \theta(x))$. As $h(x) = x$ if $\|x - x_0\| = \delta$, it follows, by Brower theorem, that there is $\bar{x}$ such that $h(\bar{x}) = x_0$. This leads to the desired contradiction. For, $\eta(\bar{x}, \theta(\bar{x})) = x_0$ forces $\theta(\bar{x}) = 0$. This is readily seen by the fact that

$$f(x_0) = f(\eta(\bar{x}, \theta(\bar{x})) \in f(\bar{x}) - \sigma\theta(\bar{x})P_0 - P$$

and that x_0 is a local Pareto maximum. But this implies $\bar{x} = x_0$ (as $\eta(u, 0) = u$ for any u), which contradicts the relation $\theta(x_0) > 0$. The proof is completed. ■

We showed our main theorem by using an appropriate real valued function (the function g) aimed at finding critical points of the given vector valued function f. A natural question is then what kind of relations exist between the critical points of g and f. Step 3 in the proof shows that actually every critical point of g is automatically critical for f. Can this statement be made more precise? For instance, is a local maximum for g automatically a local Pareto maximum for f? This is the case for local *strict* maxima for g, as it can be easily shown. But in general the answer is negative, as the following example shows.

EXAMPLE 4.1 Let $X = \mathbb{R}^2$, $P = \mathbb{R}^2_+$, $P_0 = (1, 1) + P$, $C = (-1, -1) - P$ and let

$$f(x_1, x_2) = (\min\{0, x_1\}, \min\{0, x_2\}).$$

Then all the points (x_1, x_2) such that $\max\{x_1, x_2\} \geq 0$ are maxima for g, while only the points (x_1, x_2) with $\min\{x_1, x_2\} \geq 0$ are Pareto maxima for f. The points of the form$(0, x_2)$ and $(x_1, 0)$, $x_1, x_2 < 0$, are critical points for f which are not local minima/maxima.

On the other hand, suppose there is a closed convex pointed cone $\widehat{P}$ such that $P \setminus \{0\} \subset \operatorname{Int} \widehat{P}$. Then, if we take as set C, $C := z - \widehat{P}$, for some $z \in X$, in this case local maxima/minima for g do correspond to local vector maxima/minima for f, as the following proposition shows. We do not know if a critical point for g of saddle point type could again correspond to some vector minimum/maximum for f, or it must be something different.

PROPOSITION 4.2 *With the above P, $\widehat{P}$ and C, let $\bar{x}$ be a local maximum/mini-*

Proof. We prove the statement for a maximum, and for easy notation we suppose it is a global maximum. So, suppose $\bar{x}$ is not a maximum for f. Then there is x such that $f(x) \in f(\bar{x}) + P$, and $f(x) \neq f(\bar{x})$. Thus

$$f(\bar{x}) \in f(x) - \operatorname{Int} \widehat{P}.$$

As

$$f(x) \in z - \widehat{P} + g(x)e,$$

then

$$f(\bar{x}) \in z - \widehat{P} + g(x)e - \operatorname{Int} \widehat{P} \subset \operatorname{Int} C + g(x)e.$$

It follows that $g(\bar{x}) < g(x)$: a contradiction. ■

Finally, in the following example we see that a Pareto maximum for f does not need to be a local maximum for g. Even more, we exhibit a point with weak slope zero for f which is not critical for the associated real-valued function g.

EXAMPLE 4.2 Let $X = \mathbb{R}^2$, $P = \mathbb{R}^2_+$, $P_0 = (1,1) + P$, let $C = (-1,-1) - P$ and

$$f(x_1, x_2) = (x_1, \min\{0, x_2\} + \min\{0, -x_1\}).$$

It is not difficult to show that $f(X) = \{(x_1, x_2) : x_2 \leq \min\{0, -x_1\}\}$ and thus that $(0,0)$ is a Pareto maximum for f. A direct calculation, or reference to Proposition 4.1, show that actually $|d_{P_0} f|(0,0) = 0$. However, g does not have a critical point at $(0,0)$. For, it can be shown without much effort that $g(x_1, x_2) = 1 + \max\{x_1, x_2\}$ and that $|dg|(0,0) = 1$.

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