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Calculus of second-order subdifferentials in infinite dimensions

by

Boris S. Mordukhovich

Department of Mathematics, Wayne State University,
Detroit, MI 48202, e-mail: boris@math.wayne.edu

Abstract: We study two types of second-order subdifferentials of extended-real-valued functions on Banach spaces that are important for applications in variational analysis, especially to sensitivity issues and second-order optimality conditions. The main concern of the paper is to derive extended sum and chain rules for these subdifferentials in the case of Asplund and general Banach spaces, which provide the basis for further theory and applications.

Keywords: variational analysis, nonsmooth functions, second-order subdifferentials, sum and chain rules, Banach and Asplund spaces.

1. Introduction

Second-order generalized differentiation of nonsmooth functions is a rapidly growing area of analysis, especially related to its variational aspects, optimization, and sensitivity under perturbations. There is a variety of second-order generalized differential constructions useful in optimization and variational analysis. The book of Rockafellar and Wets (1998) and Bonnans and Shapiro (2000) contain systematic expositions and references on second-order theories of generalized differentiation and their applications to optimization-related problems.

The classical analysis offers the two possibilities of defining the second-order derivatives: via derivatives of derivatives and via Taylor-like expansions of the original function. It is well known that these two approaches are generally not equivalent. Both of them have counterparts in nonsmooth analysis, where various second-order constructions are defined in these ways; see the books mentioned above and their references. Regarding the "derivative-of-derivative" approach, there are various possibilities of defining the second-order derivatives of nonsmooth functions depending on what is used as an analogue of the first-order

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derivative and what is further employed for the differentiation/approximation of first-order constructions.

Motivated by applications to sensitivity analysis in nonsmooth optimization, a notion of "second-order subdifferential" was introduced in Mordukhovich (1992) for extended-real-valued functions on finite-dimensional spaces. This construction was defined as the coderivative of the first-order subdifferential mapping and can be viewed as a realization of the *dual* "derivative-of-derivative" approach in nonsmooth analysis, since the first-order subdifferential is a natural counterpart of the classical gradient for nonsmooth functions while the coderivative provides a dual-space approximation of the set-valued subgradient mapping; see Section 2 for more details.

The second-order subdifferential and associated constructions were successfully applied to a range of problems in optimization and variational analysis. This particularly includes: the study of robust Lipschitzian stability of solutions maps to parametric variational and hemivariational inequalities in Mordukhovich (1994b, 1994b); complete characterizations of strong regularity for variational inequalities over convex polyhedra in Dontchev and Rockafellar (1996); second-order characterizations of stable optimal solutions to nonsmooth optimization problems in Poliquin and Rockafellar (1998) and Levy, Poliquin and Rockafellar (2000); necessary optimality conditions for mathematical programs with equilibrium constraints in Outrata (1999, 2000), Treiman (1999), Ye (2000), Ye and Ye (1997), and Zhang (1994); sensitivity analysis for mechanical equilibria in Mordukhovich and Outrata (2001), etc.

This paper deals with extensions of the second-order subdifferential to functions defined on Banach spaces. We propose two extensions depending on what kind of coderivatives is applied to the first-order subdifferential mapping. Our main concern is to develop the basic calculus (sum and chain) rules for the second-order subdifferentials important for the theory and applications. Known results in this directions are available only in finite-dimensional spaces; see Mordukhovich and Outrata (2001). We extend these results to the case of infinite-dimensional spaces and obtain two kinds of calculus rules. One part of the calculus results hold in arbitrary Banach spaces imposing stronger requirements on functions and mappings involved in compositions. The other part requires the Asplund structure of the spaces in question (in particular, their reflexivity) imposing essentially more general assumptions on functions and mappings. Note that some of the important results presented below are new even in finite dimensions.

The rest of the paper is organized as follows. Section 2 contains basic definitions and properties of the second-order subdifferentials and related constructions used in the sequel. Section 3 is devoted to second-order sum rules. In the final Section 4 we present the main results of the paper on chain rules for the second-order subdifferentials.

Throughout the paper we use standard notation of variational analysis; see

$F: X \rightrightarrows X^*$ between a Banach space X and its topological dual,

$$\operatorname*{Lim\,sup}_{x \to \bar{x}} F(x) := \big\{x^* \in X^* \,\big|\, \exists \text{sequences } \ x_k \to \bar{x} \text{ and } x_k^* \xrightarrow{w^*} x^* \\ \text{with } x_k^* \in F(x_k) \text{ for all } k \in \mathbb{N}\big\}$$

denotes the *sequential Painlevé–Kuratowski upper (outer) limit* with respect to the norm topology in X and the weak* topology in X^*. Depending on context, the symbols $x \xrightarrow{\Omega} \bar{x}$ and $x \xrightarrow{\varphi} \bar{x}$ for sets Ω and functions φ mean, respectively, that $x \to \bar{x}$ with $x \in \Omega$ and $x \to \bar{x}$ with $\varphi(x) \to \varphi(\bar{x})$.

2. Basic definitions and some properties

In this section we introduce the basic second-order constructions of our study, discuss some of their properties, and present the necessary preliminaries used in what follows. We begin with the first-order constructions of generalized normals to sets, coderivatives of set-valued mappings, and subdifferentials of extended-real-valued functions. Regarding these objects, the reader can find more details and references in the books of Mordukhovich (1988) and Rockafellar and Wets (1998) for finite-dimensional spaces and in the papers of Mordukhovich (1997) and Mordukhovich and Shao (1996a, 1996b) in infinite dimensions.

We start geometrically with the *set of ε-normals* to $\Omega \subset X$ at $x \in \Omega$ defined by

$$\widehat{N}_\varepsilon(x;\Omega) := \left\{x^* \in X^* \,\middle|\, \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x\rangle}{\|u - x\|} \le \varepsilon\right\} \tag{2.1}$$

for any subset Ω of a Banach space and any $\varepsilon \ge 0$. As usual, we put $\widehat{N}_\varepsilon(x;\Omega) = \emptyset$ if $x \notin \Omega$ and denote (2.1) by $\widehat{N}(x;\Omega)$ for $\varepsilon = 0$. The *basic normal cone* to Ω at $\bar{x} \in \Omega$ is defined via the sequential Painlevé-Kuratowski limit

$$N(\bar{x};\Omega) := \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ \varepsilon \downarrow 0}} \widehat{N}_\varepsilon(x;\Omega), \tag{2.2}$$

that is, $x^* \in N(\bar{x};\Omega)$ if and only if there are sequences $\varepsilon_k \downarrow 0$, $x_k \to \bar{x}$, and $x_k^* \xrightarrow{w^*} x^*$ such that $x_k \in \Omega$ and $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k;\Omega)$ for all $k \in \mathbb{N}$. If Ω is locally closed and X is an *Asplund space* (i.e., such a Banach space on which every convex continuous functions is generically Fréchet differentiable, in particular, any reflexive space; see Phelps, 1993 for more information), then one can equivalently put $\varepsilon = 0$ in (2.2) due to Theorem 2.9 from Mordukhovich and Shao (1996a). Note that, despite the nonconvexity of this normal cone, it enjoys a fairly rich calculus in both finite and infinite dimensions.

Considering a set-valued mapping $F: X \rightrightarrows Y$ between Banach spaces, we define its *ε-coderivative* $\widehat{D}_\varepsilon^* F(x,y): Y^* \rightrightarrows X^*$ at (x,y) through the set of ε-normals (2.1) to the graph by and denote it by $\widehat{D}^*F(x,y)(y^*)$ when $\varepsilon = 0$. We may omit y in (2.3) for $(x,y) \in \operatorname{gph} F$ if F is single-valued at x. Based on (2.3), we now define two limiting coderivatives of F at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ depending on the sequential convergence (weak* or norm) in Y^*; in X^* we always use the weak* convergence. The *normal coderivative* of F at $(\bar{x},\bar{y})$ is a multifunction $D_N^*(\bar{x},\bar{y}): Y^* \rightrightarrows X^*$ with the values

$$\begin{aligned} D_N^*F(\bar{x},\bar{y})(\bar{y}^*) &:= \operatorname*{Lim\,sup}_{\substack{(x,y)\to(\bar{x},\bar{y}) \\ y^* \xrightarrow{w^*} \bar{y}^* \\ \varepsilon\downarrow 0}} \widehat{D}_\varepsilon^*F(x,y)(y^*) \\ &= \left\{ x^* \in X^* \,\middle|\, (x^*,-y^*) \in N((\bar{x},\bar{y}); \operatorname{gph} F) \right\}, \end{aligned} \tag{2.4}$$

while the *mixed coderivative* is defined by

$$D_M^*F(\bar{x},\bar{y})(\bar{y}^*) := \operatorname*{Lim\,sup}_{\substack{(x,y)\to(\bar{x},\bar{y}) \\ y^* \to \bar{y}^* \\ \varepsilon\downarrow 0}} \widehat{D}_\varepsilon^*F(x,y)(y^*). \tag{2.5}$$

Obviously, $D_M^*F(\bar{x},\bar{y})(\bar{y}^*) \subset D_N^*F(\bar{x},\bar{y})(\bar{y}^*)$ for all $y^* \in Y^*$, and both coderivatives agree when $\dim Y < \infty$ (in this case we denote them by $D^*F(\bar{x},\bar{y})$). However, (2.5) may be strictly smaller than (2.4) even for single-valued Lipschitzian mappings with values in Hilbert spaces that are Fréchet differentiable at the point in question; see Example 2.9 in Mordukhovich and Shao (1998). Note that $\varepsilon > 0$ may be equivalently removed from the definitions (2.4) and (2.5) if both X and Y are Asplund and F is locally closed-graph.

Let $\varphi: X \to \overline{\mathbb{R}} := [-\infty,\infty]$ be an extended-real-valued function finite at $\bar{x}$, and let $E_\varphi: X \rightrightarrows \mathbb{R}$ be the epigraphical multifunction

$$E_\varphi(x) := \{\mu \in \mathbb{R} \mid \mu \geq \varphi(x)\}$$

associated with φ. The (first-order) *basic subdifferential* and *singular subdifferential* of φ at $\bar{x}$ are defined, respectively, by

$$\partial\varphi(\bar{x}) := D^*E_\varphi(\bar{x},\varphi(\bar{x}))(1), \quad \partial^\infty\varphi(\bar{x}) := D^*E_\varphi(\bar{x},\varphi(\bar{x}))(0). \tag{2.6}$$

Recall that $\partial^\infty\varphi(\bar{x}) = \{0\}$ if φ is locally Lipschitzian around $\bar{x}$. If φ is lower semicontinuous (l.s.c.) around $\bar{x}$, then the basic subdifferential in (2.6) admits the representation

$$\partial\varphi(\bar{x}) = \operatorname*{Lim\,sup}_{\substack{x \xrightarrow{\varphi} \bar{x} \\ \varepsilon\downarrow 0}} \widehat{\partial}_\varepsilon\varphi(x)$$

in terms of the ε-subdifferentials

$$\widehat{\partial}_\varepsilon\varphi(x) := \left\{ x^* \in X^* \,\middle|\, \liminf \frac{\varphi(u) - \varphi(x) - \langle x^*, u - x\rangle}{} \geq -\varepsilon \right\}$$

where ε can be omitted if X is Asplund. It is well known that both $\partial\varphi(x)$ and $\widehat{\partial}\varphi(x) := \widehat{\partial}_0\varphi(x)$ are nonempty on dense subsets of their domains, for any l.s.c. function φ, if and only if X is Asplund. Moreover, $\partial\varphi(\bar{x}) \neq \emptyset$ if φ is Lipschitz continuous around $\bar{x}$. Similar characterizations hold in terms of Fréchet normals and basic normals at boundary points of closed sets; see Fabian and Mordukhovich (1998).

The function φ is called *lower regular* at $\bar{x}$ if $\partial\varphi(\bar{x}) = \widehat{\partial}\varphi(\bar{x})$. It happens, in particular, when φ is convex and also for some important classes of non-convex functions; see Rockafellar and Wets (1998), where a stronger notion of subdifferential regularity has been considered in finite dimensions.

Now let us introduce the main objects of our study: *second-order subdifferentials* of extended-real-valued functions on Banach spaces. Generally we construct them as *coderivatives of first-order subdifferentials*. In this paper we apply this scheme to the basic first-order subdifferential defined in (2.6) and the two kinds of coderivatives defined in (2.4) and (2.5).

DEFINITION 2.1 *Let X be a Banach space, let $\varphi: X \to \overline{\mathbb{R}}$ be finite at $\bar{x}$, and let $\bar{y} \in \partial\varphi(\bar{x})$. Then the mapping $\partial^2_N\varphi(\bar{x},\bar{y}): X^{**} \rightrightarrows X^*$ with the values*

$$\partial^2_N\varphi(\bar{x},\bar{y})(u) := (D^*_N\partial\varphi)(\bar{x},\bar{y})(u), \quad u \in X^{**}, \tag{2.7}$$

is called the **normal second-order subdifferential** *of φ at $\bar{x}$ relative to $\bar{y}$. Similarly, the mapping $\partial^2_M\varphi(\bar{x},\bar{y}): X^{**} \rightrightarrows X^*$ with the values*

$$\partial^2_M\varphi(\bar{x},\bar{y})(u) := (D^*_M\partial\varphi)(\bar{x},\bar{y})(u), \quad u \in X^{**}, \tag{2.8}$$

is the **mixed second-order subdifferential** *of φ at $\bar{x}$ relative to $\bar{y}$.*

When X is finite-dimensional, both constructions (2.7) and (2.8) reduce to the second-order subdifferential $\partial^2\varphi(\bar{x},\bar{y})$ introduced in Mordukhovich (1992). The following result shows that, for C^2 (and for slightly more general) functions on arbitrary Banach spaces, values of both second-order subdifferential mappings in Definition 2.1 are singletons coinciding with images of the adjoint operator to the classical second-order derivative.

PROPOSITION 2.2 *Let X be a Banach space, and let $\varphi: X \to \mathbb{R}$ be continuously differentiable around $\bar{x}$. Assume also that the derivative operator $\nabla\varphi: X \to X^*$ is strictly differentiable at $\bar{x}$ with its strict derivative denoted by $\nabla^2\varphi(\bar{x})$. Then one has*

$$\partial^2_M\varphi(\bar{x})(u) = \partial^2_N\varphi(\bar{x})(u) = \{\nabla^2\varphi(\bar{x})^*u\} \text{ for all } u \in X^{**}.$$

Proof. It follows from the proof of Theorem 3.5 in Mordukhovich and Shao (1996b) that

$$D^*_N f(\bar{x})(y^*) = D^*_M f(\bar{x})(y^*) = \{\nabla f(\bar{x})^* y^*\}, \quad y^* \in Y^*$$

for any mapping $f: X \to Y$ strictly differentiable at $\bar{x}$. If $\varphi \in C^1$ around $\bar{x}$, then $\partial\varphi(x) = \{\nabla\varphi(x)\}$ for all x near $\bar{x}$. Applying the above coderivative result to the mapping $f: X \to X^*$ with $f(x) := \nabla\varphi(x)$ and using Definition 2.1, we arrive at the desired conclusion. ■

Next we present useful descriptions of the second-order subdifferentials for locally $C^{1,1}$ functions $\varphi: X \to \mathbb{R}$, i.e., such functions φ that are continuously differentiable around $\bar{x}$ with the derivative $\nabla\varphi$ Lipschitz continuous around this point. The following proposition gives an unconditional formula for the mixed second-order subdifferential (2.8) of any $C^{1,1}$ function on an arbitrary Banach space, while the corresponding representation of the normal one (2.7) holds under additional requirements on X and φ.

According to Mordukhovich and Shao (1996a), $f: X \to Y$ is *strictly Lipschitzian* around $\bar{x}$ if it is Lipschitz continuous around this point and the sequence $\{[f(x_k + t_k h) - f(x_k)]/t_k\}$ admits a norm convergent subsequence whenever $x_k \to \bar{x}$, $t_k \downarrow 0$, and $h \in X$. It is proved by Thibault (1997) that the latter property is actually equivalent to the basic version of his original concept of *compactly Lipschitzian mappings* introduced in Thibault (1978). Note that this class includes, in particular, Fredholm integral operators with Lipschitzian kernels, which are important for applications to optimal control.

PROPOSITION 2.3 *Let X be a Banach space, and let $\varphi: X \to \mathbb{R}$ be a $C^{1,1}$ function around $\bar{x}$. Then*

$$\partial^2_M\varphi(\bar{x})(u) = \partial\langle u, \nabla\varphi\rangle(\bar{x}), \quad u \in X^{**}. \tag{2.9}$$

If, in addition, X is Asplund and $\nabla\varphi$ is strictly/compactly Lipschitzian around $\bar{x}$, then

$$\partial^2_N\varphi(\bar{x})(u) = \partial\langle u, \nabla\varphi\rangle(\bar{x}), \quad u \in X^{**}. \tag{2.10}$$

Moreover, both sets $\partial^2_M\varphi(\bar{x})(u)$ and $\partial^2_N\varphi(\bar{x})(u)$ are nonempty for any $u \in X^{**}$ if $\varphi \in C^{1,1}$ around $\bar{x}$ and X is Asplund.

Proof. It is proved in Theorem 5.2 from Mordukhovich and Shao (1996a) that

$$D^*_N f(\bar{x})(y^*) = \partial\langle y^*, f\rangle(\bar{x}) \text{ for all } y^* \in Y^* \tag{2.11}$$

if $f: X \to Y$ is strictly Lipschitzian around $\bar{x}$ and X is Asplund. This immediately implies (2.10) for $f = \nabla\varphi: X \to X^*$. A slight modification of the latter proof allows us to conclude that the mixed coderivative analogue of (2.11) holds for every Lipschitzian mapping $f: X \to Y$ between Banach spaces. This yields (2.9). The last conclusion of the proposition follows from (2.9) and the fact that $\partial\varphi(\bar{x}) \neq \emptyset$ for any locally Lipschitzian function on an Asplund space; see Corollary 3.9 in Mordukhovich and Shao (1996a). ■

The next result contains an important formula for computing the basic nor-

Banach spaces. Its proof is rather involved and is given in Mordukhovich and Wang (2002).

PROPOSITION 2.4 *Let $f: X \to Y$ be a mapping between Banach spaces, and let $\Lambda \subset Y$ with $\bar{y} := f(\bar{x}) \in \Lambda$. Assume that f is strictly differentiable at $\bar{x}$ with the surjective derivative $\nabla f(\bar{x}): X \to Y$. Then one has*

$$N(\bar{x}; f^{-1}(\Lambda)) = \nabla f(\bar{x})^* N(\bar{y}; \Lambda),$$

where $f^{-1}(\Lambda) := \{x \in X \mid f(x) \in \Lambda\}$.

Finally in this section, let us consider compactness-like properties of the set-valued mappings and extended-real-valued functions that are automatic in finite dimensions while playing an essential role in infinite-dimensional variational analysis. We say that a set-valued mapping $F: X \rightrightarrows Y$ between Banach spaces is *partially sequentially normally compact* (PSNC) at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if for any sequence $(\varepsilon_k, x_k, y_k, x_k^*, y_k^*) \in [0, \infty) \times (\operatorname{gph} F) \times X^* \times Y^*$ satisfying

$$(x_k^*, y_k^*) \in \widehat{N}_{\varepsilon_k}((x_k, y_k); \operatorname{gph} F), \ \varepsilon_k \downarrow 0,$$

$$(x_k, y_k) \to (\bar{x}, \bar{y}), \ \|y_k^*\| \to 0, \ \text{and } x_k^* \xrightarrow{w^*} 0$$

one has $\|x_k^*\| \to 0$ as $k \to \infty$. This property was formulated in Mordukhovich and Shao (1996b) with $\varepsilon_k = 0$ for all $k \in \mathbb{N}$, which is equivalent to the above definition if both spaces X and Y are Asplund and F is locally closed-graph; see also Ioffe (2000) and the references therein for related compactness-like properties, their genesis and characterizations.

The PSNC property obviously holds if X is finite-dimensional. In general Banach spaces this property is ensured by some Lipschitz-like behavior of multifunctions. In particular, F is PSNC at $(\bar{x}, \bar{y})$ if it is partially compactly epi-Lipschitzian at this point in the sense of Jourani and Thibault (1995). Another important property ensuring the PSNC in the Banach space framework is the so-called "pseudo-Lipschitzian" property introduced by Aubin (1984); see Mordukhovich (1997), Rockafellar and Wets (1998), and the references therein for characterizations of the latter property and their applications in finite and infinite dimensions.

Employing the PSNC property in the case of a constant mapping $F(x) \equiv \Omega$, we get the *sequential normal compactness* (SNC) property of a set. We say that a function $\varphi: X \to \overline{\mathbb{R}}$ is *sequentially normally epi-compact* (SNEC) at $\bar{x}$ if its epigraph is SNC at $(\bar{x}, \varphi(\bar{x}))$. It always happens when φ is compactly epi-Lipschitzian in the sense of Borwein and Strojwas (1985); in particular, when either φ is locally Lipschitzian around $\bar{x}$ or X is finite-dimensional.

3. Second-order sum rules

The main concern of this paper is to develop principal calculus (sum and chain) rules for the second-order subdifferentials from Definition 2.1 in infinite-space setting as well as in Asplund spaces. To derive second-order sum and chain rules, we proceed similarly to Mordukhovich (1994a) and Mordukhovich and Outrata (2001) in the case of finite-dimensional spaces and apply calculus rules for the normal and mixed coderivatives to set-valued mappings generated by the basic first-order subdifferential (2.6). In this way we have to restrict ourselves to favorable classes of functions for which the corresponding first-order subdifferential (but not coderivative) calculus rules hold as *equalities*, since neither normal nor mixed coderivatives enjoy monotonicity properties that may allow one to use a more developed inclusion-type subdifferential calculus for (2.6). The results obtained are new in infinite-dimensional spaces; some of them are new even in finite dimensions.

In this section we present *sum rules* for second-order subdifferentials (2.7) and (2.8). The first theorem contains sum rules for two functions provided that one of them is smooth.

THEOREM 3.1 *Let X be a Banach space, let $\varphi_i: X \to \overline{\mathbb{R}}$, $i = 1, 2$, be finite at $\bar{x}$, and let $\bar{y} \in \partial(\varphi_1 + \varphi_2)(\bar{x})$. Suppose that $\varphi_1 \in C^1$ around $\bar{x}$ with $\bar{y}_1 := \nabla\varphi_1(\bar{x})$ while φ_2 is an arbitrary extended-real-valued function with $\bar{y}_2 := \bar{y} - \bar{y}_1 \in \partial\varphi_2(\bar{x})$. The following assertions hold for both normal ($\partial^2 = \partial^2_N$) and mixed ($\partial^2 = \partial^2_M$) second-order subdifferentials.*

(i) *Assume that $\nabla\varphi_1$ is strictly differentiable at $\bar{x}$ with the strict derivative $\nabla^2\varphi_1(\bar{x})$ (this is automatic when $\varphi_1 \in C^2$). Then*

$$\partial^2(\varphi_1 + \varphi_2)(\bar{x}, \bar{y})(u) = \nabla^2\varphi_1(\bar{x})^* u + \partial^2\varphi_2(\bar{x}, \bar{y}_2)(u) \textit{ for all } u \in X^{**}.$$

(ii) *Assume that both X and X^* are Asplund, that the graph of $\partial\varphi_2$ is norm-closed around $(\bar{x}, \bar{y}_2)$, and that either $\varphi_1 \in C^{1,1}$ around $\bar{x}$, or $\partial\varphi_2$ is PSNC at $(\bar{x}, \bar{y}_2)$ and*

$$\partial^2_M\varphi_1(\bar{x}, \bar{y}_1)(0) \cap (-\partial^2_M\varphi_2(\bar{x}, \bar{y}_2)(0)) = \{0\}. \tag{3.1}$$

Then for all $u \in X^{**}$ one has

$$\partial^2(\varphi_1 + \varphi_2)(\bar{x}, \bar{y})(u) \subset \partial^2\varphi_1(\bar{x}, \bar{y}_1)(u) + \partial^2\varphi_2(\bar{x}, \bar{y}_2)(u). \tag{3.2}$$

Proof. It is easy to see that if $\varphi_1 \in C^1$ around $\bar{x}$, then there is a neighborhood U of $\bar{x}$ such that the equality

$$\partial(\varphi_1 + \varphi_2)(x) = \nabla\varphi_1(x) + \partial\varphi_2(x) \text{ for all } x \in U \tag{3.3}$$

holds whenever $\varphi_2: X \to \overline{\mathbb{R}}$. Applying to (3.3) Theorem 3.5 from Mordukhovich and Shao (1996b) and its counterpart for the mixed coderivative, we arrive at the equality sum rule (i) for both second-order subdifferentials in arbitrary Banach spaces.

If the spaces X and X^* are Asplund, we apply to (3.3), in the case when $F_1 = \nabla\varphi_1$ is single-valued, the inclusion sum rule that holds for both normal and mixed coderivatives of set-valued mappings between Asplund spaces under the *mixed* qualification condition

$$D_M^* F_1(\bar{x}, \bar{y}_1)(0) \cap (-D_M^* F_2(\bar{x}, \bar{y}_2)(0)) = \{0\}$$

and the assumption that one of the mappings F_i is PSNC at $(\bar{x}, \bar{y}_i)$, $i = 1, 2$; see Mordukhovich (1997) and Mordukhovich and Shao (1998). In the case of second-order subdifferentials, the latter mixed qualification condition reduces to (3.1). Note that $\partial_M^2 \varphi_1(\bar{x}, \bar{y}_1)(0) = \{0\}$ and $\nabla \varphi_1$ is automatically PSNC at $(\bar{x}, \bar{y}_1)$ if $\varphi_1 \in C^{1,1}$ around $\bar{x}$. Thus we have justified (3.2) under the assumptions made in (ii). ■

Next, let us derive second-order sum rules in the case when both φ_i are nonsmooth. In contrast to Theorem 3.1, we now impose symmetric assumptions on φ_i to ensure the equality in the first-order subdifferential sum rule needed to begin with.

Recall that a set-valued mapping $S: X \rightrightarrows Y$ between Banach spaces is *inner semicontinuous* at $(\bar{x}, \bar{y}) \in \operatorname{gph} S$ if for every sequence $x_k \to \bar{x}$ with $S(x_k) \neq \emptyset$ there is a sequence $y_k \in S(x_k)$ converging to $\bar{y}$. The mapping S is *inner semicompact* at $\bar{x}$ if for every such $x_k \to \bar{x}$ there are $y_k \in S(x_k)$ converging to some $\bar{y} \in S(\bar{x})$ along a subsequence of $k \to \infty$; the requirement $\bar{y} \in S(\bar{x})$ is redundant if the graph of S is closed. Note that the inner semicompactness property always holds if S is closed-graph and locally compact around $\bar{x}$ (locally bounded in finite dimensions).

The two second-order sum rules in the following theorem are distinguished by which of the above properties is imposed on the the multifunction $S: X \times X^* \rightrightarrows X^* \times X^*$ with the values

$$\begin{aligned} &S(x, y) \\ &:= \{(y_1, y_2) \in X^* \times X^* \mid y_1 \in \partial\varphi_1(x),\ y_2 \in \partial\varphi_2(x),\ y_1 + y_2 = y\}. \end{aligned} \tag{3.4}$$

THEOREM 3.2 *Let X and X^* be Asplund spaces, let $\varphi_1: X \to \overline{\mathbb{R}}$, $i = 1, 2$, be l.s.c. around $\bar{x}$, and let $\bar{y} \in \partial(\varphi_1 + \varphi_2)(\bar{x})$. Assume that there is a neighborhood U of $\bar{x}$ such that*

$$\partial^\infty \varphi_1(x) \cap (-\partial^\infty \varphi_2(x)) = \{0\} \ \textit{for all}\ x \in U$$

and that one of the functions φ_i is SNEC at each $x \in U$ (both assumptions are fulfilled when one of φ_i is Lipschitz continuous around $\bar{x}$). Suppose also that both functions φ_i are lower regular at each $x \in U$. Then the following hold, where $\partial^2\varphi$ stands for either (2.7) *or* (2.8).

(i) *Fix $(\bar{y}_1, \bar{y}_2) \in S(\bar{x}, \bar{y})$ in* (3.4) *and suppose that S is inner semicontinuous at $(\bar{x}, \bar{y}, \bar{y}_1, \bar{y}_2)$. Assume also that the graphs of both $\partial\varphi_i$ are norm-closed around $(\bar{x}, \bar{y}_i)$, that one of the mappings $\partial\varphi_i$ is PSNC at $(\bar{x}, \bar{y}_i)$, and that the qualification condition* (3.1) *is fulfilled. Then the sum rule* (3.2) *holds for all*

(ii) *Suppose that S is inner semicompact at $(\bar{x}, \bar{y})$ and that the assumptions in* (i) *are fulfilled for any $(\bar{y}_1, \bar{y}_2) \in S(\bar{x}, \bar{y})$. Then for all $u \in X^{**}$ one has*

$$\partial^2(\varphi_1 + \varphi_2)(\bar{x}, \bar{y})(u) \subset \bigcup_{(\bar{y}_1, \bar{y}_2) \in S(\bar{x}, \bar{y})} [\partial^2\varphi_1(\bar{x}, \bar{y}_1)(u) + \partial^2\varphi_2(\bar{x}, \bar{y}_2)(u)].$$

Proof. By employing the Corollary 3.4 from Mordukhovich and Shao (1996b), we get the equality

$$\partial(\varphi_1 + \varphi_2)(x) = \partial\varphi_1(x) + \partial\varphi_2(x) \text{ for all } x \in U \tag{3.5}$$

under the general assumptions of the theorem. Now following the proof of Theorem 3.1(ii), we apply to (3.5) the coderivative sum rules from Theorem 4.2 in Mordukhovich (1997). In this way we arrive at the conclusions of (i) and (ii) under the assumptions made therein. ∎

4. Second-order chain rules

In this section we obtain several chain rules for the second-order subdifferentials of compositions

$$\varphi(x) := (\psi \circ h)(x) := \psi(h(x)) \tag{4.1}$$

involving inner mappings $h: X \to Z$ between Banach spaces and extended-real-valued outer functions $\psi: Z \to \overline{\mathbb{R}}$. The first result holds with no restrictions on X, Z, and ψ in (4.1), while imposing strong assumptions on the inner mapping.

THEOREM 4.1 *Let X and Z be Banach spaces, let the composition $\varphi = \psi \circ h$ be finite at $\bar{x} \in X$, and let $\bar{z} := h(\bar{x})$ Assume that $h \in C^1$ around $\bar{x}$ with the derivative ∇h strictly differentiable at $\bar{x}$, and that the operator $\nabla h(\bar{x}): X \to Z$ is surjective. Given $\bar{y} \in \partial\varphi(\bar{x})$, we find the unique vector $\bar{v} \in Z^*$ satisfying the relations*

$$\bar{y} = \nabla h(\bar{x})^* \bar{v}, \quad \bar{v} \in \partial\psi(\bar{z}). \tag{4.2}$$

Then one has the inclusion

$$\begin{aligned} &\partial^2\varphi(\bar{x}, \bar{y})(u) \subset \nabla^2\langle\bar{v}, h\rangle(\bar{x})^* u \\ &+ \nabla h(\bar{x})^* \partial^2_N \psi(\bar{z}, \bar{v})(\nabla h(\bar{x})^{**} u), \quad u \in X^{**}, \end{aligned} \tag{4.3}$$

for both second-order subdifferentials $\partial^2 = \partial^2_N$ and $\partial^2 = \partial^2_M$. Moreover, (4.3) *holds as equality for $\partial^2 = \partial^2_N$ if the kernel of $\nabla h(\bar{x})$ is complemented in X, i.e., there is a closed subspace $L \subset X$ with $L \bigoplus \ker \nabla h(\bar{x}) = X$.*

Proof. It is sufficient to prove inclusion (4.3) for $\partial^2 = \partial^2_N$. To furnish this, we follow the proof of Theorem 3.4 in Mordukhovich and Outrata (2001) developed in the case of finite-dimensional spaces, with some changes allowing us to

We begin with the observation that the surjectivity of $\nabla h(\bar{x})$ implies the coderivative and first-order subdifferential chain rules

$$
\begin{aligned}
&D_N^*(F\circ h)(x,y)(y^*) = \nabla h(x)^* D_N^* F(h(x),y)(y^*),\\
&x\in U,\ y\in (F\circ h)(x),\ y^*\in Y^*, &(4.4)\\
&\partial(\psi\circ h)(x) = \nabla h(x)^*\partial\psi(h(x)),\quad x\in U, &(4.5)
\end{aligned}
$$

where $F: Z \rightrightarrows Y$ is an arbitrary multifunction between Banach spaces, $\psi: Z \to \overline{\mathbb{R}}$, and U is a neighborhood of $\bar{x}$. Indeed, (4.4) follows from Proposition 2.4 applied to the mapping $(h, I): X \times Y \to Z \times Y$ with the identity operator $I: Y \to Y$; we obviously have $(h, I)^{-1}(\operatorname{gph} F) = \operatorname{gph}(F \circ h)$. The first-order subdifferential rule (4.5) is a special case of (4.4) for $F = E_\psi: Z \rightrightarrows \mathbb{R}$.

Observe further that due to (4.5) the composite function (4.1) admits the representation

$$\partial\varphi(x) = (f\circ G)(x),\quad x\in U, \tag{4.6}$$

where the mappings $f: X \times Z^* \to X^*$ and $G: X \rightrightarrows X \times Z^*$ are defined by

$$f(x,v) := \nabla h(x)^* v,\quad G(x) := (x, \partial\psi(h(x))). \tag{4.7}$$

Taking into account that f in (4.7) is smooth and the operator $\nabla h(\cdot)$ is surjective near $\bar{x}$, we apply to (4.6) the coderivative chain rule from Theorem 4.6 in Mordukhovich and Shao (1996b) the proof of which ensures the inclusion

$$
\begin{aligned}
&D_N^*(f\circ G)(\bar{x},\bar{y})(u)\\
&\subset D_N^* G(\bar{x},\bar{x},\bar{v})(\nabla^2\langle\bar{v},h\rangle(\bar{x})^* u, \nabla h(\bar{x})^{**}u),\quad u\in X^{**}, &(4.8)
\end{aligned}
$$

under the assumptions made, where $\bar{v}$ is uniquely defined by (4.2). It easily follows from the construction of G in (4.7) and the coderivative definition (2.4) that

$$
\begin{aligned}
&D_N^* G(\bar{x},\bar{x},\bar{v})(x^*,v^*) = x^* + D_N^*(\partial\psi\circ h)(\bar{x},\bar{v})(v^*),\\
&x^*\in X^*,\ v^*\in Z^{**}. &(4.9)
\end{aligned}
$$

Now, using the coderivative chain rule (4.4) for $F = \partial\psi$ and combining it with (4.8) and (4.9), we arrive at (4.3).

If $\nabla h(\bar{x})$ is invertible (i.e., one-to-one in addition to its surjectivity), then the well-known theorem of Leach (1961) ensures that the inverse mapping h^{-1} is locally single-valued and strictly differentiable at $h(\bar{x})$ with the strict derivative $\nabla h(\bar{x})^{-1}$ at this point. Applying inclusion (4.3) to the composition $\psi = \varphi \circ h^{-1}$, we get the opposite inclusion in this case, which justifies the equality in (4.3) for invertible $\nabla h(\bar{x})$. Finally, using the procedure suggested in Exercises 6.7 and 10.7 of Rockafellar and Wets (1998) for first-order chain rules in finite dimensions, one can reduce the general case of surjective $\nabla h(\bar{x})$ with the comequality in (4.3) for $\partial^2 = \partial^2_N$ under the assumptions made, which ends the proof of the theorem. ∎

The next theorem contains second-order subdifferential chain rules for compositions (4.1) that do not require the surjectivity of $\nabla h(\bar{x})$ while imposing more assumptions on the outer function ψ and the spaces in question under first-order and second-order qualification conditions.

THEOREM 4.2 *Let $\varphi = \psi \circ h$ be composition (4.1) of $h: X \to Z$ and $\psi: Z \to \overline{\mathbb{R}}$, where the spaces X, Z, and Z^* are Asplund. Given $\bar{x} \in X$, we assume that $h \in C^1$ around $\bar{x}$ with the derivative ∇h strictly differentiable at $\bar{x}$, that ψ is lower semicontinuous and lower regular around $\bar{z} := h(\bar{x})$, and that h^{-1} is PSNC at $(\bar{z}, \bar{x})$. Assume also that ψ is sequentially normally epi-compact around $\bar{z}$ and that the first-order qualification condition*

$$\partial^\infty \psi(h(x)) \cap \ker \nabla h(x)^* = \{0\} \tag{4.10}$$

is satisfied around $\bar{x}$ (the last two conditions are automatic when ψ is locally Lipschitzian around $\bar{x}$). Then the following assertions hold for both second-order subdifferentials $\partial^2 = \partial^2_N$ and $\partial^2 = \partial^2_M$.

(i) *Given $\bar{y} \in \partial\varphi(\bar{x})$, we assume that the mapping $S: X \times X^* \rightrightarrows Z^*$ with the values*

$$S(x,y) := \{v \in Z^* \mid v \in \partial\psi(h(x)),\ \nabla h(x)^* v = y\} \tag{4.11}$$

is inner semicontinuous at $(\bar{x}, \bar{y}, \bar{v})$ for some $\bar{v} \in S(\bar{x}, \bar{y})$, that the graph of $\partial\psi$ is norm-closed around $(\bar{z}, \bar{v})$, and that the mixed second-order qualification condition

$$\partial^2_M \psi(h(\bar{x}), v)(0) \cap \ker \nabla h(\bar{x})^* = \{0\} \tag{4.12}$$

is satisfied for $v = \bar{v}$. Then, (4.3) holds.

(ii) *Given $\bar{y} \in \partial\varphi(\bar{x})$, we assume that the mapping S in (4.11) is inner semicompact at $(\bar{x}, \bar{y})$ and that the other assumptions in (i) are satisfied whenever $v \in S(\bar{x}, \bar{y})$. Then*

$$\partial^2\varphi(\bar{x},\bar{y})(u) \subset \bigcup_{v \in S(\bar{x},\bar{y})} [\nabla^2\langle v, h\rangle(\bar{x})^* u + \nabla h(\bar{x})^* \partial^2_N \psi(\bar{z}, v)(\nabla h(\bar{x})^{**} u)],$$
$$u \in X^{**}. \tag{4.13}$$

Proof. It suffices to prove (i) for $\partial^2 = \partial^2_N$, which implies the other statements due to the definitions. We begin with application of Corollary 4.5 from Mordukhovich and Shao (1996b) that ensures the first-order chain rule equality (4.5), in some neighborhood U of $\bar{x}$, under the general assumptions of the theorem. This allows us to represent $\partial\varphi$ in the composition form (4.6) with the holds, we conclude from the proof of Theorem 4.6 in Mordukhovich and Shao (1996b) that

$$\begin{aligned}&\partial^2_N\varphi(\bar{x},\bar{y})(u)\subset\nabla^2\langle\bar{v},h\rangle(\bar{x})^*(u)\\&+D^*_N(\partial\psi\circ h)(\bar{x},\bar{v})(\nabla h(\bar{x})^{**}u),\quad u\in X^{**},\end{aligned}\tag{4.14}$$

if the mapping S in (4.11) is inner semicontinuous at $(\bar{x},\bar{y},\bar{v})$. It remains to compute the normal coderivative of the composition $\partial\psi\circ h$ in (4.14). To furnish this, we use Theorem 4.5 from Mordukhovich (1997) that ensures the coderivative chain rule

$$D^*_N(\partial\psi\circ h)(\bar{x},\bar{v})(v^*)\subset\nabla h(\bar{x})^*\circ(D^*_N\partial\psi)(\bar{z},\bar{v})(v^*),\quad v^*\in Z^{**},\tag{4.15}$$

under the PSNC assumption on h^{-1} and the mixed qualification condition

$$(D^*_M\partial\psi)(\bar{z},\bar{v})(0)\cap\ker\nabla h(\bar{x})^*=\{0\}.$$

Now substituting (4.15) into (4.14) and using Definition 2.1, we arrive at the second-order chain rule (4.3) under the qualification condition (4.12). ■

When Z is finite-dimensional (X may be not), some of the assumptions in Theorem 4.2 either are satisfied automatically or can be simplified. In this way we get the following result, where $\partial^2\psi$ stands for the common second-order subdifferential of $\psi:\mathbb{R}^m\to\overline{\mathbb{R}}$ while $\partial^2\varphi$ is the same as in the theorem.

COROLLARY 4.3 *Let $\varphi=\psi\circ h$ with $h:X\to\mathbb{R}^m$, $\psi:\mathbb{R}^m\to\overline{\mathbb{R}}$, and $\bar{y}\in\partial\varphi(\bar{x})$. Assume that X is Asplund, that $h\in C^1$ around $\bar{x}$ with the derivative strictly differentiable at $\bar{x}$, and that ψ is l.s.c. and lower regular around $\bar{z}=h(\bar{x})$ with closed graphs of $\partial\psi$ and $\partial^\infty\psi$ near $\bar{z}$; the latter holds, in particular, when ψ is either continuous or convex. Assume also that* (4.10) *is satisfied at $x=\bar{x}$ and that*

$$\begin{aligned}&\partial^2\psi(\bar{z},v)(0)\cap\ker\nabla h(\bar{x})^*=\{0\}\quad\textit{if}\\&v\in\partial\psi(\bar{z})\quad\textit{with}\quad\nabla h(\bar{x})^*v=\bar{y}.\end{aligned}\tag{4.16}$$

Then one has the second-order chain rule (4.13).

Proof. The SNEC property is automatic for functions on finite-dimensional spaces. Further, one can easily check that if (4.10) holds at $\bar{x}$ while Z is finite-dimensional, it also holds in a neighborhood of $\bar{x}$. Indeed, assuming the contrary and taking into account that $\partial^\infty\psi(\cdot)$ is a cone, we get sequences of $x_k\to\bar{x}$ and $z^*_k\in\partial^\infty\psi(h(x_k))$ with $\nabla h(x_k)^*z^*_k=0$ and $\|z^*_k\|=1$ for all $k\in\mathbb{N}$. Then $z^*\in\partial^\infty\psi(\bar{z})$ with $\nabla h(\bar{x})^*z^*=0$ and $\|z^*\|=1$ for a cluster point z^* of $\{z^*_k\}$ due to the closedness of $\operatorname{gph}\partial^\infty\psi$ near $\bar{z}$; this contradicts (4.10) at $\bar{x}$. Similarly we can check that the mapping $S:X\times X^*\rightrightarrows\mathbb{R}^m$ from (4.11) is always inner semicompact at $(\bar{x},\bar{y})$ when the qualification condition (4.10) is satisfied at $\bar{x}$. Thus we get (4.13) from Theorem 4.2(ii). ■

REMARK 4.4 If both spaces X and Z are finite-dimensional and the second-order qualification condition (4.12) holds, then all the other assumptions of Corollary 4.3 are automatically satisfied for the class of *strongly amenable* functions φ, i.e., for such $\varphi: \mathbb{R}^n \to \overline{\mathbb{R}}$ that are locally represented in the composition form (4.1) with $h \in C^2$ and a proper, l.s.c., convex function $\psi: \mathbb{R}^m \to \overline{\mathbb{R}}$ under the first-order qualification condition (4.10) at $\bar{x}$; see Section 10.F in Rockafellar and Wets (1998). Terry Rockafellar (personal communication) developed another proof of the second-order chain rule (4.13) in this case based on quadratic penalties.

Note that if ψ is $C^{1,1}$ and lower regular around $\bar{z}$, then all the assumptions of Theorem 4.2 involving ψ are automatically satisfied. In this case the second-order chain rule (4.13) reduces to the form (4.3) and can be simplified due to the scalarization formula (2.10). On the other hand, if h is Lipschitz continuous around $\bar{x}$ and ψ is continuously differentiable around $\bar{z}$, then there is a neighborhood U of $\bar{x}$ such that

$$\partial(\psi \circ h)(x) = \partial\langle \nabla\psi(h(x)), h\rangle(x) = D_M^* h(x) \circ \nabla\psi(h(x)), \quad x \in U, \tag{4.17}$$

with no other assumptions on the composition data in (4.1); see Theorem 6.5 in Mordukhovich and Shao (1996a), the proof of which holds for any Banach spaces, and the mixed scalarization formula (2.9). This helps us to evaluate the second-order subdifferentials of compositions involving smooth outer functions and nonsmooth inner mappings. For the formulation of the next theorem it is convenient to use the *second-order coderivative* sets to $h: X \to Z$ at $(\bar{x}, \bar{v}, \bar{y}) \in X \times Z^* \times X^*$ with $\bar{y} \in \partial\langle \bar{v}, h\rangle(\bar{x})$ defined as follows

$$D^2 h(\bar{x}, \bar{v}, \bar{y})(u) := (D^* \partial\langle \cdot, h\rangle)(\bar{x}, \bar{v}, \bar{y})(u), \quad u \in X^{**}, \tag{4.18}$$

where D^* stands for either *normal* ($D^* = D_N^*$, then $D^2 = D_N^2$) or *mixed* ($D^* = D_M^*$, then $D^2 = D_M^2$) coderivative of the mapping $(x, v) \to \partial\langle v, h\rangle(x)$. One easily has

$$\begin{aligned} &D_N^2 h(\bar{x}, \bar{v}, \bar{y})(u) = D_M^2 h(\bar{x}, \bar{v}, \bar{y})(u) \\ &= (\nabla^2\langle \bar{v}, h\rangle(\bar{x})^* u, \nabla h(\bar{x})^{**} u), \quad u \in X^{**}, \end{aligned} \tag{4.19}$$

when h is C^1 around $\bar{x}$ with ∇h strictly differentiable at this point.

THEOREM 4.5 *Let $h: X \to Z$ be a mapping between Banach spaces that is Lipschitz continuous around $\bar{x}$, let $\psi: Z \to \overline{\mathbb{R}}$ be continuously differentiable around $\bar{z} := h(\bar{x})$, and let $\bar{v} := \nabla\psi(\bar{z})$. The following assertions hold for both second-order subdifferentials $\partial^2 = \partial_N^2$ and $\partial^2 = \partial_M^2$ of the composition $\varphi = \psi \circ h$ at $(\bar{x}, \bar{y})$ with $\bar{y} \in \partial\varphi(\bar{x})$, where D^2 stands for the corresponding second-order*

(i) *Assume that h and $\nabla\psi$ are strictly differentiable at $\bar{x}$ and $\bar{z}$, respectively, and that the operator $\nabla^2\psi(\bar{z})\nabla h(\bar{x})\colon X \to Z^*$ is surjective. Then*

$$\partial^2\varphi(\bar{x},\bar{y})(u) = \bigcup_{(x^*,v^*)\in D^2h(\bar{x},\bar{v},\bar{y})(u)} [x^* + \nabla h(\bar{x})^*\nabla^2\psi(\bar{z})^*v^*],$$
$$u \in X^{**}. \tag{4.20}$$

The chain rule (4.20) *also holds, with D^2h computed in* (4.19) *and without the above surjectivity assumption, if ∇h is strictly differentiable at $\bar{x}$.*

(ii) *Assume that $\psi \in C^{1,1}$ around $h(\bar{x})$, that all the spaces X, X^*, Z, and Z^* are Asplund, and that the graph of $(x,v) \to \partial\langle v,h\rangle(x)$ is closed in $X \times Z^* \times X^*$ whenever (x,v) are near $(\bar{x},\bar{v})$. Then*

$$\partial^2\varphi(\bar{x},\bar{y})(u)$$
$$\subset \bigcup_{(x^*,v^*)\in D^2h(\bar{x},\bar{v},\bar{y})(u)} [x^* + D_N^*h(\bar{x})\circ\partial_N^2\psi(\bar{z})(v^*)], \quad u \in X^{**}. \tag{4.21}$$

Moreover, (4.21) *holds for an arbitrary Banach space Z if $\nabla\psi$ is strictly differentiable at $\bar{z}$.*

Proof. Due to the first equality in (4.17) we locally represent $\partial\varphi$ as the composition

$$\partial\varphi(x) = (F\circ g)(x), \quad x \in U, \tag{4.22}$$

of the mappings $F\colon X\times Z^* \rightrightarrows X^*$ and $g\colon X \to X\times Z^*$ defined by

$$F(x,v) := \partial\langle v,h\rangle(x), \quad g(x) := (x, \nabla\psi(h(x))). \tag{4.23}$$

If g is strictly differentiable at $\bar{x}$ with the surjective derivative operator, then

$$D^*(F\circ g)(\bar{x},\bar{y})(u) = \nabla g(\bar{x})^*D^*F(\bar{x},\bar{v},\bar{y})(u), \quad u \in X^{**},$$

for both normal and mixed coderivatives; see (4.4) and Mordukhovich and Wang (2002) for more details. Note that $\nabla^2(\psi\circ h)(\bar{x}) = \nabla^2\psi(\bar{z}))\nabla h(\bar{x})$ under the common assumptions of (i), and that the surjectivity of the latter operator implies the surjectivity of $\nabla g(\bar{x})$. This ensures (4.20) due to the structure of F, g in (4.23) and the second-order constructions in (2.7), (2.8), and (4.18). The last claim in (i) easily follows from the above procedure; this is actually a classical second-order chain rule for strict derivatives.

To prove (ii), we apply to (4.22) the coderivative chain rule from Theorem 4.5 in Mordukhovich (1997) and get

$$D^*(F\circ g)(\bar{x},\bar{y})(u) \subset D_N^*g(\bar{x})\circ D^*F(\bar{x},\bar{v},\bar{y})(u), \quad u \in X^{**}, \tag{4.24}$$

for both normal and mixed coderivatives under the assumptions in the first part of (ii) except that Z may be an arbitrary Banach space. If, in addition, Z is Asplund, one has from the same theorem. Combining this with (4.24), we arrive at (4.21).

Finally, let $\nabla\psi$ be strictly differentiable at $\bar{z}$. Then (4.25) holds in any Banach spaces, which follows from Theorem 4.6 in Mordukhovich and Shao (1996b). This gives the last statement in (ii) and concludes the proof of the theorem. ∎

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