Control and Cybernetics

vol. **31** (2002) No. 3

On $\alpha(\cdot)$-monotone multifunctions and differentiability of strongly $\alpha(\cdot)$-paraconvex functions

by

S. Rolewicz

Institute of Mathematics of the Polish Academy of Sciences,
Śniadeckich 8, 00-950 Warszawa, P.O.Box 137,
e-mail: rolewicz@impan.gov.pl

Abstract: Let (X, d) be a metric space. Let Φ be a family of real-valued functions defined on X. The paper provides the sufficient conditions, warranting that on $\alpha(\cdot)$-monotone multifunction $\Gamma : X \to 2^{\Phi}$ is single-valued and continuous on a weakly angle-small set. As an application it is shown that a strongly $\alpha(\cdot)$-paraconvex (i.e. uniformly approximate convex) function defined on an open convex subset of a Banach space having separable dual is Fréchet differentiable on a residual set.

Keywords: Fréchet Φ-differentiability, strongly paraconvex functions, $\alpha(\cdot)$-monotone multifunctions.

1. Introduction

Let $(E, \|\cdot\|)$ be a separable real Banach space. Let $f(\cdot)$ be a real-valued convex continuous function defined on an open convex subset $\Omega \subset E$. Mazur (1933) proved that there is a subset $A_G \subset \Omega$ of the first Baire category such that on $\Omega \setminus A_G$ the function $f(\cdot)$ is Gateaux differentiable. Asplund (1968) showed that if additionally the space E has the separable dual, then there is a subset $A_F \subset \Omega$ of the first Baire category such that on $\Omega \setminus A_F$ the function $f(\cdot)$ is Fréchet differentiable.

Observe that the results of Mazur and Asplund trivially extend to the so called DC-functions (the functions which can be represented as differences of convex continuous functions), which play an essential role in nonconvex analysis.

The proof of Asplund's result consists of two parts. In the first part one shows that if $f(\cdot)$ is a convex function defined on an open convex set X, which is a subset of a Banach space E, then $f(\cdot)$ has a subgradient at each point of X. The functions with this property are called *subdifferentiable*. In the second part of the proof one shows that for a convex subdifferentiable function $f(\cdot)$ there is a subset $A_F \subset \Omega$ of the first Baire category such that on $\Omega \setminus A_F$ the function

The extension of the second part of the proof to the case of metric spaces was done by the author, Rolewicz (1994) (see also the book by Pallaschke and Rolewicz, 1997). Let (X, d_X) be a metric space. In the paper we shall assume that Φ is a set consisting of real-valued Lipschitz functions defined on X. The properties of weakly Φ-subdifferentiable functions are investigated. In particular, sufficient conditions warranting that each weakly Φ-subdifferentiable function is Fréchet differentiable on a set of the second Baire category, are given. In Banach spaces those results give us an extension of the Asplund theorem to a larger (than convex) class of functions called strongly $\alpha(\cdot)$-paraconvex (or uniformly approximate convex) functions.

The paper is organized as follows.

Section 2 contains the definitions of Φ-convex functions, Φ-subgradients and Φ-subdifferentials in general structures and in metric spaces. Also localizations of those notions are given. In Section 3, $\alpha(\cdot)$-Φ-subgradients and $\alpha(\cdot)$-Φ-subdifferentials are introduced. It is shown that $\alpha(\cdot)$-Φ-subdifferentials are $\alpha(\cdot)$-monotone multifunctions. Section 4 contains extensions of the famous results of Mazur (1933) and Asplund (1968) about the differentiability of convex functions to the case of $\alpha(\cdot)$-Φ-subdifferentiable functions. In Section 5 the notions $\alpha(\cdot)$-paraconvex functions and strongly $\alpha(\cdot)$-paraconvex of real-valued functions defined on convex subsets contained in normed spaces are introduced. The Fréchet differentiability of strongly $\alpha(\cdot)$-paraconvex functions on a residual set is also shown. The relations between strongly $\alpha(\cdot)$-paraconvex functions and uniformly approximate convex functions are discussed. The proof that those two notions are equivalent is presented. As a consequence we show that each uniformly approximate convex function $f(\cdot)$ is Fréchet differentiable on a residual set.

2. Φ-subgradients and Φ-subdifferentials

Let X be an arbitrary set. Let Φ be a family of real-valued functions defined on X. Let $\Phi + \mathbb{R} = \{\phi + c : \phi \in \Phi, c \in \mathbb{R}\}$.

A real-valued function $f(\cdot)$ defined on X is called Φ-*convex* if it can be represented as

$$f(x) = \sup_{\psi \in \Phi_f} \psi(x), \tag{2.1}$$

supremum being taken over a subfamily $\Phi_f \subset \Phi + \mathbb{R}$.

A function $\phi(\cdot) \in \Phi$ is called a Φ-*subgradient* of a function $f(\cdot)$ at a point x_0 if

$$f(x) - f(x_0) \geq \phi(x) - \phi(x_0), \tag{2.2}$$

for all $x \in X$ (see for example Pallaschke and Rolewicz, 1997).

The set of all Φ-subgradients of the function $f(\cdot)$ at the point x_0 is called

Pallaschke and Rolewicz, 1997, Singer, 1997, Rubinov, 2000). If $\partial_\Phi f|_{x_0} \neq \emptyset$ for all $x_0 \in X$ we say that that $f(\cdot)$ is Φ-*subdifferentiable.*

Till now we have not used the fact that X is a metric space. Suppose now that (X, d_X) is a metric space and let Φ be a family of real-valued functions defined on X. There are several Φ and the natural choices of it and of a metric on Φ induced by the metric d_X. The first one is to consider continuous bounded functions and the norm $\|f\|_{\sup} = \sup_{t\in X} |f(t)|$. This approach, however, uses more the topology of the space than the metric itself.

Therefore, we consider another approach, more related to the metric. Namely we shall restrict ourselves to Φ, which is a family of Lipschitz functions. Let $\mathcal{L}^0$ be the space of all Lipschitz functions defined on X. We define on $\mathcal{L}^0$ a quasinorm

$$\|\phi\|_L = \sup_{\substack{x_1,x_2\in X,\\ x_1\neq x_2}} \frac{|\phi(x_1) - \phi(x_2)|}{d_X(x_1,x_2)}. \tag{2.3}$$

Observe that, if $\|\phi_1 - \phi_2\|_L = 0$, then the difference of ϕ_1 and ϕ_2 is a constant function. Therefore, we consider the quotient space $\mathcal{L} = {}^{\mathcal{L}^0}/_{\mathbb{R}}$. The quasinorm $\|\phi\|_L$ induces a norm in $\mathcal{L}$. Since it will not lead to any misunderstanding, this norm will also be denoted by $\|\phi\|_L$ and we shall call it the *Lipschitz norm.* This approach seems to be proper from the point of view of subdifferentiability, since if ϕ is a Φ-subgradient of a function $f(\cdot)$ at a point x_0, then so is $\phi + c$ for all real c. It is not difficult to show that $(\mathcal{L}, \|\phi\|_L)$ is a Banach space. Observe that in the classical case of X being a normed space and Φ consisting of the linear continuous functionals, $\Phi = X^*$, the norm $\|\cdot\|_L$ coincides with the norm of functionals ($\|x^*\|_L = \|x^*\|^*$ for all $x^* \in X^*$).

It is easy to give examples of Φ-convex functions defined on an open set which do not have Φ-subgradients at certain points.

EXAMPLE 2.1 *Let $X = \mathbb{R}$. Let Φ be the set of linear functions with rational coefficients. It is easy to see that a function $f(\cdot)$ is Φ-convex if and only if it is convex. In particular the function $f(x) = x^2$ is Φ-convex. On the other hand it is easy to see that it does not have a Φ-subgradient at any irrational point x_0.*

Of course the class Φ considered in Example 2.1 consists of Lipschitz functions, but ${}^{\Phi}/_{\mathbb{R}}$ is not complete in the topology induced by the Lipschitz norm $\|\cdot\|_L$. The following example shows that the linearity and completeness of ${}^{\Phi}/_{\mathbb{R}}$ are not enough.

EXAMPLE 2.2 *Let $H = \ell^2$. Let $X = \{x = (x_n) : \sum_{n=1}^{\infty} n^2 x_n^2 \leq C\}$, where $C = \sum_{n=1}^{\infty} \frac{1}{n^2}$. Let Φ be the set of linear continuous functionals on H restricted to X. Of course, Φ is linear and complete. Let $f(x) = \sum_{n=1}^{\infty} n^2 x_n^2$. Let $x^0 = (x_n^0) = (\frac{1}{4}, \dots, \frac{1}{n^2}, \dots)$. Of course, $x^0 \in X$. The function $f(\cdot)$ is convex and it* *derivative in the directions e_n at the point x^0 we see that such Φ-subgradient would be of the form $y = (2, 2, \ldots)$. But $y \notin \ell^2$.*

Modifying Example 2.1 we can obtain a metric space X and a family of Lipschitz functions Φ such that there is a Φ-convex function which does not have a Φ-subgradient at any point. Indeed

EXAMPLE 2.3 *Let $X = (-1, 1)$ with the standard metric. Let Φ be the set of linear functions with rational coefficients. Let $f(x) = cx$, where c is irrational. Of course, $f(x)$ is Φ-convex. On the other hand it is easy to see that it does not have a Φ-subgradient at any point $x_0 \in X$.*

For the class Φ given in Example 2.3, $\Phi/_{\mathbb{R}}$ is not complete in the Lipschitz norm. A natural question arises: suppose that X is complete and $\Phi/_{\mathbb{R}}$ is complete in Lipschitz norm. Does there exist for every Φ-convex function $f(\cdot)$ a point x_0 such that $\partial_\Phi f|_{x_0} \neq \emptyset$?

In the case when X is a compact set and Φ consists of continuous functions we have

PROPOSITION 2.4 *Let X be a compact set and Φ consist of continuous functions defined on X. Let $f(\cdot)$ be a continuous Φ-convex function. Then for each $\phi \in \Phi$ there is $x_\phi \in X$ such that $\phi \in \partial_\Phi f|_{x_\phi}$.*

Proof. Since $f(\cdot)$, $\phi(\cdot)$ are continuous and X is compact, there is a point $x_\phi \in X$ such that

$$f(x_\phi) - \phi(x_\phi) = \min_{x \in X}[f(x) - \phi(x)].$$

Thus, for all $x \in X$,

$$f(x) - \phi(x) \geq f(x_\phi) - \phi(x_\phi),$$

which implies

$$f(x) - f(x_\phi) \geq \phi(x) - \phi(x_\phi),$$

i.e. ϕ is a Φ-subgradient of $f(\cdot)$ at x_ϕ. ■

In the case of metric spaces (or even more general topological spaces) we can introduce the notion of a local Φ-subgradient. Namely $\phi \in \Phi$ is called a *local Φ-subgradient of a function $f(\cdot)$ at a point $x_0 \in X$* if there is a neighbourhood $U(\phi, x_0)$ of x_0 such that

$$f(x) - f(x_0) \geq \phi(x) - \phi(x_0) \tag{2.2}$$

for $x \in U(\phi, x_0)$. If a function $f(\cdot)$ has local Φ-subgradients for all $x_0 \in X$ we

3. $\alpha(\cdot)$-Φ-subdifferentials and $\alpha(\cdot)$-monotonicity

Let $\alpha(t)$ be a nondecreasing function mapping the interval $[0,+\infty)$ into $[0,+\infty]$ such that

$$\lim_{t\downarrow 0}\frac{\alpha(t)}{t}=0. \tag{3.1}$$

A function $\phi(\cdot)\in\Phi$ is called an $\alpha(\cdot)$-Φ-*subgradient* of the function $f(\cdot)$ at a point x_0 if for all $x\in X$,

$$f(x)-f(x_0)\geq \phi(x)-\phi(x_0)-\alpha(d(x,x_0)). \tag{3.2}$$

In the particular case $\alpha(t)\equiv 0$ we obtain the definition of Φ-subgradient (see for example Pallaschke and Rolewicz, 1997).

The set of all $\alpha(\cdot)$-Φ-subgradients of the function $f(\cdot)$ at the point x_0 is called the $\alpha(\cdot)$-Φ-*subdifferential* of $f(\cdot)$ at x_0 and it is denoted by $\partial^\alpha_\Phi f|_{x_0}$. In the particular case $\alpha(t)\equiv 0$ we obtain the definition of Φ-subdifferential (see for example Pallaschke and Rolewicz, 1997).

In the case when X is a normed space, $\Phi=X^*$ and $\alpha(t)=t^\gamma$, $1<t\leq 2$, we obtain the definitions of γ-subgradients and γ-subdifferentials introduced by Jourani (1996).

If $\partial^\alpha_\Phi f|_x\neq\emptyset$ for all $x\in X$ we say that $f(\cdot)$ is $\alpha(\cdot)$-Φ-*subdifferentiable.*

Now we shall localize the notions given above. If for a function $\phi(\cdot)\in\Phi$ there is a neighbourhood $U(\phi,x_0)$ of a point x_0 such that (3.2) holds for all $x\in U(\phi,x_0)$ we say that $\phi(\cdot)$ is a *local* $\alpha(\cdot)$-Φ-*subgradient* of $f(\cdot)$ at x_0. The set of all local $\alpha(\cdot)$-Φ-subgradients of function $f(\cdot)$ at x_0 it is called the *local* $\alpha(\cdot)$-Φ-*subdifferential* of $f(\cdot)$ at x_0 and it is denoted by $\partial^{\alpha,\mathrm{loc}}_\Phi f|_{x_0}$. If $\partial^{\alpha,\mathrm{loc}}_\Phi f|_{x_0}\neq\emptyset$ for all $x_0\in X$, we say that $f(\cdot)$ is *locally* $\alpha(\cdot)$-Φ-*subdifferentiable.*

Let, as before, $\alpha(t)$ be a nondecreasing function mapping $[0,+\infty)$ into $[0,+\infty]$ such that (3.1) holds. We say that a multifunction Γ mapping X into 2^Φ is $\alpha(\cdot)$-*monotone* if for all $\phi_x\in\Gamma(x)$, $\phi_y\in\Gamma(y)$ we have

$$\phi_x(x)+\phi_y(y)-\phi_x(y)-\phi_y(x)+\alpha(d(x,y))\geq 0. \tag{3.3}$$

In the particular case $\alpha(t)\equiv 0$ we obtain the definition of *monotone multifunctions* (see for example the book Pallaschke and Rolewicz, 1997).

In the case when X is a normed space, $\Phi=X^*$ and $\alpha(t)=t^\gamma$, $1<t\leq 2$, we obtain the definition of γ-monotone multifunctions introduced by Jourani (1996).

Just from the definitions we trivially obtain

PROPOSITION 3.1 (Rolewicz, 1999). *Let X be a metric space and let Φ be a family of real-valued functions. If a function $f(\cdot)$ is $\alpha(\cdot)$-Φ-subdifferentiable, then its $\alpha(\cdot)$-Φ-subdifferential $\partial^\alpha_\Phi f|_x$ considered as a multifunction of x is $2\alpha(\cdot)$-*

We can localize the notion of $\alpha(\cdot)$-monotone multifunctions. Namely, we say that a multifunction Γ mapping X into 2^{Φ} is *locally $\alpha(\cdot)$-monotone* if for all $x_0 \in X$ there is a neighbourhood U of x_0 such that for all $x, y \in U$, $\phi_x \in \Gamma(x)$, $\phi_y \in \Gamma(y)$ we have

$$\phi_x(x) + \phi_y(y) - \phi_x(y) - \phi_y(x) + \alpha(d(x,y)) \geq 0. \tag{3.4}$$

Since in the definition of local Φ-subgradient the neighbourhood $U(\phi, x_0)$ depends on ϕ, it is easy to construct examples showing that the local $\alpha(\cdot)$-Φ-subdifferential of a function $f(\cdot)$, $\partial_{\Phi}^{\alpha,\mathrm{loc}} f|_x$ need not be a locally $\alpha(\cdot)$-monotone multifunction.

4. Differentiability of $\alpha(\cdot)$-Φ-subdifferentiable functions

We shall say that a function $f(\cdot)$ mapping a metric space (X, d_X) into $\mathbb{R}$ is *Fréchet Φ-differentiable* at a point x_0 if there is a function $\phi \in \Phi$ such that

$$\lim_{x \to x_0} \frac{|[f(x) - f(x_0)] - [\phi(x) - \phi(x_0)]|}{d_X(x, x_0)} = 0. \tag{4.1}$$

The function ϕ will be called a *Fréchet Φ-gradient* of $f(\cdot)$ at x_0. The set of all Fréchet Φ-gradients of $f(\cdot)$ at x_0 is called the *Fréchet Φ-differential* of $f(\cdot)$ at x_0 and it is denoted by $\partial_{\Phi}^{F} f|_{x_0}$.

In general a Φ-subdifferentiable function may not be Fréchet Φ-differentiable at any point, as follows from

EXAMPLE 4.1 *Let $X = \mathbb{R}$ and let $\Phi = \{\phi(x) = -|x - x_0| : x_0 \in \mathbb{R}\}$. It is easy to see that a function $f(\cdot)$ is Φ-convex if and only if it is a Lipschitz function with Lipschitz constant not greater than* 1. *Thus, the function $f(x) \equiv 0$ is Φ-subdifferentiable. It is easy to see that it is not Fréchet Φ-differentiable at any point.*

However, under appropriate assumptions we can obtain an extension of the Asplund theorem to the case of metric spaces.

The assumptions are as follows:

(a) Φ is an additive group,

(sL) Φ is a set of Lipschitz functions; moreover the space $\Phi/_{\mathbb{R}}$ is separable in the Lipschitz norm $\|\phi\|_L$,

(wm) the family Φ has the *weak monotonicity property* with constant k, i.e. there is a constant k, $0 < k < 1$, such that for all $x \in X$ and all $\phi \in \Phi$,

$$\limsup_{y \to x} \frac{|\phi(y) - \phi(x)|}{d(x,y)} \geq k\|\phi\|_L. \tag{4.2}$$

In other words for all $x \in X$, all $\phi \in \Phi$ and all $t > 0$, there is a $y \in X$ such that $0 < d(x,y) < t$ and

In the previous papers (Rolewicz, 1994, 1995, 1995b, 1999) and in the book by Pallaschke and Rolewicz (1997) the assumption (wm) was formulated in a stronger way, namely

(m) the family Φ has the *monotonicity property* with constant k, i.e. there is a constant k, $0 < k < 1$, such that for all $x \in X$ and all $\phi \in \Phi$,

$$\limsup_{y \to x} \frac{\phi(y) - \phi(x)}{d(x,y)} \geq k\|\phi\|_L. \tag{4.3}$$

In other words for all $x \in X$, all $\phi \in \Phi$ and all $t > 0$, there is a $y \in X$ such that $0 < d(x,y) < t$ and

$$\phi(y) - \phi(x) \geq k\|\phi\|_L d(y,x). \tag{4.3$_1$}$$

Observe that if X is a compact set, then condition (m) is never satisfied, but the condition (wm) can hold (Rolewicz, 1999b).

For any $\phi \in \Phi$, $0 < \beta < 1$, $x \in X$, write (Pallaschke and Rolewicz, 1997, sec. 2.4, see Preiss and Zajíček, 1984, for the linear case)

$$K(\phi, \beta, x) = \{y \in X : \phi(y) - \phi(x) \geq \beta\|\phi\|_L d(y,x)\}. \tag{4.4}$$

The set $K(\phi, \beta, x)$ will be called the *β-cone with vertex at x and direction ϕ*. Of course, it may happen that $K(\phi, \beta, x) = \{x\}$. However, if Φ has the monotonicity property with constant k and $\beta < k$, then $K(\phi, \beta, x)$ has a nonempty interior and, even more,

$$x \in \overline{\operatorname{Int} K(\phi, \beta, x)}. \tag{4.5}$$

If Φ has the weak monotonicity property with constant k and $\beta < k$ it still may happen that $K(\phi, \beta, x) = \{x\}$. But we have the following obvious

PROPOSITION 4.2 *If Φ has the weak monotonicity property with constant k and $\beta < k$, then either the set $K(\phi, \beta, x)$ has a nonempty interior and*

$$x \in \overline{\operatorname{Int} K(\phi, \beta, x)}, \tag{4.5}$$

or $K(-\phi, \beta, x)$ has a nonempty interior and

$$x \in \overline{\operatorname{Int} K(-\phi, \beta, x)}. \tag{4.5$_1$}$$

Now we shall extend a little the definition of a cone. Namely the set

$$K(\phi, \beta, x, \varrho) = K(\phi, \beta, x) \cap \{y : d(x,y) < \varrho\} \tag{4.6}$$

will be called the *(β, ϱ)-cone with vertex at x and direction ϕ*.

Observe that just from the definition it follows that if $\beta_1 \leq \beta_2$ and $\varrho_1 \geq \varrho_2$, then $K(\phi, \beta_1, x, \varrho_1) \supset K(\phi, \beta_2, x, \varrho_2)$.

We recall that $M \subset X$ is said to be *β-cone meagre* if for every $x \in M$ and $\varepsilon > 0$ there are $z \in X$ with $d(x,z) < \varepsilon$ and $\phi \in \Phi$ such that

$$M \cap \operatorname{Int} K(\phi, \beta, z) = \emptyset \tag{4.7}$$

A set $M \subset X$ is said to be *(β, ϱ)-cone meagre* if for every $x \in M$ and arbitrary $\varepsilon > 0$ there are $z \in X$, $d(x, z) < \varepsilon$ and $\phi \in \Phi$ such that

$$M \cap \operatorname{Int} K(\phi, \beta, z, \varrho) = \emptyset. \tag{4.8}$$

The arbitrariness of ε and (4.8) imply that a (β, ϱ)-cone meagre set M is nowhere dense.

A simple example shows that those two notions do not coincide (see Rolewicz, 1999).

We recall that a set $M \subset X$ is called *angle-small* if it can be represented as a union of a countable number of β-cone meagre sets M_n,

$$M = \bigcup_{n=1}^{\infty} M_n. \tag{4.9}$$

We say that a set $M \subset X$ is *weakly angle-small* if it can be represented as a union of a countable number of (β, ϱ_n)-cone meagre sets M_n,

$$M = \bigcup_{n=1}^{\infty} M_n \tag{4.9$_w$}$$

for some $\beta > 0$ and $\varrho_n > 0$, $n = 1, 2, \ldots$

Of course, every angle-small set M is weakly angle-small. The converse is also true in the case of separable X.

PROPOSITION 4.3 *Let X be a separable metric space. Let Φ be a fixed family of functions. Then each weakly angle-small set M is angle-small.*

Proof. By the definition, the set M can be represented as a union of (β, ϱ_n)-cone meagre sets M_n. Since X is separable, we can cover it by a family of sets X_k such that the diameter of X_k is smaller than ϱ_n. Let $M_{n,k} = M_n \cap X_k$. Since M_n is a (β, ϱ_n)-cone meagre set, the sets $M_{n,k}$ are also (β, ϱ_n)-cone meagre. This means that for $x \in M_{n,k}$ and $\varepsilon > 0$, there are $z \in X, d_X(x, z) < \varepsilon$ and $\phi \in \Phi$ such that

$$M_{n,k} \cap \operatorname{Int} K(\phi, \beta, z, \varrho_n) = \emptyset. \tag{4.10}$$

Since the diameter of X_k is smaller than ϱ_n, then the diameter of $M_{n,k}$ is also smaller than ϱ_n. This trivially implies that

$$M_{n,k} \cap \operatorname{Int} K(\phi, \beta, z) = \emptyset,$$

i.e. $M_{k,n}$ is a β-cone meagre set. Hence

$$M = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} M_{n,k} \tag{4.11}$$

It is not clear what happens when the space X is not separable. In particular, let $X = \ell^\infty$, let $\Phi = \ell^1$. Let $A = \{(x_0, x_1, \ldots) : x_i \text{ is an integer}\} \subset \ell^\infty$. It is easy to see that A is a (β, ϱ)-cone meagre set for $0 < \beta, 0 < \varrho < 1$. Is the set A angle-small?

Adapting the method of Preiss and Zajíček (1984) and the proof of Rolewicz (1994) (see also proof of Theorem 2.4.11 of Pallaschke and Rolewicz, 1997) we can obtain

THEOREM 4.4 (compare Rolewicz, 1999). *Let X be a metric space. Let Φ be a family of Lipschitz functions satisfying assumptions* (a), (sL) *and* (wm). *Let a multifunction Γ mapping X into 2^Φ be $\alpha(\cdot)$-monotone and such that* $\operatorname{dom}\Gamma = X$ *(i.e., $\Gamma(x) \neq \emptyset$ for all $x \in X$). Then there exists a weakly angle-small set A such that Γ is single-valued and continuous (i.e. simultaneously lower semi-continuous and upper semi-continuous) at each point of $X \setminus A$.*

Proof. The proof is almost the same as the proof of Theorem 4 in Rolewicz (1999). There is only one difference. In Rolewicz (1999) we have assumed that (m) holds. By Proposition 4.2 and the fact that $\Phi = -\Phi$ (which follows from (a)) it is easy to observe that the assumption (wm) is sufficient. ■

Since the subdifferential $\partial^\alpha_\Phi f|_x$ of an $\alpha(\cdot)$-Φ-subdifferentiable function is a $2\alpha(\cdot)$-monotone multifunction of x, we immediately obtain

COROLLARY 4.5 (compare Rolewicz, 1999). *Let X be a metric space. Let Φ be a family of Lipschitz functions satisfying assumptions* (a), (sL) *and* (wm). *Let $f(\cdot)$ be an $\alpha(\cdot)$-Φ-subdifferentiable function. Then, there is a weakly angle-small set A such that at each point of $X \setminus A$ the $\alpha(\cdot)$-Φ-subdifferential $\partial^\alpha_\Phi f|_x$ is single-valued and continuous in the metric d_L.*

By Proposition 4.3 in the case of X separable we can replace "weakly angle-small" in Theorem 4.4 by "angle-small" and obtain a

THEOREM 4.6 *Let X be a separable metric space. Let Φ be a family of Lipschitz functions satisfying assumptions* (a), (sL) *and* (wm). *Let a multifunction Γ mapping X into 2^Φ be $\alpha(\cdot)$-monotone and such that* $\operatorname{dom}\Gamma = X$. *Then there exists an angle-small set A such that $\Gamma(\cdot)$ is single-valued and continuous at each point of $X \setminus A$.*

COROLLARY 4.7 *Let X be a separable metric space. Let Φ be a family of Lipschitz functions satisfying assumptions* (a), (sL) *and* (wm). *Let $f(\cdot)$ be an $\alpha(\cdot)$-Φ-subdifferentiable function. Then there exists an angle-small set A such that the $\alpha(\cdot)$-Φ-subdifferential $\partial^\alpha_\Phi f|_x$ is single-valued and continuous at each point of $X \setminus A$ in the metric d_L.*

We recall that a set B of the second Baire category is called *residual* if its complement is of the first Baire category. Since weakly angle-small sets are

COROLLARY 4.8 (compare Rolewicz, 1999). *Let X be a metric space of the second Baire category in itself (in particular, let X be a complete metric space). Let Φ be a family of Lipschitz functions satisfying assumptions* (a), (sL) *and* (wm). *Let Γ be an $\alpha(\cdot)$-monotone multifunction mapping X into 2^{Φ} such that $\Gamma(x) \neq \emptyset$ for all $x \in X$. Then, there exists a residual set B such that $\Gamma(\cdot)$ is single-valued and continuous at each point of $X \setminus B$.*

COROLLARY 4.9 *Let X be a metric space which is of the second Baire category in itself (in particular, let X be a complete metric space). Let Φ be a family of Lipschitz functions satisfying assumptions* (a), (sL) *and* (wm). *Let $f(\cdot)$, an $\alpha(\cdot)$-Φ-subdifferentiable function. Then there exists a residual set B such that the $\alpha(\cdot)$-Φ-subdifferential $\partial^{\alpha}_{\Phi} f|_x$ is single-valued and continuous at each point of $X \setminus B$.*

Recall that in the case of normed spaces Gateaux differentiability of a convex continuous function $f(\cdot)$ at a point x is equivalent to the fact that the subdifferential $\partial f|_x$ consists of one point only. Moreover the continuity of a Gateaux differential in the norm operator topology implies that it is the Fréchet differential. Similarly, we have an extension of this fact to metric spaces (Rolewicz, 1995c, 1996). In Rolewicz (1999) we have extended this result to $\alpha(\cdot)$-monotone operators.

We recall that the subdifferential $\partial f|_x$ is lower semi-continuous at x_0 in the Lipschitz norm if for any $\phi_{x_0} \in \partial_{\Phi} f|_{x_0}$ there is a function $\mu(t)$ such that $\mu(0) = 0$ and $\mu(t) > 0$ for $t > 0$ and

$$\lim_{t \downarrow 0} \mu(t) = 0 \tag{4.12}$$

and such that for all $x \in X$ there is $\phi_x \in \partial_{\Phi} f|_x$ such that

$$\|\phi_x - \phi_{x_0}\|_L \leq \mu(d(x, x_0)). \tag{4.13}$$

PROPOSITION 4.10 (Rolewicz, 1999). *Let X be a metric space. Let Φ be a family of Lipschitz functions defined on X satisfying* (a) . *Let $f(\cdot)$ be an $\alpha(\cdot)$-Φ-subdifferentiable function. If the subdifferential $\partial f|_x$ is lower semi-continuous at x_0 in the Lipschitz norm, then it is the Fréchet Φ-differential of $f(\cdot)$ at x_0, and of course it is also lower semi-continuous at x_0 in the Lipschitz norm.*

Let X be a metric space which is of the second Baire category in itself. Let Ω_0 be a residual set in X. Let Ω be a residual set in Ω_0. Then, trivially, Ω is a residual set in X. Thus, as a consequence of Theorem 4.4 and Proposition 4.10, we obtain

THEOREM 4.11 (compare Rolewicz, 1999). *Let X be a metric space which is of the second Baire category in itself (in particular, let X be a complete metric* *and* (wm). *Let $f(\cdot)$ be a continuous $\alpha(\cdot)$-Φ-subdifferentiable function. Then there is a weakly angle-small set A such that $f(\cdot)$ is Fréchet Φ-differentiable at every point $x_0 \in \Omega = X \setminus A$. Moreover, on Ω the Fréchet Φ-gradient is unique and it is continuous there in the metric d_L.*

Suppose that X is an open subset of a Banach space Y having separable dual Y^*. Let Φ be the family of continuous linear functionals on Y restricted to X. It is easy to see that Φ satisfies assumptions (a), (sL) and (wm). Thus, Theorem 4.11 can be rewritten in this case in the following way:

THEOREM 4.11$_B$. *Let X be an open subset of a Banach space Y having separable dual Y^*. Let Φ be the family of linear continuous functionals on Y restricted to X, $\Phi = Y^*|_X$. Let $f(\cdot)$ be a continuous $\alpha(\cdot)$-Φ-subdifferentiable function. Then there is a weakly angle-small set A such that $f(\cdot)$ is Fréchet differentiable at every point $x_0 \in \Omega = X \setminus A$. Moreover, on Ω the Fréchet Φ-gradient is unique and it is continuous there in the conjugate norm $\|\cdot\|^*$.*

5. $\alpha(\cdot)$-paraconvex and strongly $\alpha(\cdot)$-paraconvex functions

Theorem 4.11$_B$ has a certain disadvantage. Namely it is difficult to check whether $f(\cdot)$ is a continuous $\alpha(\cdot)$-Φ-subdifferentiable function. Therefore it is a natural question to describe classes of functions which have this property. In this section $(X, \|\cdot\|)$ will be a normed space and X^* will be its dual. Let $\Omega \subset X$. By Φ we shall denote the restriction to Ω of the elements of X^*. Since it does not lead to misunderstanding, in this section we shall omit Φ when speaking of $\alpha(\cdot)$-Φ-subdifferentiability and $\alpha(\cdot)$-Φ-monotonicity.

Let $\alpha(t)$ be a nondecreasing function mapping from $[0, +\infty)$ into $[0, +\infty]$ such that

$$\lim_{t \downarrow 0} \frac{\alpha(t)}{t} = 0. \tag{3.1}$$

Let Ω be a convex subset of X. Let $f(\cdot)$ be a real-valued function defined on Ω. We say that $f(\cdot)$ is $\alpha(\cdot)$-*paraconvex** *with constant* $C > 0$ if for all $x, y \in \Omega$ and $0 \le t \le 1$,

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + C\alpha(\|x - y\|). \tag{5.1}$$

We say that the function $f(\cdot)$ is $\alpha(\cdot)$-*paraconvex* if there is a constant $C > 0$ such that $f(\cdot)$ is $\alpha(\cdot)$-paraconvex with constant C. For $\alpha(t) = t^2$ this definition

* In general in the definitions of $\alpha(\cdot)$-paraconvex and strongly $\alpha(\cdot)$-paraconvex functions the assumption (3.1) is replaced by the weaker assumption

$$\limsup_{t \downarrow 0} \frac{\alpha(t)}{t} < +\infty,$$

was introduced in Rolewicz (1979) and the t^2-paraconvex functions were called simply paraconvex. In Rolewicz (1979b) the notion was extended to the case of $\alpha(t) = t^\gamma, 1 \leq \gamma \leq 2$, and the t^γ-paraconvex functions were called γ-paraconvex.

We say that the function $f(\cdot)$ is *strongly* $\alpha(\cdot)$*-paraconvex with constant* $C_1 > 0$ if for all $x, y \in \Omega$ and $0 \leq t \leq 1$,

$$\begin{aligned} f(tx + (1-t)y) &\leq tf(x) + (1-t)f(y) \\ &+ C_1 \min[t, (1-t)]\alpha(\|x-y\|). \end{aligned} \tag{5.2}$$

If there is a constant $C_1 > 0$ such that $f(\cdot)$ is strongly $\alpha(\cdot)$-paraconvex with constant C_1, we say that $f(\cdot)$ is *strongly* $\alpha(\cdot)$*-paraconvex* .

Of course, every $f(\cdot)$ strongly $\alpha(\cdot)$-paraconvex with constant C_1 is also $\alpha(\cdot)$-paraconvex with the constant C_1.

It was shown in Rolewicz (1979, 1979b) that for $\alpha(t) = t^\gamma, 1 < \gamma \leq 2$, any $\alpha(\cdot)$-paraconvex function is strongly $\alpha(\cdot)$-paraconvex.

There are $\alpha(\cdot)$-paraconvex functions $f(\cdot) : X \to \mathbb{R}$ which are not strongly $\alpha(\cdot)$-paraconvex. Conditions warranting that each $\alpha(\cdot)$-paraconvex functions is automatically strongly $\alpha(\cdot)$-paraconvex can be found in Rolewicz (2000). In particular, using those conditions we find that every γ-paraconvex (i.e. t^γ-paraconvex) function is strongly γ-paraconvex if and only if $1 < \gamma$ (Jourani, 1996).

PROPOSITION 5.1 *Let* $(X, \|\cdot\|)$ *be a normed space. Let* Ω *be a convex set in* X. *Let* $f(\cdot)$ *be a real-valued function defined on* Ω. *If* $f(\cdot)$ *is* $\alpha(\cdot)$*-subdifferentiable, then it is* $\alpha(\cdot)$*-paraconvex with constant* 1. *If, additionally,*

$$\alpha(ts) \leq t\alpha(s) \tag{5.3}$$

for $0 < t < 1$ *and* $s > 0$, *then* $f(\cdot)$ *is strongly* $\alpha(\cdot)$*-paraconvex with constant* 2.

Proof. Take arbitrary $x, y \in \Omega$. Let $0 \leq t \leq 1$ and let $z = tx + (1-t)y$. Since the function $f(\cdot)$ is $\alpha(\cdot)$-subdifferentiable, there is a linear continuous functional ℓ such that for h such that $z + h \in \Omega$ we have

$$f(z+h) - f(z) \geq \ell(h) - \alpha(\|h\|). \tag{5.4}$$

Put $h_1 := x - z = (1-t)(x-y)$ and $h_2 := y - z = -t(x-y)$. Thus, we obtain

$$f(x) - f(z) \geq (1-t)\ell(x-y) - \alpha((1-t)\|x-y\|) \tag{5.5}$$

and

$$f(y) - f(z) \geq -t\ell(x-y) - \alpha(t\|x-y\|). \tag{5.6}$$

Multiplying inequality (5.5) by t, inequality (5.6) by $1-t$ and adding we obtain

$$tf(x) + (1-t)f(y) - f(tx + (1-t)y)$$

The function $\alpha(\cdot)$ is nondecreasing, thus

$$\begin{aligned}&t\alpha((1-t)\|x-y\|)+(1-t)\alpha(t\|x-y\|)\\ &\le t\alpha(\|x-y\|)+(1-t)\alpha(\|x-y\|)=\alpha(\|x-y\|).\end{aligned} \tag{5.8}$$

Inequalities (5.7) and (5.8) imply that $f(\cdot)$ is $\alpha(\cdot)$-paraconvex with constant 1.

If (5.3) holds, then $\alpha(t\|x-y\|)\le t\alpha(\|x-y\|)$ and $\alpha((1-t)\|x-y\|)\le(1-t)\alpha(\|x-y\|)$. Thus we have

$$\begin{aligned}&t\alpha((1-t)\|x-y\|)+(1-t)\alpha(t\|x-y\|)\\ &\le t(1-t)\alpha(\|x-y\|)+(1-t)t\alpha(\|x-y\|)\\ &=2t(1-t)\alpha(\|x-y\|)\le 2\min[t,(1-t)]\alpha(\|x-y\|).\end{aligned} \tag{5.9}$$

Inequalities (5.7) and (5.9) imply that $f(\cdot)$ is strongly $\alpha(\cdot)$-paraconvex with constant 2. ■

For the purpose of further considerations we shall localize the notions of $\alpha(\cdot)$-paraconvex and strongly $\alpha(\cdot)$-paraconvex functions.

First we recall the classical notion of locally convex sets. We say that a set $A\subset X$ is *locally convex* if for each $x\in X$ there is a neighbourhood U of x such that $A\cap U$ is convex. In particular it is easy to see that each open set is locally convex. If a locally convex set is connected and closed, then it is convex (see Tietze, 1928, Matsumura, 1928 for $\mathbb{R}^n$, Klee, 1951, for general linear topological spaces).

We say that a set $A\subset X$ is *uniformly locally convex* if there is a neighbourhood V of 0 such that for each $x\in X$ the set $A\cap(x+V)$ is convex. Observe that if A is uniformly locally convex, then its closure $\overline{A}$ is also uniformly locally convex. Thus, by the above mentioned Tietze–Matsumura–Klee theorem its interior is a union of open convex sets $A_\gamma,\gamma\in\Gamma$, such that their closures are disjoint.

We say that a real-valued function $f(\cdot)$, defined on a locally convex set $\Omega\subset X$, is *locally (strongly)* $\alpha(\cdot)$*-paraconvex with constant* $C>0$ if for each $x_0\in\Omega$ there is a neighbourhood U of x_0 such that $f(\cdot)$ restricted to U is (strongly) $\alpha(\cdot)$-paraconvex with constant C. We say that $f(\cdot)$ is *locally (strongly)* $\alpha(\cdot)$*-paraconvex* if there is C, such that it is $\alpha(\cdot)$-paraconvex with constant C.

We say that a real-valued function $f(\cdot)$, defined on a uniformly locally convex set $\Omega\subset X$, is *uniformly locally (strongly)* $\alpha(\cdot)$*-paraconvex with constant* $C>0$ if there is a neighbourhood V of 0 such that for each $x_0\in\Omega$ the function $f(\cdot)$ restricted to $\Omega\cap(x_0+V)$ is (strongly) $\alpha(\cdot)$-paraconvex with constant C. If there is a constant $C>0$, such that $f(\cdot)$ is uniformly locally (strongly) $\alpha(\cdot)$-paraconvex with the constant C we say that $f(\cdot)$ is *uniformly locally (strongly)* $\alpha(\cdot)$*-paraconvex*.

In general a local $\alpha(\cdot)$-Φ-subgradient need not be an $\alpha(\cdot)$-Φ-subgradient. However in the case of strongly $\alpha(\cdot)$-paraconvex functions defined on open con-

THEOREM 5.2 (Rolewicz, 2001; in the case of $\alpha(t) = t^\gamma$, $1 < t \leq 2$, Jourani, 1996). *Let $(X, \|.\|)$ be a real Banach space, and let $f(\cdot)$ be a strongly $\alpha(\cdot)$-paraconvex function defined on an open convex subset $\Omega \subset X$. Then, each local $\alpha(\cdot)$-subgradient of $f(\cdot)$ at a point x_0, is automatically an $\alpha(\cdot)$-subgradient of $f(\cdot)$ at x_0.*

Thus we have

PROPOSITION 5.3 *Let $(X, \|\cdot\|)$ be a real Banach space, and let $f(\cdot)$ be a locally strongly $\alpha(\cdot)$-paraconvex function defined on an open subset $\Omega \subset X$. Then, the local $\alpha(\cdot)$-X^*-subdifferential of $f(\cdot)$, $\partial^{\alpha,\mathrm{loc}}_{X^*} f|_x$, is a locally $2\alpha(\cdot)$-monotone multifunction.*

Proof. Let $x_0 \in \Omega$. Since $f(\cdot)$ is locally strongly $\alpha(\cdot)$-paraconvex, there is a convex open neighbourhood U of x_0 such that the restriction $f|_U(x)$ is strongly $\alpha(\cdot)$-paraconvex. Thus by Theorem 5.2, $\partial^{\alpha,\mathrm{loc}}_{X^*} f|_U = \partial^{\alpha}_{X^*} f|_U$. Hence by Proposition 3.1 $\partial^{\alpha,\mathrm{loc}}_{X^*} f|_U$ is a $2\alpha(\cdot)$-monotone multifunction. Therefore, $\partial^{\alpha}_{X^*} f|_U$ is a locally $2\alpha(\cdot)$-monotone multifunction. ∎

As a simple consequence of Propositions 5.1 and 5.3 we get

PROPOSITION 5.4 *Let $(X, \|\cdot\|)$ be a normed space. Let Ω be a (uniformly) locally convex set in X. Let $f(\cdot)$ be a real-valued function defined on Ω. If $f(\cdot)$ is $\alpha(\cdot)$-subdifferentiable, then it is (uniformly) locally $\alpha(\cdot)$-paraconvex with constant 1. If, additionally*

$$\alpha(ts) \leq t\alpha(s) \tag{5.3}$$

for $0 < t < 1$ and $s > 0$, then the function $f(\cdot)$ is (uniformly) locally strongly $\alpha(\cdot)$-paraconvex with constant 2.

Let $f(\cdot)$ be a real-valued function defined on a uniformly locally convex set Ω. If for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$tf(x) + (1-t)f(y) - f(tx + (1-t)y) \geq -\varepsilon t(1-t)\|x - y\| \tag{5.10}$$

for all $x, y \in \Omega$ such that $\|x - y\| \leq \delta$ we say that the function $f(\cdot)$ is *uniformly approximate convex* (Rolewicz, 2001b). This is a uniformization of the notion of approximate convex functions introduced in Luc-Ngai-Théra (2000).

PROPOSITION 5.5 *Let $(X, \|\cdot\|)$ be a normed space. Let Ω be a uniformly locally convex set in X. Let $f(\cdot)$ be a real-valued function defined on Ω. Suppose that (3.1) holds. If the function $f(\cdot)$ is $\alpha(\cdot)$-subdifferentiable, then it is uniformly*

Proof. First we shall show that there is a function $\beta(\cdot)$ such that $\beta(t) \geq \alpha(t)$ for all $0 < t$ and (5.3) holds for $\beta(t)$.

Let $\beta_0(t) = \sup_{0<s\leq t} \frac{\alpha(s)}{s}$. Of course $\beta_0(t)$ is nondecreasing and by (3.1), $\lim_{t\downarrow 0} \beta_0(t) = 0$. We put $\beta(t) = t\beta_0(t)$. Of course $\beta(t) = t\beta_0(t) \geq t\frac{\alpha(t)}{t} = \alpha(t)$. The function $\beta_0(t)$ is nondecreasing, hence for $0 < s < 1$ we have

$$\beta(st) = st\beta_0(st) \leq st\beta_0(t) = s\beta(t). \tag{5.3'}$$

Since $\beta(t) \geq \alpha(t)$ the function $f(\cdot)$ is $\beta(\cdot)$-subdifferentiable.

Since $\lim_{t\downarrow 0} \beta_0(t) = 0$ for every $\varepsilon > 0$, there is a $\delta > 0$ such that for $0 < s < \delta$ we have $\beta_0(s) < \frac{\varepsilon}{4}$. Let $B_\delta = \{x : \|x\| < \delta\}$. By Proposition 5.4 there is a neighbourhood V of 0, $V \subset B_\delta$, such that for each $x_0 \in \Omega$ the function $f(\cdot)$ restricted to the set $x_0 + V$ is strongly $\beta(\cdot)$-paraconvex with constant 2. Thus

$$\begin{aligned} &tf(x) + (1-t)f(y) - f(tx + (1-t)y) \geq -2\min[t,(1-t)]\beta(\|x-y\|) \\ &= -2\min[t,(1-t)]\beta_0(\|x-y\|)\|x-y\| \\ &\geq -2\min[t,(1-t)]\frac{\varepsilon}{4}\|x-y\| \geq -\varepsilon t(1-t)\|x-y\|. \end{aligned}$$

The arbitrariness of ε implies that $f(\cdot)$ is uniformly approximate convex. ■

Between uniformly approximate convex and strongly $\alpha(\cdot)$-paraconvex functions the following relations are true.

Proposition 5.6 (Rolewicz, 2002). *Let $(X, \|\cdot\|)$ be a normed space. Let Ω be a uniformly locally convex set in X. Let $f(\cdot)$ be a real-valued function defined on Ω. If $f(\cdot)$ is uniformly locally strongly $\alpha(\cdot)$-paraconvex, then it is uniformly approximately convex.*

Conversely, if $f(\cdot)$ is uniformly approximate convex, then there is a nondecreasing function $\alpha(\cdot)$ mapping $[0, +\infty)$ into $[0, +\infty]$ satisfying (3.1) such that $f(\cdot)$ is uniformly locally strongly $\alpha(\cdot)$-paraconvex.

The converse of Proposition 5.1 requires the openness of the set Ω.

We start with

Proposition 5.7 (see Rolewicz, 2000). *Let $(X, \|\cdot\|)$ be a normed space. Let a real-valued function $f(\cdot)$ defined on a (locally) convex set $\Omega \subset X$ be (locally) strongly $\alpha(\cdot)$-paraconvex. If $f(\cdot)$ is locally bounded, then it is locally Lipschitz.*

Using category methods we can obtain

Proposition 5.8 (see Rolewicz, 2000). *Let $(X, \|\cdot\|)$ be a Banach space. Let a real-valued function $f(\cdot)$ defined on an open (locally) convex set $\Omega \subset X$ be*

PROPOSITION 5.9 (see Rolewicz, 2001; in the case of $\alpha(t) = t^{\gamma}$, $1 < t \leq 2$, Jourani, 1996). *Let $f(\cdot)$ be a locally strongly $\alpha(\cdot)$-paraconvex function defined on an open subset Ω of a Banach space X. Then the local $\alpha(\cdot)$-subdifferentials and Clarke subdifferentials of $f(\cdot)$ coincide.*

COROLLARY 5.10 (see Rolewicz, 2001). *Let $f(\cdot)$ be a (locally) strongly $\alpha(\cdot)$-paraconvex function defined on an open (locally) convex set Ω of a Banach space X. Then $f(\cdot)$ is (locally) $\alpha(\cdot)$-subdifferentiable.*

Proof. By Proposition 5.8 $f(\cdot)$ is locally Lipschitz. Thus, at each point its Clarke subdifferential is not empty. Hence, by Proposition 5.9 its (local) $\alpha(\cdot)$-subdifferential is also not empty. ∎

Since every open set is locally convex, we do not need to assume convexity of Ω in the local versions of Propositions 5.8, 5.9 and Corollary 5.10.

As a consequence of Corollary 5.10 and Theorem 4.11_B we get the following extension of the Asplund (1968) theorem:

THEOREM 5.11 *Let $(X, \|.\|)$ be a real Banach space, with separable dual X^*. Let $f(\cdot)$ be a locally strongly $\alpha(\cdot)$-paraconvex function defined on an open subset $\Omega \subset X$. Then there is a subset $A_f \subset \Omega$ of the first Baire category such that on $\Omega \setminus A_f$ the function $f(\cdot)$ is Fréchet differentiable. Moreover, the Fréchet Φ-gradient is continuous in the conjugate norm $\|\cdot\|^*$.*

It is of interest to find the relation between t^{γ}-paraconvex functions and DC-functions.

Firtst we shall consider the relation between DC and t^2-paraconvex functions.

We have

PROPOSITION 5.12 (Rolewicz, 1980). *Let $(X, \|\cdot\|)$ be a normed space. Then, each t^2-paraconvex function is a difference of a convex function and a quadratic function. Thus, each t^2-paraconvex function is a DC-function.*

Unfortunately, for $1 < \gamma < 2$ the situation is not so nice.

PROPOSITION 5.13 *Let $X = \mathbb{R}$. Let $1 < \gamma < 2$. Then there is a t^{γ}-subdifferentiable (i.e. γ-paraconvex with constant 1) function which is not a DC-function.*

The proof is based on the following

LEMMA 5.14 *The function $f(t) = -t^{\gamma}$, $1 < \gamma < 2$, defined on $(0, +\infty)$ is t^{γ}-*

Proof. Take an arbitrary $t_0 > 0$. The function $f(t) = -t^\gamma$ is differentiable and its derivative at t_0 is equal to $-\gamma {t_0}^{\gamma-1}$. By definition, the function $f(t) = -t^\gamma$ is t^γ-subdifferentiable with constant 1 if and only if

$$-(t_0 + h)^\gamma \geq -{t_0}^\gamma - \gamma {t_0}^{\gamma-1} h - |h|^\gamma. \tag{5.11}$$

for all $t_0 > 0$ and h such that $h > -t_0$. This holds if and only if

$$g(h) = -(t_0 + h)^\gamma + {t_0}^\gamma + \gamma {t_0}^{\gamma-1} h + |h|^\gamma \geq 0. \tag{5.12}$$

Observe that $g(0) = 0$. We shall show that $g(h)$ is increasing for $h > 0$ and decreasing for $h < 0$.

Indeed, let $h > 0$. Then

$$g(h) = -(t_0 + h)^\gamma + {t_0}^\gamma + \gamma {t_0}^{\gamma-1} h + h^\gamma.$$

Let us calculate its derivative:

$$g'(h) = -\gamma(t_0 + h)^{\gamma-1} + \gamma {t_0}^{\gamma-1} + \gamma h^{\gamma-1}.$$

Since $0 < \gamma - 1 < 1$, $g'(h) > 0$.

Now let $h < 0$. Then

$$g(h) = -(t_0 + h)^\gamma + {t_0}^\gamma + \gamma {t_0}^{\gamma-1} h + (-h)^\gamma$$

and

$$\begin{aligned} g'(h) &= -\gamma(t_0 + h)^{\gamma-1} + \gamma {t_0}^{\gamma-1} - \gamma(-h)^{\gamma-1} \\ &= \gamma(t_0 - |h|)^{\gamma-1} + {t_0}^{\gamma-1} - (|h|)^{\gamma-1}. \end{aligned}$$

Since $0 < \gamma - 1 < 1$ in this case, $g'(h) < 0$. Thus (5.11) holds and the function $f(t) = -t^\gamma$, $1 < \gamma < 2$, is t^γ-subdifferentiable with constant 1. ∎

Proof of Proposition 5.13. Let $X = \mathbb{R}$. Let

$$t_n = \sum_{k=1}^{n} \frac{1}{k^\gamma}.$$

Since $1 < \gamma$, the sequence $\{t_n\}$ is bounded. Let $g(t) = \max[-|t - t_n|^{\gamma-1}]$ and let $f(t) = \int_0^{|t|} g(s)ds$. We shall show that $f(t)$ is locally weakly t^γ-subdifferentiable with constant 1. Indeed, if $t_n - \frac{1}{2n^\gamma} < t < t_n$, $n = 1, 2, \ldots$, it is obvious, since the function $f(\cdot)$ is locally convex in the neighbourhood of t. If $t_n < t < t_n + \frac{1}{2n^\gamma}$, $n = 0, 1, 2, \ldots$, then it follows from Lemma 5.14.

At $t = t_n$, 0 is a local t^γ-subgradient with constant 1, since $f(t_n + h) - f(t_n) \geq -\int_0^{|h|} s^{\gamma-1} ds = -\frac{1}{\gamma}|h|^\gamma$. Recall that by Jourani (1996) this shows that $f(t)$ is t^γ-subdifferentiable.

On the other hand, since $g(\cdot)$ is not of bounded variation, $f(\cdot)$ is not a DC-function. ∎

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