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Three boundary value problems for second order differential inclusions in Banach spaces

by

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Abstract: The paper studies, in the context of Banach spaces, the problem of three boundary conditions for both second order differential inclusions and second order ordinary differential equations. The results are obtained in several new settings of Sobolev-type spaces involving Bochner and Pettis integrals. Some classes of second order multivalued evolution equations associated with m-accretive operators are also considered. Applications to some control problems are provided with the help of narrow convergence for Young measures.

Keywords: boundary, multifunction, extreme point, relaxation, Pettis integral, second derivative, Young measure.

1. Introduction

The pioneering work concerning second order ordinary differential equations with two boundary conditions seems to go back to Hartman (1964). Later several authors (see Gomaa, 2000; Gupta, 1992; Marano, 1992, 1994) studied second order differential equations and inclusions with three boundary conditions. All those results deal with finite dimensional spaces. The aim of our paper is to provide new existence results for problems of three boundary conditions associated with differential inclusions or ordinary differential equations in the general context of Banach spaces. Properties of the set of solutions are also investigated. The results are achieved in several new settings involving some Sobolev-like spaces and the use of weak compactness results in $L^1_E([0,1])$ and $P^1_E([0,1])$ (the space of Pettis integrable functions with values in E). The narrow convergence for Young measures is also used in the application to a relaxation of some optimal control problems governed by a second order differential equation

After the Introduction, and the Preliminaries, in Section 3 we present existence and uniqueness of $W_E^{2,1}([0,1])$-solution for ordinary differential equation with three boundary conditions. We suppose that E is a finite dimensional space. Let $f : [0,1] \times E \times E \to E$ be a mapping such that f is Lebesgue measurable on $[0,1]$ and continuous on $E \times E$ satisfying a Lipschitzean-type condition, that is, there are Lipschitz constants λ_1, λ_2 satisfying $\lambda_1 + \lambda_2 < \frac{1}{2}$ such that

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \leq \lambda_1\|x_1 - x_2\| + \lambda_2\|y_1 - y_2\|$$

for all $t \in [0,1]$ and for all $(x_1,y_1),(x_2,y_2) \in E \times E$, and a growth-type condition

$$\|f(t,x,y)\| \leq c(1 + \|x\| + \|y\|), \forall (t,x,y) \in [0,1] \times E \times E$$

for some $c > 0$. Then the differential equation

$$\begin{cases} \ddot{u}(t) = f(t,u(t),\dot{u}(t)), \text{ a.e. } t \in [0,1], \\ u(0) = 0; \ u(\theta) = u(1), \end{cases}$$

has a unique solution $u \in W_E^{2,1}([0,1])$. As an application we present a new Bolza type problem and a new relaxation property in Optimal Control for a second order differential equation where the controls are Young measures.

In Section 4 we study the differential inclusion of the form

$$\begin{cases} \ddot{u}(t) \in F(t,u(t),\dot{u}(t)) \subset \Gamma(t), \text{ a.e. } t \in [0,1], \\ u(0) = 0; \ u(\theta) = u(1), \end{cases}$$

where $F : [0,1] \times E \times E \to E$ is a convex compact valued multifunction, Lebesgue-measurable on $[0,1]$ and upper semicontinuous on $E \times E$, and $\Gamma : [0,1] \rightrightarrows E$ is a convex compact valued, Lebesgue-measurable and *integrably bounded* multifunction, that is, the scalar function $t \mapsto |\Gamma(t)| := \sup\{\|x\| : x \in \Gamma(t)\}$ is Lebesgue-integrable on $[0,1]$. In particular, we show a relaxation property for a second order differential inclusion. Namely, we show that the $W_E^{2,1}([0,1])$-solutions set of the differential inclusion

$$\begin{cases} \ddot{u}(t) \in \text{ext}(\Gamma(t)), \text{ a.e. } t \in [0,1], \\ u(0) = 0; \ u(\theta) = u(1), \end{cases}$$

where $\text{ext}(\Gamma(t))$ is the set of extreme points of $\Gamma(t)$, is a G_δ-dense subset for the topology of uniform convergence of the $W_E^{2,1}([0,1])$-solutions set of the differential inclusion

$$\begin{cases} \ddot{u}(t) \in \Gamma(t), \text{ a.e. } t \in [0,1], \\ u(0) = 0; \ u(\theta) = u(1), \end{cases}$$

via a lower semicontinuity result for integral functionals.

We end this section by giving a new existence result of $W_E^{2,1}([0,1])$-solutions for a second order evolution inclusion governed by a class of m-accretive operator (see e.g. Vrabie, 1987) $A(t) : E \to 2^E$ depending on $t \in [0,1]$ in a finite dimensional space E with convex compact valued perturbation

$$\begin{cases} -\ddot{u}(t) \in A(t)u(t) + F(t,u(t),\dot{u}(t)), \text{ a.e. } t \in [0,1], \end{cases}$$

$F : [0,1] \times E \times E \to E$ is a convex compact valued multifunction, Lebesgue-measurable on $[0,1]$ and upper semicontinuous on $E \times E$, under the assumption that, for every $t \in [0,1]$, $D(A(t))$ contains a closed ball of center 0 with radius γ, for some $\gamma > 0$. Here $D(A(t)) = \{x \in E : A(t)x \neq \emptyset\}$.

A new variant for the second order differential inclusion given above is obtained in Section 5 when $\Gamma(.)$ is convex compact valued, Lebesgue-measurable, and *scalarly Pettis uniformly integrable*, that is,

$$\{\delta^*(x', \Gamma(.)) : \|x'\| \leq 1\} \text{ is uniformly integrable in } L^1_{\mathbb{R}}([0,1], dt),$$

here $\delta^*(x', \Gamma(t))$ denotes the support function of the convex compact set $\Gamma(t)$ ($t \in [0,1]$). In this new setting the solutions set is in $W^{2,1}_{P,E}([0,1])$. It is easy to check that if $\Gamma(.)$ is a convex compact valued, measurable, and integrably bounded mutifunction, then $\Gamma(.)$ is scalarly Pettis integrable, because in this particular case, $\delta^*(x', \Gamma(t)) \leq |\Gamma(t)|$ for all $x' \in \overline{B}_{E'}$ and for all $t \in [0,1]$.

2. Preliminaries and notations

Throughout, E is a separable Banach space and E' is its topological dual, $\mathcal{L}([0,1])$ is the σ-algebra of Lebesgue-measurable sets of $[0,1]$, $\lambda = dt$ is the Lebesgue measure on $[0,1]$ and θ is a given number in $]0,1[$. By $L^1_E([0,1], dt)$ we denote the space of all Lebesgue–Bochner integrable E-valued functions defined on $[0,1]$. We recall some preliminary results. Let $f : [0,1] \to E$ be a scalarly integrable function, that is, for every $x' \in E'$, the scalar function $t \mapsto \langle x', f(t)\rangle$ is Lebesgue-integrable on $[0,1]$. A scalarly integrable function $f : [0,1] \to E$ is Pettis-integrable if, for every Lebesgue-measurable set A in $[0,1]$, the weak integral $\int_A f(t)dt$ defined by $\langle x', \int_A f(t)dt\rangle = \int_A \langle x', f(t)\rangle\, dt$ for all $x' \in E'$, belongs to E. We denote by $P^1_E([0,1], dt)$ the space of all Pettis integrable E-valued functions defined on $[0,1]$. The Pettis norm of any element $f \in P^1_E([0,1], dt)$ is defined by $\|f\|_{Pe} = \sup_{x' \in \overline{B}_{E'}} \int_{[0,1]} |\langle x', f(t)\rangle| dt$, where $\overline{B}_{E'}$ is the closed unit ball of E' (Geitz, 1981; Huff, 1986; Musiał, 1987, 1991). The space $P^1_E([0,1], dt)$ endowed with $\|.\|_{Pe}$ is a normed space. A subset $\mathcal{H} \subset P^1_E([0,1], dt)$ is *Pettis uniformly integrable* (PUI for short) if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\lambda(A) \leq \delta \Longrightarrow \sup_{f \in \mathcal{H}} \|1_A f\|_{Pe} \leq \varepsilon.$$

If $f \in P^1_E([0,1]), dt)$, the singleton $\{f\}$ is PUI since the set $\{\langle x', f\rangle : \|x'\| \leq 1\}$ is uniformly integrable (Geitz, 1981; Huff, 1986). More generally, a subset $\mathcal{H} \subset P^1_E([0,1], dt)$, is *scalarly Pettis uniformly integrable* if the set $\{\langle x', f\rangle : f \in \mathcal{H}, \|x'\| \leq 1\}$ is uniformly integrable in the space $L^1_{\mathbb{R}}([0,1], dt)$. If $\mathcal{H}$ is scalarly Pettis uniformly integrable, then it is PUI. Indeed, we have

$$\lim \sup \quad \sup \int |\langle x', f\rangle|\, dt = 0.$$

For any $x' \in \overline{B}_{E'}$, one has

$$(*) \int_A |\langle x', f\rangle|\, dt = \int_{A\cap[|\langle x', f\rangle|\le a]} |\langle x', f\rangle|\, dt + \int_{A\cap[|\langle x', f\rangle|>a]} |\langle x', f\rangle|\, dt.$$

Let a be large enough in order to ensure

$$\forall x' \in \overline{B}_{E'},\ \forall f \in \mathcal{H},\ \int_{A\cap[|\langle x', f\rangle|>a]} |\langle x', f\rangle|\, dt \le \varepsilon/2.$$

Thus, the last term of $(*)$ is $\le \varepsilon/2$. Now, if δ is small enough in order to ensure $a\delta \le \varepsilon/2$, we obtain

$$\int_{A\cap[|\langle x', f\rangle|\le a]} |\langle x', f\rangle|\, dt \le a\lambda(A) \le \varepsilon/2$$

as soon as $\lambda(A) \le \delta$. In the following, only the scalarly Pettis uniformly integrable notion is used.

Let $\mathcal{C}_E([0,1])$ be the Banach space of all continuous functions u from $[0,1]$ into E equipped with the sup-norm. By $W^{2,1}_{B,E}([0,1])$ (resp. $W^{2,1}_{P,E}([0,1])$) we denote the space of all continuous functions in $\mathcal{C}_E([0,1])$ such that their first derivatives (resp. weak derivatives) are continuous and their second weak derivatives belong to $L^1_E([0,1])$ (resp. $P^1_E([0,1])$). It is obvious that $W^{2,1}_{B,E}([0,1]) \subset W^{2,1}_{P,E}([0,1])$. When E is finite dimensional, $L^1_E([0,1]) = P^1_E([0,1])$, and hence we put $W^{2,1}_E([0,1]) := W^{2,1}_{B,E}([0,1]) = W^{2,1}_{P,E}([0,1])$.

3. Existence results in $W^{2,1}_{B,E}([0,1])$ for ordinary differential equation

We begin with a lemma which summarizes some properties of some Hartman-type function (see Gomaa, 2000; Ibrahim and Gomaa, 2000; Hartman, 1964; Marano, 1992, 1994). Such a function was first introduced by Hartman (1964) to study two boundary problems for ordinary differential equations. The following Hartman-type function seems to be introduced by Marano (1992, 1994). It is useful in the study of three boundary problems for differential equations. We include a complete proof for the convenience of the reader

Lemma 1 *Let E be a separable Banach space and let $G : [0,1]\times[0,1] \to \mathbb{R}$ be the function defined by*

if $0 \le t < \theta$,

$$G(t,s) = \begin{cases} -s & \text{if } 0 \le s \le t, \\ -t & \text{if } t < s \le \theta, \end{cases} \tag{3.1}$$

if $\theta \le t \le 1$,

$$G(t,s) = \begin{cases} -s & \text{if } 0 \le s < \theta, \\ (\theta(s-t) + s(t-1))/(1-\theta) & \text{if } \theta \le s \le t, \\ t(s-1)/(1-\theta) & \text{if } t < s \le 1. \end{cases} \tag{3.2}$$

Then the following assertions hold:

1) If $u \in W^{2,1}_{B,E}([0,1])$ *with* $u(0) = 0$ *and* $u(\theta) = u(1)$, *then*

$$u(t) = \int_0^1 G(t,s)\ddot{u}(s)\,ds, \quad \forall t \in [0,1], \tag{3.3}$$

2) $G(.,s)$ *is derivable on* $[0,1]$, *for every* $s \in [0,1]$, *that is,* $G(.,s)$ *is right derivable on* $[0,1[$ *and left derivable on* $]0,1]$. *Its derivative is given by*

if $0 \le t < \theta$,

$$\frac{\partial G}{\partial t}(t,s) = \begin{cases} 0 & \text{if } 0 \le s \le t, \\ -1 & \text{if } t < s \le \theta, \\ (s-1)/(1-\theta) & \text{if } \theta \le s \le 1, \end{cases} \tag{3.4}$$

if $\theta \le t \le 1$,

$$\frac{\partial G}{\partial t}(t,s) = \begin{cases} 0 & \text{if } 0 \le s \le \theta, \\ (s-\theta)/(1-\theta) & \text{if } \theta \le s \le t, \\ (s-1)/(1-\theta) & \text{if } t < s \le 1. \end{cases} \tag{3.5}$$

3) $G(.,.)$ *and* $\frac{\partial G}{\partial t}(.,.)$ *satisfies*

$$\sup_{t,s\in[0,1]} |G(t,s)| \le 1, \qquad \sup_{t,s\in[0,1]} |\frac{\partial G}{\partial t}(t,s)| \le 1. \tag{3.6}$$

4) Let $f \in L^1_E([0,1])$ *and let* $u_f : [0,1] \to E$ *be the function defined by*

$$u_f(t) = \int_0^1 G(t,s)f(s)\,ds, \quad \forall t \in [0,1],$$

then,

$$u_f(0) = 0, \;\; u_f(\theta) = u_f(1). \tag{3.7}$$

Further, the function u_f *is derivable, and its derivative* $\dot{u}_f$ *satisfies*

$$\lim_{h\to 0} \frac{u_f(t+h) - u_f(t)}{h} = \dot{u}_f(t) = \int_0^1 \frac{\partial G}{\partial t}(t,s)f(s)\,ds \tag{3.8}$$

5) The function $\dot{u}_f$ is scalarly derivable, that is, for every $x' \in E'$, the scalar function $\langle x', \dot{u}_f(.)\rangle$ is derivable, and its weak derivative $\ddot{u}_f$ is equal to f a.e.

Proof. 1) Let $x' \in E'$. Let $0 \le t < \theta$. We have

$$\begin{aligned}
&\langle x', \int_0^1 G(t,s)\ddot{u}(s)\,ds\rangle = \int_0^t -\langle x', s\ddot{u}(s)\rangle\,ds \\
&+\int_t^\theta -\langle x', t\ddot{u}(s)\rangle\,ds + \frac{t}{1-\theta}\int_\theta^1 \langle x', (s-1)\ddot{u}(s)\rangle\,ds \\
&= [\langle x', -s\dot{u}(s)\rangle]_0^t + \int_0^t \langle x', \dot{u}(s)\rangle\,ds - \langle x', t(\dot{u}(\theta)-\dot{u}(t))\rangle \\
&+\frac{t}{1-\theta}[\langle x', (s-1)\dot{u}(s)\rangle]_\theta^1 - \frac{t}{1-\theta}\int_\theta^1 \langle x', \dot{u}(s)\rangle\,ds \\
&= \langle x', -t\dot{u}(t) + u(t) - u(0) - t\dot{u}(\theta) + t\dot{u}(t) + t\dot{u}(\theta) - \frac{t}{1-\theta}(u(1)-u(\theta))\rangle \\
&= \langle x', u(t)-u(0)\rangle.
\end{aligned}$$

Thus, $\langle x', u(t)\rangle = \langle x', \int_0^1 G(t,s)\ddot{u}(s)\,ds\rangle$. Let $\theta \le t \le 1$. We have

$$\begin{aligned}
&\langle x', \int_0^1 G(t,s)\ddot{u}(s)\,ds\rangle = \int_0^\theta -\langle x', s\ddot{u}(s)\rangle\,ds + \int_\theta^t \langle x', \frac{\theta(s-t)+s(t-1)}{1-\theta}\ddot{u}(s)\rangle\,ds \\
&+\int_t^1 \langle x', \frac{t(s-1)}{1-\theta}\ddot{u}(s)\rangle\,ds \\
&= [\langle x', -s\dot{u}(s)\rangle]_0^\theta + \int_0^\theta \langle x', \dot{u}(s)\rangle\,ds + \frac{1}{1-\theta}[\langle x', (\theta(s-t)+s(t-1))\dot{u}(s)\rangle]_\theta^t \\
&-\frac{1}{1-\theta}\int_\theta^t \langle x', (\theta+t-1)\dot{u}(s)\rangle\,ds + \frac{t}{1-\theta}[\langle x', (s-1)\dot{u}(s)\rangle]_t^1 \\
&-\frac{t}{1-\theta}\int_t^1 \langle x', \dot{u}(s)\rangle\,ds = \langle x', u(t)-u(0) + \frac{t}{1-\theta}(u(1)-u(\theta))\rangle \\
&= \langle x', u(t)-u(0)\rangle.
\end{aligned}$$

Therefore $\langle x', u(t)\rangle = \langle x', \int_0^1 G(t,s)\ddot{u}(s)\,ds\rangle$ for all $t \in [0,1]$. Since the preceding equalities hold for every $x' \in E'$, we get $u(t) = \int_0^1 G(t,s)\ddot{u}(s)\,ds$ for all $t \in [0,1]$.

2) Let $t \in [0,\theta[$. For every fixed $s \in [0,1]$ and for every small $h > 0$ with $t < t+h < \theta$, we have

$$\begin{aligned}
&\frac{G(t+h,s)-G(t,s)}{h} \\
&= \begin{cases} 0 & \text{if } 0 \le s \le t < t+h, \\ [-t-h+t]/h & \text{if } t+h \le s \le \theta, \end{cases}
\end{aligned}$$

Hence

$$\lim_{h\to 0}\frac{G(t+h,s)-G(t,s)}{h}=\frac{\partial G}{\partial t}(t,s)=\begin{cases}0 & \text{if } 0\le s\le t,\\ -1 & \text{if } t\le s\le \theta,\\ (s-1)/(1-\theta) & \text{if } \theta\le s\le 1.\end{cases}$$

Let $t\in[\theta,1]$. For every fixed $s\in[0,1]$ and for every small $h>0$ with $\theta\le t<t+h<1$, we have

$$\frac{G(t+h,s)-G(t,s)}{h}=\begin{cases}0 & \text{if } 0\le s\le \theta,\\ (-\theta h+sh)/h(1-\theta) & \text{if } \theta\le s\le t<t+h,\\ [h(s-1)]/h(1-\theta) & \text{if } t+h\le s\le 1.\end{cases}$$

Hence

$$\lim_{h\to 0}\frac{G(t+h,s)-G(t,s)}{h}=\frac{\partial G}{\partial t}(t,s)=\begin{cases}0 & \text{if } 0\le s\le \theta,\\ (s-\theta)/(1-\theta) & \text{if } \theta\le s\le t,\\ (s-1)/(1-\theta) & \text{if } t<s\le 1.\end{cases}$$

Thus $G(.,s)$ is right derivable on $[0,1[$ and its value is given by (3.4) and (3.5), accordingly. Similarly, by analogous computations, it is not difficult to check that $G(.,s)$ is left derivable on $]0,1]$ and its value is given by (3.4) and (3.5).

3) From the definition of G, for $0\le t<\theta<s\le 1$

$$\begin{aligned}|G(t,s)| &= |t(s-1)/(1-\theta)|\\ &= t(1-s)/(1-\theta)\le t(1-\theta)/(1-\theta)=t\le 1,\end{aligned}$$

for $\theta\le s\le t\le 1$

$$\begin{aligned}|G(t,s)| &= |\theta(s-t)+s(t-1)|/(1-\theta)\\ &\le \theta(t-s)/(1-\theta)+s(1-t)/(1-\theta)\\ &\le (t-s)/(1-\theta)+(1-t)/(1-\theta)\le (1-s)/(1-\theta)\le 1,\end{aligned}$$

and for $\theta\le t<s\le 1$

$$|G(t,s)|=t|(s-1)|/(1-\theta)\le (1-s)/(1-\theta)\le 1.$$

From (3.4) and (3.5) it is easy to check that

$$\sup_{t,s\in[0,1]}\left|\frac{\partial G}{\partial t}(t,s)\right|\le 1.$$

4) Let $u_f(t)=\int_0^1 G(t,s)f(s)\,ds$ for every $t\in[0,1]$ with $f\in L^1_E([0,1])$. Then $u_f(0)=0$ and by the definition of G

$$u_f(1)=\int^{\theta} -s f(s)\,ds+\int^{1}\frac{\theta(s-1)}{\ldots}f(s)\,ds=u_f(\theta)$$

From 3) and Lebesgue's theorem, u_f is continuous on $[0,1]$. We claim that u_f is derivable. Indeed, from 2), the function $G(.,s)$ is derivable for every fixed $s\in[0,1]$, so is the function $G(.,s)f(s)$. As $\|G(t,s)f(s)\|\le\|f(s)\|$ for all $(t,s)\in[0,1]\times[0,1]$, it follows that u_f is derivable and its derivative $\dot u_f$ is given by

$$\dot u_f(t)=\int_0^1\frac{\partial G}{\partial t}(t,s)f(s)\,ds,\quad \forall t\in[0,1]. \tag{3.9}$$

5) It remains to check that $\dot u_f$ is scalarly derivable a.e on $[0,1]$ and its weak derivative $\ddot u_f$ is equal to f a.e. Indeed by (3.4) we have

$$\dot u_f(t)=\int_\theta^t f(s)\,ds+\frac{1}{1-\theta}\int_\theta^1(s-1)f(s)\,ds$$

for $0\le t<\theta$. Hence for each $x'\in E'$, $\langle x',\ddot u_f(t)\rangle=\langle x',f(t)\rangle$ for almost every $t\in[0,\theta[$. Let (e'_n) be a sequence in E' which separates the points of E. Then we have $\langle e'_n,\ddot u_f(t)\rangle=\langle e'_n,f(t)\rangle$ for all $n\in\mathbb{N}$ and for almost every $t\in[0,\theta[$. So we conclude that $\ddot u_f=f$ on $[0,\theta[$. By (3.5)

$$\dot u_f(t)=\frac{1}{1-\theta}\int_\theta^t(s-\theta)f(s)\,ds+\frac{1}{1-\theta}\int_t^1(s-1)f(s)\,ds$$

for $\theta\le t\le 1$. So, for each $x'\in E'$ we have

$$\langle x',\ddot u_f(t)\rangle=\langle x',\frac{t-\theta}{1-\theta}f(t)+\frac{1-t}{1-\theta}f(t)\rangle=\langle x',f(t)\rangle$$

for almost every $t\in[\theta,1]$ and hence, as above, $\ddot u_f(t)=f(t)$ a.e $t\in[\theta,1]$. ■

Let us mention a useful consequence of Lemma 1.

PROPOSITION 1 *Let E be a separable Banach space and let $f:[0,1]\to E$ be a continuous mapping (respectively a mapping in $L^1_E([0,1])$). Then the function*

$$u_f(t)=\int_0^1 G(t,s)f(s)\,ds,\qquad \forall t\in[0,1]$$

is the unique $\mathcal{C}^2_E([0,1])$-solution (respectively the $W^{2,1}_{B,E}([0,1])$-solution) to the differential equation

$$\begin{cases}\ddot u(t)=f(t),\ t\in[0,1],\\ u(0)=0;\ u(\theta)=u(1).\end{cases}$$

The following is a three boundary version of a result due to Hartman (1964). For the sake of completeness we give the proof in full details since several notations and results in the proof are necessary in further applications.

PROPOSITION 2 *Let E be a separable Banach space and let $f:[0,1]\times E\times E\to E$ be a mapping satisfying: (i) there exist Lipschitz constants λ_1,λ_2 satisfying $\lambda_1+\lambda_2<\frac{1}{2}$ such that*

$$\|f(t,x_1,y_1)-f(t,x_2,y_2)\|\le\lambda_1\|x_1-x_2\|+\lambda_2\|y_1-y_2\| \tag{3.10}$$

(ii) $f(.,x,y)$ *is continuous on* $[0,1]$ *for every fixed* $(x,y) \in E \times E$.

Then the differential equation

$$\begin{cases} \ddot{u}(t) = f(t, u(t), \dot{u}(t)), & t \in [0,1], \\ u(0) = 0; \ u(\theta) = u(1), \end{cases}$$

has a unique solution $u \in \mathcal{C}_E^2([0,1])$.

Proof. We will follow the arguments of the proof given in Hartman (1964, pp. 423–424). Fix $\beta \in]0,1[$ such that $\lambda_1 + \lambda_2 < (1-\beta)/2$. Let $m := \max_{t\in[0,1]} \|f(t,0,0)\|$ and $r > 0$ satisfy

$$m < r[1 - 2(\lambda_1 + \lambda_2)/(1-\beta)].$$

Let us denote by $\mathcal{C}_E^1([0,1])$ the Banach space of all continuous mappings $h : [0,1] \to E$ with continuous derivative, equipped with the norm

$$\|h\|_{\mathcal{C}_E^1([0,1])} = \max\{\max_{t\in[0,1]} \|h(t)\|, \max_{t\in[0,1]} \|\dot{h}(t)\|\}.$$

Let $h \in \mathcal{C}_E^1([0,1])$ with $\|h\|_{\mathcal{C}_E^1([0,1])} \leq r$. By Proposition 1, there is a unique solution u_h in $\mathcal{C}_E^2([0,1])$ of the differential equation

$$\begin{cases} \ddot{u}(t) = f(t, h(t), \dot{h}(t)), & t \in [0,1], \\ u(0) = 0; \ u(\theta) = u(1). \end{cases}$$

Since u_h has the form $u_h(t) = \int_0^1 G(t,s) f(s, h(s), \dot{h}(s))\, ds$ for all $t \in [0,1]$ and since $|G(t,s)| \leq 1$ for all $t, s \in [0,1]$, we have the estimate

$$\|u_h(t)\| \leq \max_{s\in[0,1]} \|f(s, h(s), \dot{h}(s))\|, \quad \forall t \in [0,1].$$

In particular, if $h(t) = 0$, for all $t \in [0,1]$, we have

$$\|u_0(t)\| \leq m, \quad \forall t \in [0,1].$$

Using the equality $\dot{u}_h(t) = \int_0^1 \frac{\partial G}{\partial t}(t,s) f(s, h(s), \dot{h}(s))\, ds$ for all $t \in [0,1]$, and the inequality $|\frac{\partial G}{\partial t}(t,s)| \leq 1$ for all $t, s \in [0,1]$, we also have the estimate

$$\|\dot{u}_0(t)\| \leq m, \quad \forall t \in [0,1].$$

Let $\overline{B}_{\mathcal{C}_E^1([0,1])}(0,r) = \{h \in \mathcal{C}_E^1([0,1]) : \|h\|_{\mathcal{C}_E^1([0,1])} \leq r\}$ be the closed ball of center 0 and radius r in the Banach space $\mathcal{C}_E^1([0,1])$ equipped with the corresponding norm. For the mapping A from $\overline{B}_{\mathcal{C}_E^1([0,1])}(0,r)$ into $\mathcal{C}_E^1([0,1])$ given by $A(h) = u_h$, we can write with $0 < \alpha := 2(\lambda_1 + \lambda_2)/(1-\beta) < 1$. Fixing $h_i \in \overline{B}_{\mathcal{C}^1_E([0,1])}(0,r)$ for $i = 1,2$, we have

$$\begin{aligned}
&u_{h_1}(t) - u_{h_2}(t)\\
&= \int_0^1 G(t,s) f(s, h_1(s), \dot h_1(s))\, ds - \int_0^1 G(t,s) f(s, h_2(s), \dot h_2(s))\, ds\\
&= \int_0^1 G(t,s)[f(s, h_1(s), \dot h_1(s)) - f(s, h_2(s), \dot h_2(s))]\, ds
\end{aligned}$$

and hence

$$\begin{aligned}
&\|u_{h_1}(t) - u_{h_2}(t)\| \le \max_{s\in[0,1]} \|f(s, h_1(s), \dot h_1(s)) - f(s, h_2(s), \dot h_2(s))\|\\
&\le \lambda_1 \max_{s\in\in[0,1]} \|h_1(s) - h_2(s)\| + \lambda_2 \max_{s\in[0,1]} \|\dot h_1(s) - \dot h_2(s)\|\\
&\le (\lambda_1+\lambda_2)\|h_1 - h_2\|_{\mathcal{C}^1_E([0,1])} < \frac{2}{1-\beta}(\lambda_1+\lambda_2)\|h_1 - h_2\|_{\mathcal{C}^1_E([0,1])}\\
&= \alpha \|h_1 - h_2\|_{\mathcal{C}^1_E([0,1])}. \qquad (3.12)
\end{aligned}$$

In the same way, the equality

$$\dot u_{h_1}(t) - \dot u_{h_2}(t) = \int_0^1 \frac{\partial G}{\partial t}(t,s)[f(s, h_1(s), \dot h_1(s)) - f(s, h_2(s), \dot h_2(s))]\, ds$$

ensures

$$\|\dot u_{h_1}(t) - \dot u_{h_2}(t)\| \le \alpha \|h_1 - h_2\|_{\mathcal{C}^1_E([0,1])}$$

for all $t \in [0,1]$, which gives, together with (3.12)

$$\|A(h_1) - A(h_2)\|_{\mathcal{C}^1_E([0,1])} \le \alpha \|h_1 - h_2\|_{\mathcal{C}^1_E([0,1])}.$$

So, by Theorem 0.1 in Hartman (1964), the mapping A admits a unique fixed point, that is, the unique $\mathcal{C}^2_E([0,1])$-solution of the differential equation under consideration. ■

Using Proposition 2 we are able to produce the following variant.

THEOREM 1 *Suppose that E is a finite dimensional space. Let $f : [0,1] \times E \times E \to E$ be a mapping satisfying the following conditions:*

(i) *for any fixed $(x,y) \in E \times E$, $f(.,x,y)$ is Lebesgue measurable on $[0,1]$,*

(ii) *there is a constant $c > 0$ such that $\|f(t,x,y)\| \le c(1 + \|x\| + \|y\|)$ for all $(t,x,y) \in [0,1] \times E \times E$,*

(iii) *f satisfies condition (3.10) of Proposition 2, that is, there exist Lipschitz constants λ_1, λ_2 satisfying $\lambda_1 + \lambda_2 < \frac{1}{2}$ such that*

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \le \lambda_1\|x_1 - x_2\| + \lambda_2\|y_1 - y_2\|$$

Then the differential equation

$$\begin{cases} \ddot{u}(t) = f(t, u(t), \dot{u}(t)), & a.e.\ t \in [0,1], \\ u(0) = 0; \ u(\theta) = u(1), \end{cases}$$

has a unique solution $u \in W_E^{2,1}([0,1])$. *Further, for some constant* $m > 0$ *which depends only on* c, λ_1, λ_2, *one has* $\|\ddot{u}(t)\| \leq m$ *for almost all* $t \in [0,1]$.

Proof. Fix $\beta \in]0,1[$ such that $\lambda_1 + \lambda_2 < (1-\beta)/2$.

a) Existence. Step 1. By Scorza-Dragoni theorem, for every $\varepsilon > 0$ there is a compact set $J_\varepsilon \subset [0,1]$ such that the Lebesgue measure of $[0,1] \setminus J_\varepsilon$ is less than ε and the restriction $f|J_\varepsilon \times E \times E$ is continuous. So, there exists an increasing sequence of compact sets (J_n) in $[0,1]$ such that the Lebesgue measure of $[0,1] \setminus J_n$ tends to 0 when $n \to \infty$ and the restriction of f to $J_n \times E \times E$ is continuous. Let $\widetilde{f}_n$ be the Dugundji continuous extension of $f|J_n \times E \times E$ to $[0,1] \times E \times E$. Note that $\widetilde{f}_n$ satisfies (ii) and (iii), namely

$$\|\widetilde{f}_n(t,x,y)\| \leq c(1 + \|x\| + \|y\|), \ \forall (t,x,y) \in [0,1] \times E \times E, \tag{3.13}$$

and

$$\|\widetilde{f}_n(t,x_1,y_1) - \widetilde{f}_n(t,x_2,y_2)\| \leq \lambda_1 \|x_1 - x_2\| + \lambda_2 \|y_1 - y_2\|, \tag{3.14}$$

for all $(t,x_1,y_1), (t,x_2,y_2) \in [0,1] \times E \times E$. By (3.13) we have

$$\|\widetilde{f}_n(t,0,0)\| \leq c, \ \forall t \in [0,1].$$

So $m_n := \max_{t \in [0,1]} \|\widetilde{f}_n(t,0,0)\| \leq c$ for all $n \in \mathbb{N}$. (Note that the results in this step hold when E is a separable Banach space). It is obvious that $\widetilde{f}_n$ satisfies the hypotheses of Proposition 2. Choosing $r > 0$ with

$$c < r[1 - 2(\lambda_1 + \lambda_2)/(1-\beta)],$$

we apply the arguments of the proof of Proposition 2 to each $\widetilde{f}_n$ in order to obtain an estimate for the $C_E^2([0,1])$-solution u_n to

$$\begin{cases} \ddot{u}_n(t) = \widetilde{f}_n(t, u_n(t), \dot{u}_n(t)), & t \in [0,1], \\ u_n(0) = 0; \ u_n(\theta) = u_n(1), \end{cases}$$

so that

$$\max_{t \in [0,1]} \|u_n(t)\| \leq r \quad \text{and} \quad \max_{t \in [0,1]} \|\dot{u}_n(t)\| \leq r.$$

Step 2. Coming back to the preceding equation, we see that

$$\|\ddot{u}_n(t)\| = \|\widetilde{f}_n(t, u_n(t), \dot{u}_n(t))\| \leq c(1 + \|u_n(t)\| + \|\dot{u}_n(t)\|) \leq c(1 + 2r)$$

for all $n \in \mathbb{N}$ and for all $t \in [0,1]$. So, by extracting a subsequence, we may

$(u_n(.))$ converges to a continuous function $u(.)$ having absolutely continuous first derivative $\dot{u}$ with $\ddot{u} = w$ and satisfying $u(0) = 0$ and $u(\theta) = u(1)$. It remains to check that

$$\ddot{u}(t) = f(t, u(t), \dot{u}(t)) \text{ a.e. } t \in [0, 1].$$

By construction, there is a Lebesgue null set N_n such that

$$\ddot{u}_n(t) = f(t, u_n(t), \dot{u}_n(t))$$

for all $t \in J_n \backslash N_n$. Let $N_0 := ([0,1] \backslash \cup_n J_n) \cup (\cup_n N_n)$ which is Lebesgue-negligible. If $t \notin N_0$, there is an integer $p := p(t)$ such that

$$\ddot{u}_n(t) = f(t, u_n(t), \dot{u}_n(t))$$

for all $n \geq p$, which entails

$$\limsup_n \langle x', \ddot{u}_n(t) \rangle = \limsup_n \langle x', f(t, u_n(t), \dot{u}_n(t)) \rangle \leq \langle x', f(t, u(t), \dot{u}(t)) \rangle$$

for all $x' \in E'$ and for all $n \geq p$. It follows that, for every measurable set $A \subset [0, 1]$ and for every $x' \in E'$,

$$\lim_n \int_A \langle x', \ddot{u}_n(t) \rangle \, dt = \int_A \langle x', \ddot{u}(t) \rangle \, dt \leq \int_A \langle x', f(t, u(t), \dot{u}(t)) \rangle \, dt,$$

using Fatou's lemma. Consequently $\ddot{u}(t) = f(t, u(t), \dot{u}(t))$ for a.e. $t \in [0, 1]$.

b) Uniqueness. Let u_1 and u_2 be two $W_E^{2,1}([0,1])$-solutions to the differential equation

$$\begin{cases} \ddot{u}(t) = f(t, u(t), \dot{u}(t)), \text{ a.e. } t \in [0, 1], \\ u(0) = 0; \ u(\theta) = u(1). \end{cases}$$

For each $t \in [0, 1]$, we have

$$\begin{aligned} &\|\ddot{u}_1(t) - \ddot{u}_2(t)\| = \|f(t, u_1(t), \dot{u}_1(t)) - f(t, u_2(t), \dot{u}_2(t))\| \\ &\leq \lambda_1 \|u_1(t) - u_2(t)\| + \lambda_2 \|\dot{u}_1(t) - \dot{u}_2(t)\| \\ &= \lambda_1 \| \int_0^1 G(t,s)(\ddot{u}_1(s) - \ddot{u}_2(s)) \, ds\| + \lambda_2 \| \int_0^1 \frac{\partial G}{\partial t}(t,s)(\ddot{u}_1(s) - \ddot{u}_2(s)) \, ds\| \\ &\leq (\lambda_1 + \lambda_2) \|\ddot{u}_1 - \ddot{u}_2\|_{L_E^1([0,1])}. \end{aligned}$$

Thus

$$\|\ddot{u}_1 - \ddot{u}_2\|_{L_E^1([0,1])} \leq (\lambda_1 + \lambda_2) \|\ddot{u}_1 - \ddot{u}_2\|_{L_E^1([0,1])} < \frac{1}{2} \|\ddot{u}_1 - \ddot{u}_2\|_{L_E^1([0,1])},$$

which ensures $\ddot{u}_1 = \ddot{u}_2$, and hence by (3.3), we get $u_1 = u_2$. ∎

Now we present a Bolza-type example of an optimal control problem. Let

$\Gamma : [0,1] \to k(Z)$ be a compact valued Lebesgue measurable multifunction from $[0,1]$ to Z and $\mathcal{M}^1_+(Z)$ be the set of all probability Radon measures on Z. It is well-known that $\mathcal{M}^1_+(Z)$ is a compact metrizable space for the $\sigma(\mathcal{C}(Z)', \mathcal{C}(Z))$-topology.

Let E be a finite dimensional space. Consider a mapping $f : [0,1] \times E \times E \times Z \to E$ satisfying:

(i) for every fixed $t \in [0,1]$, $f(t,.,.,.)$ is continuous on $E \times E \times Z$;

(ii) for every $(x,y,z) \in E \times E \times Z$, $f(.,x,y,z)$ is Lebesgue-measurable on $[0,1]$;

(iii) there is a constant $c > 0$ such that $f(t,x,y,Z) \subset c(1 + \|x\| + \|y\|)\overline{B}_E(0,1)$ for all $(t,x,y) \in [0,1] \times E \times E$;

(iv) there exist Lipschitz constants λ_1, λ_2 satisfying $\lambda_1 + \lambda_2 < \frac{1}{2}$ such that

$$\|f(t,x_1,y_1,z) - f(t,x_2,y_2,z)\| \leq \lambda_1\|x_1 - x_2\| + \lambda_2\|y_1 - y_2\|$$

for all $(t,x_1,y_1,z), (t,x_2,y_2,z) \in [0,1] \times E \times E \times Z$.

We consider the $W^{2,1}_E([0,1])$-solutions set of the two following second order differential equation

$$(\mathcal{D}_{\mathcal{O}}) \quad \begin{cases} \ddot{u}_\zeta(t) = f(t, u_\zeta(t), \dot{u}_\zeta(t), \zeta(t)) \text{ a.e. } t \in [0,1], \\ u_\zeta(0) = 0, \ u_\zeta(\theta) = u_\zeta(1), \end{cases}$$

where ζ belongs to the set S_Γ of all original controls, which means that ζ is a Lebesgue-measurable mapping from $[0,1]$ into Z with $\zeta(t) \in \Gamma(t)$ for a.e. $t \in [0,1]$, and

$$(\mathcal{D}_{\mathcal{R}}) \quad \begin{cases} \ddot{u}_\nu(t) = \int_{\Gamma(t)} f(t, u_\nu(t), \dot{u}_\nu(t), z)\, \nu_t(dz) \text{ a.e. } t \in [0,1], \\ u_\nu(0) = 0, u_\nu(\theta) = u_\nu(1), \end{cases}$$

where ν belongs to the set $\mathcal{R}$ of all relaxed controls, which means that ν is a Lebesgue-measurable selection of the multifunction Σ defined by

$$\Sigma(t) := \{\sigma \in \mathcal{M}^1_+(Z) : \sigma(\Gamma(t)) = 1\}$$

for all $t \in [0,1]$. Note that the existence of $W^{2,1}_E([0,1])$-solutions for the preceding equations follows from Theorem 1, because the function

$$g : (t,x,y,\nu) \mapsto \int_Z f(t,x,y,z)\, \nu(dz),$$

$(t,x,y,\nu) \in [0,1] \times E \times E \times \mathcal{M}^1_+(Z)$, inherits the properties of the function f, namely, the following hold

(i)′ for every fixed $t \in [0,1]$, $g(t,.,.,.)$ is continuous on $E \times E \times \mathcal{M}^1_+(Z)$;

(ii)′ for every $(x, y, \nu) \in E \times E \times \mathcal{M}^1_+(Z)$, $g(., x, y, \nu)$ is Lebesgue-measurable on $[0, 1]$;

(iii)′ there is a constant $c > 0$ such that $g(t, x, y, \mathcal{M}^1_+(Z)) \subset c(1 + \|x\| + \|y\|)\overline{B}_E(0, 1)$ for all $(t, x, y) \in [0, 1] \times E \times E$;

(iv)′ there exist Lipschitz constants λ_1, λ_2 with $\lambda_1 + \lambda_2 < \frac{1}{2}$ such that

$$\|g(t, x_1, y_1, \nu) - g(t, x_2, y_2, \nu)\| \leq \lambda_1\|x_1 - x_2\| + \lambda_2\|y_1 - y_2\|$$

for all $(t, x_1, y_1, \nu), (t, x_2, y_2, \nu) \in [0, 1] \times E \times E \times \mathcal{M}^1_+(Z)$.

For the sake of completeness, let us mention a general result of convergence for Young measures that we need in the proof of next theorems.

Proposition 3 *Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, S be a Polish space and E be a separable Banach space. Let (u^n) be a sequence of $\mathcal{F}$-measurable mappings from Ω to E which converges pointwise on Ω to a $\mathcal{F}$-measurable mapping u^∞ and (ζ^n) a sequence of $\mathcal{F}$-measurable mappings from Ω to S which converges narrowly to a Young measure $\lambda^\infty \in \mathcal{Y}(\Omega, \mathcal{F}, P; \mathcal{M}^1_+(S))$. Let $J : \Omega \times E \times S \to \mathbb{R}$ be a Carathéodory integrand (that is, $J(\omega, ., .)$ is continuous on $E \times S$ for every $\omega \in \Omega$ and $J(., x, u)$ is $\mathcal{F}$-measurable on Ω, for every $(x, u) \in E \times S$) such that the sequence $(J(., u^n(.), \zeta^n(.))_n$ is uniformly integrable. Then we have*

$$\lim_{n\to\infty} \int_\Omega J(\omega, u^n(\omega), \zeta^n(\omega))P(d\omega) = \int_\Omega [\int_S J(\omega, u^\infty(\omega), s)\lambda^\infty_\omega(ds)]P(d\omega).$$

Proof. See, Castaing et al (2002), Proposition 8.1.5.

Now comes a Bolza-type optimal control problem associated with a second order-type differential equation where the controls are Young measures.

Theorem 2 *Assume that E is a finite dimensional space, $I : [0, 1] \times E \times E \times Z \to \mathbb{R}$ is a Carathéodory integrand (that is, $I(t, ., ., .)$ is continuous on $E \times E \times Z$ for every $t \in [0, 1]$ and $I(., x, y, z)$ is Lebesgue-measurable on $[0, 1]$, for every $(x, y, z) \in E \times E \times Z$) which satisfies the condition:*

$(\mathcal{C})$ For any bounded sequence (u^n) and (v^n) in $\mathcal{C}_E([0, 1])$ and for any sequence (ζ^n) in S_Γ, the sequence $(I(., u^n(.), v^n(.), \zeta^n(.)))_n$ is uniformly integrable.

Let us consider the control problems

$$(P_\mathcal{O}) : \inf_{\zeta \in S_\Gamma} \int_0^1 I(t, u_\zeta(t), \dot{u}_\zeta(t), \zeta(t))\, dt$$

and

$$(P_\mathcal{R}) : \inf \int_0^1 \Big[\int I(t, u_\nu(t), \dot{u}_\nu(t), z)\, \nu_t(dz)\Big]\, dt$$

where u_ζ (respectively u_ν) is the unique solution associated with ζ (respectively ν) to the differential equation $(\mathcal{D}_\mathcal{O})$ (respectively $(\mathcal{D}_\mathcal{R})$). Then one has $\inf(\mathcal{P}_O) = \inf(\mathcal{P}_R)$.

Proof. Let (ζ_n) be a minimizing sequence for $(\mathcal{P}_O)$, that is,

$$\lim_{n\to\infty} \int_0^1 I(t, u_{\zeta_n}(t), \dot{u}_{\zeta_n}(t), \zeta_n(t))\,dt = \inf_{\zeta(.)\in S_\Gamma} \int_0^1 I(t, u_\zeta(t), \dot{u}_\zeta(t), \zeta(t))\,dt$$

where u_ζ is the unique $W_E^{2,1}([0,1])$-solution to

$$\begin{cases} \ddot{u}_\zeta(t) = f(t, u_\zeta(t), \dot{u}_\zeta(t), \zeta(t)) \text{ a.e. } t \in [0,1], \\ u_\zeta(0) = 0,\ u_\zeta(\theta) = u_\zeta(1), \end{cases}$$

and, for each n, u_{ζ_n} is the unique $W_E^{2,1}([0,1])$-solution to

$$\begin{cases} \ddot{u}_{\zeta_n}(t) = f(t, u_{\zeta_n}(t), \dot{u}_{\zeta_n}(t), \zeta_n(t)) \text{ a.e. } t \in [0,1], \\ u_{\zeta_n}(0) = 0,\ u_{\zeta_n}(\theta) = u_{\zeta_n}(1). \end{cases}$$

As the sequence (u_{ζ_n}) is relatively compact in $C_E([0,1])$ in view of Theorem 1 and Lemma 1, we may suppose, by extracting subsequences, that (u_{ζ_n}) converges uniformly to a $W_E^{2,1}([0,1])$ function $u(.)$, $(\dot{u}_{\zeta_n})$ converges pointwise to $\dot{u}$ and $(\ddot{u}_{\zeta_n})$ $\sigma(L^1, L^\infty)$-converges to $\ddot{u}$ with $u(0) = 0, u(\theta) = u(1)$. (See Theorem 4 and Lemma 5 below for a general compactness result when E is a separable Banach space). Further, the sequence (δ_{ζ_n}) of Young measures associated with (ζ_n) is relatively narrowly compact in the space $\mathcal{Y}([0,1]; \mathcal{M}_+^1(Z))$ of Young measures, and hence by extracting a subsequence, we may suppose that (δ_{ζ_n}) converges narrowly to a Young measure ν with $\nu_t(\Gamma(t)) = 1$ a.e. Using the $\sigma(L^1, L^\infty)$-convergence of $(\ddot{u}_{\zeta_n})$ towards $\ddot{u}$ and the narrow convergence of (δ_{ζ_n}) towards ν, we get, for every Lebesgue measurable set $A \subset [0,1]$

$$\begin{aligned} &\int_A \ddot{u}(t)\,dt = \lim_{n\to\infty} \int_A \ddot{u}_{\zeta_n}(t)\,dt \\ &= \lim_{n\to\infty} \int_A f(t, u_{\zeta_n}(t), \dot{u}_{\zeta_n}(t), \zeta_n(t))\,dt = \int_A \Big[\int_Z f(t, u(t), \dot{u}(t), z)\,\nu_t(dz)\Big]\,dt, \end{aligned}$$

the last equality following from Theorem 1 and Proposition 3. So we deduce that

$$\ddot{u}(t) = \int_Z f(t, u(t), \dot{u}(t), z)\,\nu_t(dz) = \int_{\Gamma(t)} f(t, u(t), \dot{u}(t), z)\,\nu_t(dz)$$

for a.e. $t \in [0,1]$ (because $\nu_t(\Gamma(t)) = 1$ a.e.) with $u(0) = 0, u(\theta) = u(1)$. So, we have necessarily $u(.) = u_\nu(.)$, where u_ν is the unique $W_E^{2,1}([0,1])$-solution uniformly integrable assumption $(\mathcal{C})$ for the cost functional I and applying again Proposition 3 yields

$$\inf_{\zeta(.)\in S_\Gamma}\int_0^1 I(t,u_\zeta(t),\dot u_\zeta(t),\zeta(t))\,dt = \lim_{n\to\infty}\int_0^1 I(t,u_{\zeta_n}(t),\dot u_{\zeta_n}(t),\zeta_n(t))\,dt$$
$$= \int_0^1 [\int_Z I(t,u_\nu(t),\dot u_\nu(t),z)\,\nu_t(dz)]\,dt,$$

with

$$\begin{cases} \ddot u_\nu(t) = \int_{\Gamma(t)} f(t,u_\nu(t),\dot u_\nu(t),z)\,\nu_t(dz) \ \text{ a.e. } t\in[0,1],\\ u_\nu(0)=0, u_\nu(\theta)=u_\nu(1),\end{cases}$$

and hence

$$\inf(P_{\mathcal{R}}) \le \int_0^1 \Big[\int_Z I(t,u_\nu(t),\dot u_\nu(t),z)\,\nu_t(dz)\Big]\,dt = \inf(P_{\mathcal{O}}).$$

As $\inf(P_{\mathcal{O}}) \ge \inf(P_{\mathcal{R}})$, the proof is therefore complete. ■

REMARK. Theorem 2 and its proof provide new results in relaxed control theory because we deal here with a second order type differential equation.

At this point we are going to show that the $W_E^{2,1}([0,1])$-solutions set $(\mathcal{S}_{\mathcal{R}})$ of $(\mathcal{D}_{\mathcal{R}})$ is compact in $\mathcal{C}_E([0,1])$ so that $\inf(P_{\mathcal{O}}) = \min(P_{\mathcal{R}})$.

THEOREM 3 *Under the hypotheses of Theorem 2, the $W_E^{2,1}([0,1])$-solutions set $(\mathcal{S}_{\mathcal{R}})$ of $(\mathcal{D}_{\mathcal{R}})$ is compact with respect to the topology of uniform convergence and the $W_E^{2,1}([0,1])$-solutions set $(\mathcal{S}_{\mathcal{O}})$ of $(\mathcal{D}_{\mathcal{O}})$ is dense in $(\mathcal{S}_{\mathcal{R}})$ with respect to the same topology.*

Proof. Theorem 3 follows from the lemma below.

LEMMA 2 *Let $\mathcal{S}_\Gamma$ and $\mathcal{S}_\Sigma$ be the set of all Lebesgue measurable selections of Γ and Σ respectively. Then the following assertions hold:*

(a) $\mathcal{S}_\Gamma$ is dense in $\mathcal{S}_\Sigma$ with respect to the topology $\sigma(L^\infty_{C(Z)'}, L^1_{C(Z)})$;

(b) Let (ν^n) be a sequence in $\mathcal{S}_\Sigma$ which converges $\sigma(L^\infty_{C(Z)'}, L^1_{C(Z)})$ to $\nu^\infty \in \mathcal{S}_\Sigma$, and, for each $n\in\mathbb{N}\cup\{\infty\}$, let u_{ν^n} be the unique solution to

$$\begin{cases} \ddot u_{\nu^n}(t) = \int_{\Gamma(t)} f(t,u_{\nu^n}(t),\dot u_{\nu^n}(t),z)\,\nu^n_t(dz) \ \text{ a.e. } t\in[0,1],\\ u_{\nu^n}(0)=0,\ u_{\nu^n}(\theta)=u_{\nu^n}(1),\end{cases}$$

then $(u_{\nu^n}(.))$ converges uniformly to $u_{\nu^\infty}(.)$.

Proof. (a) follows from Castaing et al. (2002), Lemma 7.1.1.

(b) Fix $\beta \in]0,1[$ such that $\lambda_1+\lambda_2 < (1-\beta)/2$. Using the estimation in

(u_{ν^n}) converges uniformly to a $W_E^{2,1}([0,1])$ function $u^\infty(.)$ and $(\dot{u}_{\nu^n}(.))$ converges pointwise to $\dot{u}^\infty(.)$. For each $t \in [0,1]$ and for each $n \in \mathbb{N}$, let us write

$$u_{\nu^\infty}(t) - u_{\nu^n}(t) = \int_0^1 G(t,s)\Big[\int_Z f(s, u_{\nu^\infty}(s), \dot{u}_{\nu^\infty}(s), z)\, \nu_s^\infty(dz)\Big]\, ds$$
$$- \int_0^1 G(t,s)[\int_Z f(s, u_{\nu^\infty}(s), \dot{u}_{\nu^\infty}(s), z)\, \nu_s^n(dz)]\, ds$$
$$+ \int_0^1 G(t,s)[\int_Z f(s, u_{\nu^\infty}(s), \dot{u}_{\nu^\infty}(s), z)\, \nu_s^n(dz)]\, ds$$
$$- \int_0^1 G(t,s)[\int_Z f(s, u_{\nu^n}(s), \dot{u}_{\nu^n}(s), z)\, \nu_s^n(dz)]\, ds.$$

By assumption, we have

$$\|f(s, u_{\nu^\infty}(s), \dot{u}_{\nu^\infty}(s), z) - f(s, u_{\nu^n}(s), \dot{u}_{\nu^n}(s), z)$$
$$\leq \lambda_1 \|u_{\nu^\infty}(s) - u_{\nu^n}(s)\| + \lambda_2 \|\dot{u}_{\nu^\infty}(s) - \dot{u}_{\nu^n}(s)\|$$
$$\leq (\lambda_1 + \lambda_2)(\|u_{\nu^\infty}(s) - u_{\nu^n}(s)\| + \|\dot{u}_{\nu^\infty}(s) - \dot{u}_{\nu^n}(s)\|)$$
$$< \frac{1-\beta}{2}(\|u_{\nu^\infty}(s) - u_{\nu^n}(s)\| + \|\dot{u}_{\nu^\infty}(s) - \dot{u}_{\nu^n}(s)\|)$$

for all $s \in [0,1]$ and for all $z \in Z$. For simplicity, for each $t \in [0,1]$ and for each $n \in \mathbb{N}$, let us set

$$v^n(t) = \int_0^1 \Big[\int_Z G(t,s) f(s, u_{\nu^\infty}(s), \dot{u}_{\nu^\infty}(s), z)\, \nu_s^n(dz)\Big]\, ds$$

and

$$v^\infty(t) = \int_0^1 \Big[\int_Z G(t,s) f(s, u_{\nu^\infty}(s), \dot{u}_{\nu^\infty}(s), z)\, \nu_s^\infty(dz)]\, ds.$$

Note that the integrand $\varphi_t : (s,z) \mapsto G(t,s) f(s, u_{\nu^\infty}(s), \dot{u}_{\nu^\infty}(s), z)$ is Carathéodory integrable on $[0,1] \times Z$ because there exists a constant $M > 0$ such that

$$\sup_{(t,s,z) \in [0,1]\times[0,1]\times Z} \|G(t,s) f(s, u_{\nu^\infty}(s), \dot{u}_{\nu^\infty}(s), z)\| \leq M < +\infty$$

using the estimations obtained in Theorem 1 and Lemma 1. Since (ν^n) converges $\sigma(L^\infty_{C(Z)'}, L^1_{C(Z)})$ to ν^∞, we have that $\lim_{n\to\infty} v^n(t) = v^\infty(t)$ for every $t \in [0,1]$. Therefore, for each $t \in [0,1]$, we have the estimation

$$\|u_{\nu^\infty}(t) - u_{\nu^n}(t)\| < \|v^\infty(t) - v^n(t)\|$$
$$+ \frac{1-\beta}{2} \int_0^1 [\|u_{\nu^\infty}(s) - u_{\nu^n}(s)\| + \|\dot{u}_{\nu^\infty}(s) - \dot{u}_{\nu^n}(s)\|]\, ds. \tag{3.15}$$

Since

$$\dot{u}_{\nu^\infty}(t) = \int_0^1 \frac{\partial G}{\partial t}(t,s)\Big[\int_Z f(s, u_{\nu^\infty}(s), \dot{u}_{\nu^\infty}(s), z)\, \nu_s^\infty(dz)\Big]\, ds, \ \forall t \in [0,1]$$

and

$$\dot{u}_{\nu^n}(t) = \int_0^1 \frac{\partial G}{\partial t}(t,s)\Big[\int_Z f(s, u_{\nu^n}(s), \dot{u}_{\nu^n}(s), z)\,\nu_s^n(dz)\Big]\,ds, \qquad \forall t \in [0,1],$$

using similar computations and Lemma 1, we get the estimation

$$\|\dot{u}_{\nu^\infty}(t) - \dot{u}_{\nu^n}(t)\| < \|w^\infty(t) - w^n(t)\| + \frac{1-\beta}{2}\int_0^1 [\|u_{\nu^\infty}(s) - u_{\nu^n}(s)\| + \|\dot{u}_{\nu^\infty}(s) - \dot{u}_{\nu^n}(s)\|]\,ds, \tag{3.16}$$

where

$$w^n(t) = \int_0^1 \Big[\int_Z \frac{\partial G}{\partial t}(t,s) f(s, u_{\nu^\infty}(s), \dot{u}_{\nu^\infty}(s), z)\,\nu_s^n(dz)\Big]\,ds, \qquad \forall t \in [0,1],$$

and

$$w^\infty(t) = \int_0^1 \Big[\int_Z \frac{\partial G}{\partial t}(t,s) f(s, u_{\nu^\infty}(s), \dot{u}_{\nu^\infty}(s), z)\,\nu_s^\infty(dz)\Big]\,ds, \;\; \forall t \in [0,1],$$

with $w^n(t) - w^\infty(t) \to 0$ for every $t \in [0,1]$. Adding (3.15) and (3.16) and integrating we get the estimation

$$\int_0^1 [\|u_{\nu^\infty}(s) - u_{\nu^n}(s)\| + \|\dot{u}_{\nu^\infty}(s) - \dot{u}_{\nu^n}(s)\|]\,ds < \beta^{-1}\Big[\int_0^1 \|v^\infty(t) - v^n(t)\|\,dt + \int_0^1 \|w^\infty(t) - w^n(t)\|\,dt\Big]. \tag{3.17}$$

Taking the limits when $n \to +\infty$ in (3.17) gives

$$\int_0^1 \|u_{\nu^\infty}(s) - u^\infty(s)\|\,ds = \int_0^1 \|\dot{u}_{\nu^\infty}(s) - \dot{u}^\infty(s)\|\,ds = 0.$$

The preceding arguments show that for any subsequence of (u_{ν^n}) still denoted by (u_{ν^n}) there is a subsequence which converges uniformly to u_{ν^∞}. Thus (u_{ν^n}) converges uniformly to u_{ν^∞} and the proof is therefore complete. ■

4. Existence results in $W_{B,E}^{2,1}([0,1])$ for multivalued differential equations in Banach spaces

The following result is related to some topological properties of solutions set of a special class of multivalued differential equations with three boundary conditions in Banach spaces.

THEOREM 4 *Let E be a separable Banach space and let $\Gamma : [0,1] \rightrightarrows E$ be a convex compact valued, measurable and integrably bounded multifunction. Then*

1) The $W^{2,1}_{B,E}([0,1])$*-solutions set* $\mathcal{X}_\Gamma$ *of the differential inclusion*

$$\begin{cases} \ddot{u}(t) \in \Gamma(t), \;\; a.e.t \in [0,1], \\ u(0) = 0; \;\; u(\theta) = u(1), \end{cases}$$

is convex compact in the Banach space $\mathcal{C}_E([0,1])$ *of all continuous mappings from* $[0,1]$ *into* E *endowed with the topology of uniform convergence.*

2) Let $\mathrm{ext}(\Gamma(t))$ *be the set of extreme points of* $\Gamma(t)$*. Then the* $W^{2,1}_{B,E}([0,1])$*-solutions set* $\mathcal{X}_{\mathrm{ext}(\Gamma)}$ *of the differential inclusion*

$$\begin{cases} \ddot{u}(t) \in \mathrm{ext}(\Gamma(t)), \;\; a.e. \; t \in [0,1], \\ u(0) = 0; \;\; u(\theta) = u(1), \end{cases}$$

is a G_δ *dense subset of the convex compact set* $\mathcal{X}_\Gamma$*.*

Proof. Step 1. Let us recall that the set S^1_Γ of all measurable selections of Γ is convex and $\sigma(L^1_E, L^\infty_{E'})$-compact. For the sake of completeness we sketch the proof. Let $g \in L^\infty_{E'}([0,1])$. The measurable selection theorem and the Strassen theorem provide a measurable selection $f \in S^1_\Gamma$ such that

$$\delta^*(g, S^1_\Gamma) = \int_0^1 \delta^*(g(t), \Gamma(t))\, dt = \int_0^1 \langle g(t), f(t) \rangle\, dt.$$

It follows from the James theorem that the bounded closed convex set S^1_Γ in $L^1_E([0,1])$ is $\sigma(L^1_E, L^\infty_{E'})$ compact. Furthermore, the set-valued integral

$$\int_0^1 \Gamma(t)\, dt = \Big\{ \int_0^1 f(t)\, dt : f \in S^1_\Gamma \Big\}$$

is convex and norm-compact. This follows again from the Strassen formula

$$\delta^*(x', \int_0^1 \Gamma(t)\, dt) = \int_0^1 \delta^*(x', \Gamma(t))\, dt, \qquad \forall x' \in E'.$$

It is easily seen from the Banach-Dieudonné theorem (using Lebesgue theorem) that the function

$$x' \mapsto \delta^*\Big(x', \int_0^1 \Gamma(t)\, dt\Big) = \int_0^1 \delta^*(x', \Gamma(t))\, dt$$

is continuous on the closed unit ball $\overline{B}_{E'}$ of E' equipped with the topology of compact convergence. Hence $\int_0^1 \Gamma(t)\, dt$ is norm compact. See Castaing (1969,1972); Castaing and Valadier (1977) for a more general result. Again by Castaing and Valadier (1977) the multifunction $\mathrm{ext}(\Gamma(.))$ is measurable and the set $S^1_{\mathrm{ext}(\Gamma)}$ of measurable selections of the multifunction $\mathrm{ext}(\Gamma(.))$ is a dense

Step 2. In view of Lemma 1 and Proposition 1, the solutions set $\mathcal{X}_\Gamma$ and $\mathcal{X}_{\mathrm{ext}(\Gamma)}$ are characterized by

$$\mathcal{X}_\Gamma = \Big\{u_f : [0,1] \to E \mid u_f(t) = \int_0^1 G(t,s)f(s)\,ds, \ \ \forall t \in [0,1]; \ f \in S^1_\Gamma\Big\}$$

and

$$\mathcal{X}_{\mathrm{ext}(\Gamma)} = \Big\{u_f : [0,1] \to E \mid u_f(t) = \int_0^1 G(t,s)f(s)\,ds, \ \forall t \in [0,1]; \ f \in S^1_{\mathrm{ext}(\Gamma)}\Big\}$$

respectively. Since

$$\|u_f(t) - u_f(\tau)\| \le \int_0^1 |G(t,s) - G(\tau,s)|\,\|f(s)\|\,ds \le \int_0^1 |G(t,s) - G(\tau,s)|\,|\Gamma(s)|\,ds$$

for all $f \in S^1_\Gamma$ and for all $t, \tau \in [0,1]$, $\mathcal{X}_\Gamma$ is equicontinuous in $\mathcal{C}_E([0,1])$. Further, for each $t \in [0,1]$, the set $\mathcal{X}_\Gamma(t)$ is relatively compact in E because it is included in the norm compact set $\int_0^1 G(t,s)\Gamma(s)\,ds$ using the obvious property of G (see (3.3)) and the norm compactness of the multivalued integral of a convex norm compact valued measurable and integrably bounded multifunction mentioned above. We claim that $\mathcal{X}_\Gamma$ is compact in $\mathcal{C}_E([0,1])$. Let (f_n) be a sequence in S^1_Γ. As S^1_Γ is weakly compact in $L^1_E([0,1])$ and the sequence (u_{f_n}) is relatively compact in $\mathcal{C}_E([0,1])$ by Ascoli's theorem, we extract from (f_n) a sequence (f_m) such that (f_m) converges $\sigma(L^1_E, L^\infty_{E'})$ to a function $f \in S^1_\Gamma$ and such that the sequence (u_{f_m}) converges uniformly to a continuous function $\zeta \in \mathcal{C}_E([0,1])$. In particular, for every $x' \in E'$ and for every $t \in [0,1]$, we have

$$\lim_{m\to+\infty} \int_0^1 \langle G(t,s)x', f_m(s)\rangle\,ds = \lim_{m\to+\infty} \langle x', \int_0^1 G(t,s)f_m(s)\rangle\,ds$$
$$= \int_0^1 \langle G(t,s)x', f(s)\rangle\,ds = \langle x', \int_0^1 G(t,s)f(s)\,ds\rangle. \tag{4.1}$$

As the multivalued integral $\int_0^1 G(t,s)\Gamma(s)\,ds$ $(t \in [0,1])$ is norm compact, (4.1) shows that the sequence $(u_{f_m}(.)) = (\int_0^1 G(.,s)f_m(s)\,ds)$ converges pointwise to $u_f(.)$ for E endowed with the strong topology. Since (u_{f_m}) converges uniformly to $\zeta \in \mathcal{C}_E([0,1])$, we get $u_f = \zeta$. This shows the compactness of $\mathcal{X}_\Gamma$ in $\mathcal{C}_E([0,1])$. At this point, it is worth to mention that the sequence $(\dot u_{f_m}(.)) = (\int_0^1 \frac{\partial G}{\partial t}(.,s)f_m(s)\,ds)$ converges pointwise to $\dot u_f(.)$ for E endowed with the strong topology, using the weak convergence of (f_m) and the norm compactness of the

Step 3. Repeating the arguments given in Step 2, it is not difficult to see that the mapping $f \mapsto u_f$ from S^1_Γ into $\mathcal{X}_\Gamma$ is continuous when S^1_Γ is equipped with the topology $\sigma(L^1_E, L^\infty_{E'})$ and $\mathcal{X}_\Gamma$ is equipped with the topology of uniform convergence. Hence we may conclude that $\mathcal{X}_{\text{ext}(\Gamma)}$ is dense in $\mathcal{X}_\Gamma$ with respect to the topology of uniform convergence by recalling that the set $S^1_{\text{ext}(\Gamma)}$ of all measurable selections of ext(Γ) is $\sigma(L^1_E, L^\infty_{E'})$ dense in S^1_Γ (see e.g. Castaing and Valadier, 1977). It is worth observing that, for every $f \in S^1_\Gamma$, u_f and $\dot{u}_f$ satisfy the estimates

$$\|u_f(t)\| \leq \int_0^1 |\Gamma(s)|\, ds \ \text{ and } \ \|\dot{u}_f(t)\| \leq \int_0^1 |\Gamma(s)|\, ds, \qquad \forall t \in [0,1]. \tag{4.2}$$

Step 4. This last part follows from a sharp use of parametric Choquet function initiated in Castaing and Valadier (1977, Theorem IV.3). Let us consider the mapping $\varphi : [0,1] \times E \to [0, +\infty]$

$$\varphi(t,x) = \begin{cases} \sum_{m=1}^\infty \frac{\langle e_m, x\rangle^2}{m^2(1+h_m(t))^2} & \text{if } x \in \Gamma(t) \\ +\infty & \text{if } x \notin \Gamma(t) \end{cases}$$

where (e_m) is a sequence in the dual E' which separates the points of E and $h_m(t) := \sup\{|\langle e_m, x\rangle| : x \in \Gamma(t)\}$. The associated parametric Choquet function is

$$\widehat{\varphi}(t,x) = \inf\{\langle x, y\rangle + \beta_y(t) : y \in E'\},$$

where

$$\beta_y(t) = \sup\{\varphi(t,x) - \langle x, y\rangle : x \in \Gamma(t)\}.$$

We have $\widehat{\varphi}(t,x) \geq \varphi(t,x)$ for all $x \in \Gamma(t)$ and

$$\text{ext}(\Gamma(t)) = \{x \in \Gamma(t) : \varphi(t,x) = \widehat{\varphi}(t,x)\}.$$

So $u_f \in \mathcal{X}_{\text{ext}(\Gamma)}$ iff $u_f \in \mathcal{X}_\Gamma$ and $\varphi(t, f(t)) = \widehat{\varphi}(t, f(t))$ a.e. It turns out that $u_f \in \mathcal{X}_{\text{ext}(\Gamma)}$ iff for every $p \in \mathbb{N}$, $u_f \in \mathcal{X}_\Gamma$ and

$$\int_0^1 \widehat{\varphi}(t, f(t)) - \varphi(t, f(t))\, dt < 1/p.$$

Finally we need to check that the set

$$\mathcal{X}^p_\Gamma := \Big\{u_f \in \mathcal{X}_\Gamma : \int_0^1 \widehat{\varphi}(t, f(t)) - \varphi(t, f(t))\, dt \geq 1/p\Big\}$$

is compact. Since the integrand

$$\psi(t,x) = \begin{cases} \varphi(t,x) - \widehat{\varphi}(t,x) & \text{if } x \in \Gamma(t)

is a normal convex lower semicontinuous integrand, the convex integral functional

$$I_\psi(f) := \int_0^1 \psi(t, f(t))\, dt$$

is weakly sequentially lower semicontinuous on $L^1_E([0,1])$. So, repeating the above arguments involving the sequentially weak compactness of S^1_Γ and the compactness of $\mathcal{X}_\Gamma$, compactness of $\mathcal{X}^p_\Gamma$ follows from the lower semicontinuity property of I_Ψ. This shows that $\mathcal{X}_{\text{ext}(\Gamma)} = \cap_p \mathcal{X}_\Gamma \setminus \mathcal{X}^p_\Gamma$. ∎

COMMENTS. Differential inclusion of the form

$$\dot{u}(t) \in \text{ext } F(t, u(t)), \qquad u(0) = x_0$$

was initiated by Cellina (1980) and De Blasi-Pianigiani (Pianigiani, 1990) in a series of papers. We refer to Gomaa (2000) and the references therein for other related results concerning second order differential inclusions.

Now we proceed to the existence of $W^{2,1}_{B,E}([0,1])$-solutions for the differential inclusion

$$\begin{cases} \ddot{u}(t) \in F(t, u(t), \dot{u}(t)) \text{ a.e. } t \in [0,1], \\ u(0) = 0; \;\; u(\theta) = u(1). \end{cases}$$

THEOREM 5 *Let $F : [0,1] \times E \times E \rightrightarrows E$ be a convex compact valued multifunction, Lebesgue-measurable on $[0,1]$ and upper semicontinuous on $E \times E$ and let $\Gamma : [0,1] \rightrightarrows E$ be a convex compact valued, measurable and integrably bounded multifunction such that $F(t,x,y) \subset \Gamma(t)$ for all $(t,x,y) \in [0,1] \times E \times E$. Then the $W^{2,1}_{B,E}([0,1])$-solutions set of the above differential inclusion is nonempty and compact in $\mathcal{C}_E([0,1])$.*

Proof. Step 1. Taking the results obtained in Theorem 4 into account, a mapping $u : [0,1] \to E$ is a $W^{2,1}_{B,E}([0,1])$-solution of the preceding equation, iff there exists $f \in \mathcal{S}^1_\Gamma$ such that $u(t) := u_f(t) = \int_0^1 G(t,s) f(s)\, ds$, $\forall t \in [0,1]$ and such that $f(t) \in F(t, u_f(t), \dot{u}_f(t))$ for a.e. $t \in [0,1]$. Let us observe that, for any Lebesgue-measurable mappings $v : [0,1] \to E$ and $w : [0,1] \to E$, there is a Lebesgue-measurable selection $s \in \mathcal{S}^1_\Gamma$ such that $s(t) \in F(t, v(t), w(t))$ a.e. Indeed, there exist sequences (v_n) and (w_n) of simple E-valued functions such that (v_n) converges pointwise to v and (w_n) converges pointwise to w respectively, for E endowed with the norm topology. Notice that the multifunctions $F(., v_n(.), w_n(.))$ are Lebesgue-measurable. Let s_n be a Lebesgue-measurable selection of $F(., v_n(.), w_n(.))$. As $s_n(t) \in F(t, v_n(t), w_n(t)) \subset \Gamma(t)$, $\forall t \in [0,1]$ and $\mathcal{S}^1_\Gamma$ is $\sigma(L^1_E, L^\infty_{E'})$-compact, by Eberlein-Smulian theorem, we may extract from (s_n) a subsequence (s'_n) which converges $\sigma(L^1_E, L^\infty_{E'})$ to a function $s \in \mathcal{S}^1_\Gamma$. Here we may invoke the fact that $\mathcal{S}^1_\Gamma$ is a weakly compact metrizable set of to (s'_n) provides a sequence (z_n) with $z_n \in co\{s'_m : m \geq n\}$ such that (z_n) converges pointwise a.e. to s. Using this fact and the pointwise convergence of the sequences (v_n) and (w_n) and the upper semicontinuity of $F(t,.,.)$, it is not difficult to check that $s(t) \in F(t, v(t), w(t))$ a.e.

Step 2. For each $f \in \mathcal{S}^1_\Gamma$, let us set

$$\Phi(f) = \{g \in \mathcal{S}^1_\Gamma : g(t) \in F(t, u_f(t), \dot{u}_f(t)) \text{ a.e.}\}$$

where $u_f(t) = \int_0^1 G(t,s)f(s)\,ds$, for all $t \in [0,1]$ and $\dot{u}_f(t) = \int_0^1 \frac{\partial G}{\partial t}(t,s)f(s)\,ds$, for all $t \in [0,1]$ (see Lemma 1). In view of Step 1, $\Phi(f)$ is a nonempty set. These considerations lead us to the application of the Kakutani-Ky Fan fixed point theorem to the multifunction $\Phi(.)$. It is clear that $\Phi(f)$ is a convex weakly compact subset of $\mathcal{S}^1_\Gamma$. We need to check that $\Phi : \mathcal{S}^1_\Gamma \rightrightarrows \mathcal{S}^1_\Gamma$ is upper semicontinuous on the convex weakly compact metrisable set $\mathcal{S}^1_\Gamma$. Equivalently, we need to prove that the graph of Φ is sequentially weakly compact in $\mathcal{S}^1_\Gamma \times \mathcal{S}^1_\Gamma$. Let (f_n) be a sequence in $\mathcal{S}^1_\Gamma$. By extracting a subsequence we may suppose that (f_n) converges weakly to $f \in \mathcal{S}^1_\Gamma$. It follows that the sequences (u_{f_n}) and $(\dot{u}_{f_n})$, converge pointwise to u_f and $\dot{u}_f$ respectively, for E endowed with the norm topology. Let $g_n \in \Phi(f_n) \subset \mathcal{S}^1_\Gamma$. We may suppose that (g_n) converges weakly to some element $g \in \mathcal{S}^1_\Gamma$. As $g_n(t) \in F(t, u_{f_n}(t), \dot{u}_{f_n}(t))$ a.e., by repeating the arguments given in the end of Step 1 we obtain that $g(t) \in F(t, u_f(t), \dot{u}_f(t))$ a.e. Thus, the graph of Φ is weakly compact in the weakly compact set $\mathcal{S}^1_\Gamma \times \mathcal{S}^1_\Gamma$. Hence, Φ admits a fixed point. So, we have proved the existence of a solution in $W^{2,1}_{B,E}([0,1])$. Compactness of solutions set follows easily from the compactness in $\mathcal{C}_E([0,1])$ of

$$\mathcal{X}_\Gamma = \left\{u_f : [0,1] \to E \mid u_f(t) = \int_0^1 G(t,s)f(s)\,ds,\ \forall t \in [0,1];\ f \in S^1_\Gamma\right\}$$

given in Step 2 of the proof of Theorem 4 and the preceding arguments. ■

REMARK. It is worth to mention that Theorem 5 is valid when we only assume that, for $\gamma := \int_0^1 |\Gamma(t)|\,dt$, F is defined on $[0,1] \times \gamma\overline{B}_E(0,1) \times \gamma\overline{B}_E(0,1)$ and satisfies $F(t,x,y) \subset \Gamma(t)$ for $(t,x,y) \in [0,1] \times \gamma\overline{B}_E(0,1) \times \gamma\overline{B}_E(0,1)$.

Now we present an example of application of Theorem 5 to the existence of $W^{2,1}_E([0,1])$-solutions for a class of second order evolution inclusion of the form

$$\begin{cases} -\ddot{u}(t) \in A(t)u(t) + F(t, u(t), \dot{u}(t)), & \text{a.e. } t \in [0,1], \\ u(0) = 0 \in \overline{D(A(0))},\ u(1) = u(\theta), \end{cases}$$

where $A(t) : E \to 2^E, (t \in [0,1])$ is an *m-accretive* operator in a finite dimensional space. Recall that a multivalued operator $A(t) : E \to 2^E, (t \in [0,1])$ is *m-accretive*, if, for each $t \in [0,1]$ and each $\lambda > 0$, $R(I_E + \lambda A(t)) = E$, and for each $x_1 \in D(A(t)), x_2 \in D(A(t)), y_1 \in A(t)x_1, y_2 \in A(t)x_2$, we have

$$\|x_1 - x_2\| \leq \|(x_1 - x_2) + \lambda(y_1 - y_2)\|, \tag{j}$$

If $A(t)$ is m-accretive, then

$$\begin{aligned} &\frac{1}{\lambda}\|J_\lambda A(t)x - x\| = \|A_\lambda(t)x\| \\ &\leq |A(t)x|_0 := \inf_{y\in A(t)x} \|y\|, \ \forall x \in D(A(t)), \end{aligned} \tag{jj}$$

where $J_\lambda A(t) = (I_E + \lambda A(t))^{-1}$ is the resolvent of $A(t)$, and $A_\lambda(t) = \frac{1}{\lambda}(I_E - J_\lambda A(t))$ is the Yosida approximation of $A(t)$. We refer to Vrabie (1987) for the theory of accretive operators and evolution equations in Banach spaces.

PROPOSITION 4 *Let E be a finite dimensional space, $A(t) : E \to 2^E, (t \in [0,1])$ be an m-accretive operator and $F : [0,1] \times E \times E \rightrightarrows E$ be a convex compact valued multifunction, Lebesgue-measurable on $[0,1]$ and upper semicontinuous on $E \times E$. Suppose that the following assumptions are satisfied:*

(H_1) For every $x \in E$ and for every $\lambda > 0$ the function $t \mapsto (I_E + \lambda A(t))^{-1}x$ is Lebesgue-measurable and there exists $\overline{g} \in L^2_E([0,1])$ such that $t \mapsto (I_E + \lambda A(t))^{-1}\overline{g}(t)$ belongs to $L^2_E([0,1])$ for all $\lambda > 0$;

(H_2) There is $\gamma \in L^2_{\mathbb{R}}([0,1])$ such that $\|\gamma\|_{L^1}\overline{B}_E(0,1) \subset D(A(t))$ for all $t \in [0,1]$ and such that

$$|A(t)x|_0 + |F(t,x,y)| \leq \gamma(t)$$

for all $(t,x,y) \in [0,1] \times \|\gamma\|_{L^1}\overline{B}_E(0,1) \times \|\gamma\|_{L^1}\overline{B}_E(0,1)$.

Then, there is a $W^{2,1}_E([0,1])$-solution to the problem

$$\begin{cases} -\ddot{u}(t) \in A(t)u(t) + F(t, u(t), \dot{u}(t)), \ \ a.e. \ t \in [0,1], \\ u(0) = 0, \ u(\theta) = u(1). \end{cases}$$

Moreover, the $W^{2,1}_E([0,1])$-solutions set is compact in $\mathcal{C}_E([0,1])$.

Proof. Let (λ_n) be a decreasing sequence in $]0,1[$ such that $\lambda_n \to 0$. For each $n \in \mathbb{N}$, let us consider the multifunction

$$M_n(t,x,y) = A_{\lambda_n}(t)x + F(t,x,y),$$

for every $(t,x,y) \in [0,1] \times \|\gamma\|_{L^1}\overline{B}_E(0,1) \times \|\gamma\|_{L^1}\overline{B}_E(0,1)$. In view of (jj) and (H_2) we have

$$|M_n(t,x,y)| \leq \gamma(t), \ \forall (t,x,y) \in [0,1] \times \|\gamma\|_{L^1}\overline{B}_E(0,1) \times \|\gamma\|_{L^1}\overline{B}_E(0,1).$$

Let $\Gamma(t) = \gamma(t)\overline{B}_E(0,1)$ for all $t \in [0,1]$. Let us consider the set

Note that (H_1) implies that $(t,x) \mapsto A_{\lambda_n}(t)x$ is a Carathéodory mapping. Application of Theorem 5 and its remark gives a $W_E^{2,1}([0,1])$-solution $u_n \in \mathcal{X}_\Gamma$ to the inclusion

$$\begin{cases} -\ddot{u}_n(t) \in A_{\lambda_n}(t)u_n(t) + F(t, u_n(t), \dot{u}_n(t)), \text{ a.e. } t \in [0,1], \\ u_n(0) = 0, \; u_n(\theta) = u_n(1). \end{cases} \tag{4.3}$$

By (4.3) there is a measurable function $g_n(.)$ such that

$$\begin{cases} -\ddot{u}_n(t) = A_{\lambda_n}(t)u_n(t) + g_n(t), \text{ a.e. } t \in [0,1], \\ u_n(0) = 0, \; u_n(\theta) = u_n(1), \end{cases} \tag{4.4}$$

and such that $g_n(t) \in F(t, u_n(t), \dot{u}_n(t))$ for all $t \in [0,1]$. Using the compactness of $\mathcal{X}_\Gamma$, we may suppose, by extracting subsequences, that (u_n) converges uniformly to u, $(\dot{u}_n)$ converges pointwise to $\dot{u}$, $(\ddot{u}_n)$ converges $\sigma(L^2, L^2)$ to $\ddot{u}$ and (g_n) converges $\sigma(L^2, L^2)$ to a measurable function g with $\|g(t)\| \leq \gamma(t)$ a.e. Hence $g(t) \in F(t, u(t), \dot{u}(t))$, for almost every $t \in [0,1]$, because of the upper semicontinuity of $F(t,.,.)$. By (4.4) we have

$$-\ddot{u}_n(t) - g_n(t) = A_{\lambda_n}(t)u_n(t) \in A(t)J_{\lambda_n}A(t)u_n(t). \tag{4.5}$$

But

$$\|J_{\lambda_n}A(t)u_n(t) - u(t)\| \leq \|J_{\lambda_n}A(t)u_n(t) - u_n(t)\| \\ + \|u_n(t) - u(t)\|, \tag{4.6}$$

with

$$\|J_{\lambda_n}A(t)u_n(t) - u_n(t)\| = \lambda_n \|A_{\lambda_n}(t)u_n(t)\| \leq \lambda_n \gamma(t) \tag{4.7}$$

using (jj) and (H_2). As $\lambda_n \gamma(t) \to 0$, from (4.6) and (4.7), we see that

$$\|J_{\lambda_n}A(t)u_n(t) - u(t)\| \to 0. \tag{4.8}$$

By (4.6) and (4.7) we have the estimation

$$\|J_{\lambda_n}A(t)u_n(t) - u(t)\| \leq \gamma(t) + 2\int_0^1 \gamma(s)\, ds$$

for all $n \in \mathbb{N}$ and for all $t \in [0,1]$. Hence $J_{\lambda_n}A(.)u_n(.) \to u(.)$ in $L_E^2([0,1])$ by Lebesgue's theorem. Now let us consider the operator $\mathcal{A} : L_E^2([0,1]) \to 2^{L_E^2([0,1])}$ defined by

$$z \in \mathcal{A}y \iff z(t) \in A(t)y(t) \text{ for almost every } t \in [0,1].$$

Then $\mathcal{A}$ is an m-accretive operator in $L_E^2([0,1])$ owing to (H_1) (see the lemma below), and hence its graph is sequentially strongly-weakly closed in $L_E^2([0,1]) \times L_E^2([0,1]$, see e.g. Vrabie (1987). As $\ddot{u}_n + g_n$ converges $\sigma(L^2, L^2)$ to $\ddot{u} + g$ and $J_{\lambda_n}A(.)u_n(.) \to u(.)$ in $L_E^2([0,1])$, from (4.5) we get

$$-\ddot{u}(t) \in A(t)u(t) + g(t) \subset A(t)u(t) + F(t, u(t), \dot{u}(t) \text{ a.e.}$$

Compactness of the solutions set follows easily from the above arguments and the lemma given below. ∎

LEMMA 3 *Suppose that E is a separable Hilbert space and $A(t) : E \to 2^E (t \in [0,1])$, is an m-accretive operator satisfying the following assumption:*

($\mathcal{H}$): For every $x \in E$ and for every $\lambda > 0$ the function $t \mapsto (I_E + \lambda A(t))^{-1}x$ is Lebesgue-measurable and there exists $\overline{g} \in L^2_E([0,1])$ such that $t \mapsto (I_E + \lambda A(t))^{-1}\overline{g}(t)$ belongs to $L^2_E([0,1])$ for all $\lambda > 0$.

Let (u_n) and (v_n) be sequences in $L^2_E([0,1])$ satisfying:

(i) *(u_n) converges strongly to $u \in L^2_E([0,1])$ and (v_n) converges to $v \in L^2_E([0,1])$ with respect to the topology $\sigma(L^2_E, L^2_E)$;*

(ii) *$v_n(t) \in A(t)u_n(t)$ for all n and all $t \in [0,1]$.*

Then we have $v(t) \in A(t)u(t)$ a.e. $t \in [0,1]$.

Proof. Let $I_{L^2_E([0,1])}$ be the identity operator in $L^2_E([0,1])$. Let $\mathcal{A}$ be the operator in $L^2_E([0,1])$ defined by

$$v \in \mathcal{A}u \iff v(t) \in A(t)u(t) \text{ a.e } t \in [0,1].$$

As $A(t)$ is accretive for each $t \in [0,1]$, it is easy to check that $\mathcal{A}$ is accretive in $L^2_E([0,1])$. We claim that $\mathcal{A}$ is m-accretive. Let $\lambda > 0$ and let $g \in L^2_E([0,1])$. By ($\mathcal{H}$) there exists $\overline{g} \in L^2_E([0,1])$ such that $\overline{h} : t \mapsto (I_E + \lambda A(t))^{-1}\overline{g}(t)$ belongs to $L^2_E([0,1])$. Since $(I_E + \lambda A(t))^{-1}$ is nonexpansive, we deduce that the function $t \mapsto (I_E + \lambda A(t))^{-1}g(t)$ is Lebesgue-measurable and belongs to $L^2_E([0,1])$. It follows that, for $h(t) := (I_E + \lambda A(t))^{-1}g(t)$, for every $t \in [0,1]$, we have $h \in L^2_E([0,1])$, and furthermore,

$$\begin{aligned} & g(t) \in h(t) + \lambda A(t)h(t), \quad \forall t \in [0,1] \\ & \iff g \in h + \lambda \mathcal{A}h \\ & \iff h \in (I_{L^2_E} + \lambda\mathcal{A})^{-1}g \\ & \implies R(I_{L^2_E} + \lambda\mathcal{A}) = L^2_E. \end{aligned}$$

Thus $\mathcal{A}$ is m-accretive in the Hilbert space $L^2_E([0,1])$. Consequently, its graph is strongly-weakly sequentially closed. As, $v_n \to v$ weakly in $L^2_E([0,1])$, and $u_n \to u$ strongly, we conclude that $v \in \mathcal{A}u$, that is $v(t) \in A(t)u(t)$ a.e. The proof is therefore complete. ∎

REMARK. Lemma 3 is borrowed from a forthcoming paper, "Functional evolution equation governed by m-accretive operators", by Castaing and Ibrahim. Actually, this lemma holds when E is a separable reflexive Banach space such that its strong dual is uniformly convex.

COMMENTS. Proposition 4 is new since here we deal with a convex compact perturbation for a second order differential inclusion governed by a class of m-

most existence results for evolution equations involving accretive operators $A(t)$ depending on t, the domain $D(A(t))$ is constant. In the context of Proposition 4, it is worth to observe that, if $D(A(t))$ contains a closed ball $\overline{B}_E(0,r)$ of center 0 and of radius r for all $t \in [0,1]$ and if $A(t)x \subset \overline{B}_E(0,r)$ for all $(t,x) \in [0,1] \times \overline{B}_E(0,r)$, then the problem

$$\begin{cases} -\ddot{u}(t) \in A(t)u(t) \text{ a.e. } t \in [0,1], \\ u(0) = 0, \ u(\theta) = u(1), \end{cases}$$

has at least one $W_E^{2,1}$-solution. Compare with Attouch et. al. (2001), Azzam and Bounkhel (2001), Moreau (1977), Tolstonogov (2002), for related results. We refer to Benabdellah et al. (1996), Benabdellah (2000), Colombo and Goncharov (1999), Monteiro Marques (1993), Moreau (1977), Thibault (1999) for results on sweeping process by closed convex sets, on closed ρ-prox-regular sets – to Poliquin et al. (2000) and on φ-convex sets – to Colombo and Goncharov (1999).

5. Existence results in $W_{P,E}^{2,1}([0,1])$ for differential inclusions in Banach spaces

In this section we provide some unusual applications of Pettis integration to differential inclusions in Banach spaces with three boundary conditions. We need first some Pettis analogs of the results developed in the preceding section.

LEMMA 4 *Let $f \in P_E^1([0,1])$ and let us consider the function*

$$u_f(t) = \int_0^1 G(t,s)f(s)\,ds, \qquad \forall t \in [0,1],$$

where $G(t,s)$ is the function defined in Lemma 1. Then the following assertions hold:

(1) *$t \mapsto u_f(t)$ is a continuous mapping from $[0,1]$ into E (shortly $u_f \in \mathcal{C}_E([0,1])$);*

(2) $u_f(0) = 0, \ u_f(\theta) = u_f(1)$;

(3) *The function u_f is scalarly derivable, that is, for every $x' \in E'$, the scalar function $\langle x', u_f(.)\rangle$ is derivable, and its weak derivative $\dot{u}_f$ satisfies*

$$\lim_{h\to 0}\left\langle x', \frac{u_f(t+h)-u_f(t)}{h}\right\rangle = \langle x', \dot{u}_f(t)\rangle$$
$$= \int_0^1 \frac{\partial G}{\partial t}(t,s)\langle x', f(s)\rangle\,ds = \left\langle x', \int_0^1 \frac{\partial G}{\partial t}(t,s)f(s)\,ds\right\rangle$$

for all $t \in [0,1]$ *and for all* $x' \in E'$. *Consequently*

$$\dot{u}_f(t) = \int_0^1 \frac{\partial G}{\partial t}(t,s) f(s)\, ds, \qquad \forall t \in [0,1],$$

and $\dot{u}_f$ *is a continuous mapping from* $[0,1]$ *into* E.

(4) *The function* $\dot{u}_f$ *is scalarly derivable, that is, for every* $x' \in E'$, *the scalar function* $\langle x', \dot{u}_f(.)\rangle$ *is derivable, and its weak derivative* $\ddot{u}_f$ *is equal to* f *a.e.*

Proof. (1) As $(t,s) \mapsto G(t,s)\langle x', f(s)\rangle$ is a Carathéodory function on $[0,1]\times[0,1]$ with $|G(t,s)\langle x', f(s)\rangle| \le |\langle x', f(s)\rangle|$ and

$$\langle x', u_f(t)\rangle = \left\langle x', \int_0^1 G(t,s) f(s)\, ds \right\rangle = \int_0^1 G(t,s)\langle x', f(s)\rangle\, ds,$$

from Lebesgue's theorem, it is easily seen that $t \mapsto \langle x', u_f(t)\rangle$ is continuous on $[0,1]$ for every $x' \in E'$. So, u_f is a continuous mapping from $[0,1]$ into the weak space E_σ, shortly $u_f \in \mathcal{C}_{E_\sigma}([0,1])$. In order to prove that u_f belongs to $\mathcal{C}_E([0,1])$, we need a delicate argument. First, since f is Pettis integrable, the set $\{h_{x'}(.) := |\langle x', f(.)\rangle|;\ \|x'\| \le 1\}$ is uniformly integrable in $L^1_{\mathbb{R}}([0,1])$. Let (t_n) be a sequence in $[0,1]$ converging to $t \in [0,1]$. We have

$$\|u_f(t_n) - u_f(t)\| = \sup_{x' \in \overline{B}_{E'}} |\langle x', u_f(t_n) - u_f(t)\rangle|$$

$$\le \sup_{x' \in \overline{B}_{E'}} \int_0^1 |G(t_n,s) - G(t,s)| h_{x'}(s)\, ds. \tag{5.1}$$

As the sequence $(v_n(.)) := (|G(t_n,.) - G(t,.)|)$ is uniformly bounded and converges pointwise to 0 and the set $\{h_{x'}(.) : x' \in \overline{B}_{E'}\}$ is uniformly integrable in $L^1_{\mathbb{R}}([0,1])$, $(v_n(.))$ converges uniformly to 0 on this set in the duality $\langle L^\infty_{\mathbb{R}}, L^1_{\mathbb{R}}\rangle$ in view of a lemma due to Grothendieck (1964), see also Castaing (1980) for a more general result concerning the Mackey topology for bounded sequences in $L^\infty_{E'}$. Hence the second member of (5.1) converges to 0. This proves the strong continuity of u_f. This fact can be deduced from a general compactness result given below. We prove the continuity of $\dot{u}_f$ by using the same arguments and by taking the property of $\frac{\partial G}{\partial t}(.,.)$ given in (3) of Lemma 1 into account.

(2)–(4) follow from the same computations used for (3.7), (3.8) and (3.9) in Lemma 1 by observing that the functions $s \mapsto G(t,s)f(s)$ and $s \mapsto \frac{\partial G}{\partial t}(t,s)f(s)$ are Pettis integrable and equality "= a.e." is equivalent to "=" scalarly a.e. ∎

The following is an analogous version of Theorem 5 providing the existence of solution in $W^{2,1}_{P,E}([0,1])$ for differential inclusion of second order of the form

$$\begin{cases} \ddot{u}(t) \in F(t, u(t), \dot{u}(t)) \ \text{ a.e. } t \in [0,1], \\ u(0) = 0; \ \ u(\theta) = u(1). \end{cases}$$

LEMMA 5 *Let $\Gamma : [0,1] \rightrightarrows E$ be a convex compact valued measurable, and scalarly Pettis uniformly integrable multifunction. Then the $W^{2,1}_{P,E}([0,1])$-solutions set $\mathcal{X}_\Gamma$ of the multivalued differential equation*

$$\begin{cases} \ddot{u}(t) \in \Gamma(t), \ \ a.e.\ t \in [0,1], \\ u(0) = 0; \ \ u(\theta) = u(1), \end{cases}$$

is convex compact in the Banach space $\mathcal{C}_E([0,1])$ of all continuous mappings from $[0,1]$ into E endowed with the topology of uniform convergence. Further, if a sequence (u_n) of $\mathcal{X}_\Gamma$ converges uniformly to u, then $(\dot{u}_n)$ converges pointwise to $\dot{u}$ and $(\ddot{u}_n)$ converges $\sigma(P^1_E, L^\infty \otimes E')$ to $\ddot{u}$.

Proof. Step 1. Let us recall that the set S^{Pe}_Γ of all Pettis integrable selections of Γ is nonempty and sequentially compact for the topology of pointwise convergence on $L^\infty \otimes E'$ and the multivalued integral

$$\int_0^1 \Gamma(t)\,dt = \left\{ \int_0^1 f(t)\,dt : f \in S^{Pe}_\Gamma \right\}$$

is convex and norm compact in E (see Amrani and Castaing, 1997; Amrani et al., 1998; Castaing, 1996).

Step 2. In view of Lemma 4, the solution set $\mathcal{X}_\Gamma$ in $W^{2,1}_{P,E}([0,1])$ is characterized by

$$\mathcal{X}_\Gamma = \left\{ u_f : [0,1] \to E \mid u_f(t) = \int_0^1 G(t,s) f(s)\,ds, \ \forall t \in [0,1]; \ \ f \in S^{Pe}_\Gamma \right\}.$$

Further, we have

$$\begin{aligned} \|u_f(\tau) - u_f(t)\| &= \sup_{x' \in \overline{B}_{E'}} |\langle x', u_f(\tau) - u_f(t)\rangle| \\ &\leq \sup_{x' \in \overline{B}_{E'}} \int_0^1 |G(\tau,s) - G(t,s)|\, |\delta^*(x', \Gamma(s))|\,ds \end{aligned} \tag{5.2}$$

for all $f \in S^{Pe}_\Gamma$ and for all $t, \tau \in [0,1]$. As Γ is scalarly Pettis uniformly integrable, the set $\{|\delta^*(x', \Gamma(.))| : x' \in \overline{B}_{E'}\}$ is uniformly integrable in $L^1_{\mathbb{R}}([0,1])$. So if (t_n) is a sequence in $[0,1]$, which converges to $t \in [0,1]$, taking t_n in place of τ in (5.2) we obtain that

$$\sup_{x' \in \overline{B}_{E'}} \int_0^1 |G(t_n,s) - G(t,s)| |\delta^*(x', \Gamma(s))|\,ds \to 0$$

by applying again the Grothendieck lemma (Grothendieck, 1964) as in Lemma 4. By (5.2) again we see that $\mathcal{X}_\Gamma$ is equicontinuous in $\mathcal{C}_E([0,1])$. Further, for each $t \in [0,1]$, the set $\mathcal{X}_\Gamma(t)$ is relatively compact in E because it is included in the norm compact set $\int_0^1 G(t,s)\Gamma(s)\,ds$ because of the obvious property of G

norm compact valued, measurable and scalarly Pettis uniformly integrable multifunction mentioned in Step 1. We claim that $\mathcal{X}_\Gamma$ is compact in $\mathcal{C}_E([0,1])$. Let (f_n) be a sequence in S_Γ^{Pe}. As S_Γ^{Pe} is sequentially compact for the topology of pointwise convergence on $L^\infty \otimes E'$ and the sequence (u_{f_n}) is relatively compact in $\mathcal{C}_E([0,1])$, by Ascoli's theorem, we extract from (f_n) a sequence (f_m) such that (f_m) converges $\sigma(P_E^1, L^\infty \otimes E')$ to a function $f \in S_\Gamma^{Pe}$ and such that the sequence (u_{f_m}) converges uniformly to a continuous function $\zeta \in \mathcal{C}_E([0,1])$. So, for every $x' \in E'$ and for every $t \in [0,1]$, we have

$$\lim_{m\to+\infty} \int_0^1 \langle G(t,s)x', f_m(s)\rangle\, ds = \lim_{m\to+\infty} \Big\langle x', \int_0^1 G(t,s) f_m(s) \Big\rangle\, ds \tag{5.3}$$

$$= \int_0^1 \langle G(t,s)x', f(s)\rangle\, ds = \Big\langle x', \int_0^1 G(t,s) f(s)\, ds \Big\rangle. \tag{5.4}$$

As the multivalued integral $\int_0^1 G(t,s)\Gamma(s)\, ds$ $(t \in [0,1])$ is norm compact, (5.4) shows that the sequence $(u_{f_m}(.)) = (\int_0^1 G(.,s) f_m(s)\, ds)$ converges pointwise to $u_f(.)$, for E endowed with the norm topology. As (u_{f_m}) converges uniformly to $\zeta \in \mathcal{C}_E([0,1])$, we get $u_f = \zeta$. This shows the compactness of $\mathcal{X}_\Gamma$ in $\mathcal{C}_E([0,1])$. At this point, it is worth mentioning that the sequence $(\dot{u}_{f_m}(.)) = (\int_0^1 \frac{\partial G}{\partial t}(.,s) f_m(s)\, ds)$ converges pointwise to $\dot{u}_f(.)$, for E endowed with the norm topology, by using the $\sigma(P_E^1, L^\infty \otimes E')$ convergence of (f_m) and the norm compactness of the multivalued integral $\int_0^1 \frac{\partial G}{\partial t}(t,s)\Gamma(s)\, ds$. The last points of the lemma follow from the arguments above and the sequential $\sigma(P_E^1, L^\infty \otimes E')$ compactness of $\mathcal{S}_\Gamma^{Pe}$. ■

Now we proceed to the existence of solutions in $W_{P,E}^{2,1}([0,1])$ for the differential inclusion

$$\begin{cases} \ddot{u}(t) \in F(t, u(t), \dot{u}(t)) \ \text{a.e}\ t \in [0,1], \\ u(0) = 0;\ \ u(\theta) = u(1). \end{cases}$$

THEOREM 6 *Let $F : [0,1] \times E \times E \rightrightarrows E$ be a convex compact valued multifunction, Lebesgue-measurable on $[0,1]$, and upper semicontinuous on $E \times E$, and let $\Gamma : [0,1] \rightrightarrows E$ be a convex compact valued, measurable and scalarly Pettis uniformly integrable multifunction such that $F(t,x,y) \subset \Gamma(t)$ for all $(t,x,y) \in [0,1] \times E \times E$. Then the $W_{P,E}^{2,1}([0,1])$-solutions set of the above differential inclusion is nonempty and compact in $\mathcal{C}_E([0,1])$.*

Proof. Step 1. By virtue of Lemma 4, a mapping $u : [0,1] \to E$ is a $W_{P,E}^{2,1}([0,1])$ solution of the preceding inclusion, iff there exists $f \in \mathcal{S}_\Gamma^{Pe}$ such that $u(t) := u_f(t) = \int_0^1 G(t,s) f(s)\, ds$, for all $t \in [0,1]$ and such that $f(t) \in F(t, u_f(t), \dot{u}_f(t))$ for a.e. $t \in [0,1]$. Let us observe that, for any Lebesgue-measurable mappings $v : [0,1] \to E$ and $w : [0,1] \to E$, there is a Pettis integrable selection $s \in \mathcal{S}_\Gamma^{Pe}$ such that $s(t) \in F(t, v(t), w(t))$ a.e. Indeed, there exist two sequences to v and (w_n) converges pointwise to w respectively, for E endowed with the norm topology. As the multifunction $F(t, v_n(t), w_n(t))$ is Lebesgue-measurable, there is a Lebesgue-measurable selection s_n of $F(., v_n(.), w_n(.))$. As $s_n(t) \in F(t, v_n(t), w_n(t)) \subset \Gamma(t)$, for all $t \in [0,1]$ and $\mathcal{S}_\Gamma^{Pe}$ is sequentially $\sigma(P_E^1, L^\infty \otimes E')$-compact, we may extract from (s_n) a subsequence (s'_n) which converges $\sigma(P_E^1, L^\infty \otimes E')$ to a function $s \in \mathcal{S}_\Gamma^{Pe}$. Let $(e_k^*)_{k\in\mathbb{N}}$ be a dense sequence for the Mackey topology $\tau(E', E)$. Let $k \in \mathbb{N}$ be fixed. Applying the Mazur's trick to $(\langle e_k^*, s'_n(.)\rangle)_n$ provides a sequence (z_n) with $z_n \in co\{\langle e_k^*, s'_m(.)\rangle : m \geq n\}$ such that (z_n) converges pointwise a.e. to $\langle e_k^*, s(.)\rangle$. Using the upper semicontinuity of $F(t, ., .)$, it is not difficult to check that $\langle e_k^*, s(t)\rangle \leq \delta^*(e_k^*, F(t, v(t), w(t)))$ a.e. From Castaing and Valadier (1977, Prop. III-35), we get the above inclusion.

Step 2. For any $u \in \mathcal{X}_\Gamma$, let us consider the multifunction

$$\Psi(u) = \{v \in \mathcal{X}_\Gamma : \ddot{v}(t) \in F(t, u(t), \dot{u}(t)), \text{ a.e.}\}.$$

By Step 1 and Lemma 4, $\Psi(u)$ is a nonempty convex subset of $\mathcal{X}_\Gamma$. It is easy to check that $\Psi : \mathcal{X}_\Gamma \rightrightarrows \mathcal{X}_\Gamma$ has a closed graph. Let $(u_n, v_n) \in graph(\Psi)$ such that (u_n, v_n) converges to (u, v) in $\mathcal{X}_\Gamma \times \mathcal{X}_\Gamma$. By Lemma 5, (u_n, v_n) converges uniformly to (u, v), $\dot{u}_n$ (respectively $\dot{v}_n$) converges pointwise to $\dot{u}$ ($\dot{v}$), for E endowed with the norm topology, and $\ddot{u}_n$ (respectively $\ddot{v}_n$) converges $\sigma(P_E^1, L^\infty \otimes E')$ to $\ddot{u}$ ($\ddot{v}$). As we have

$$\ddot{v}_n(t) \in F(t, u_n(t), \dot{u}_n(t)) \text{ a.e.},$$

by applying a closure-type theorem from Castaing and Valadier (1969, 1977) or the arguments given in Step 1, we get

$$\ddot{v}(t) \in F(t, u(t), \dot{u}(t)) \text{ a.e.}$$

So, the sets $\Psi(u)$ are closed and hence compact in $\mathcal{X}_\Gamma$ and the multifunction Ψ is upper semicontinuous. Hence Ψ admits a fixed point, that is a solution of

$$\begin{cases} \ddot{u}(t) \in F(t, u(t), \dot{u}(t)) \text{ a.e. } t \in [0,1], \\ u(0) = 0; \;\; u(\theta) = u(1). \end{cases}$$

The compactness of the set of solutions follows. ∎

COMMENTS. 1) There are several alternative proofs for the weak compactness of $\mathcal{S}_\Gamma^1$ Amrani et al., 1992; Amrani and Castaing, 1997; Castaing, 1996; Castaing and Valadier, 1977; Castaing and Saadoune, 2000) while the strong compactness of the set-valued integral $\int_0^1 \Gamma(t)\,dt$ was first initiated by the second author (Castaing, 1969, 1972) via the Banach-Dieudonné theorem (see e.g. Grothendieck, 1964). This fact appeared in several places (El Amri and Hess, 2000; Castaing, 1984; Castaing and Valadier, 1977; Castaing et al., 2002). Actually, both weak compactness of $\mathcal{S}_\Gamma^1$ and strong compactness of $\int_0^1 \Gamma(t)\,dt$ can be obtained by new tools of Young measures (Castaing et al., 2002). Other contributions for

Piccinini and Valadier (1995), Roubíček (1997), Valadier (1990a, 1990b, 1994). Sequential weak compactness of $\mathcal{S}_\Gamma^{Pe}$ was demonstrated in Amrani and Castaing (1997), Castaing (1996), via Komlós convergence (Komlós, 1967). For the Bochner case we refer to Castaing (1996), Castaing and Saadoune (2000), Colombo and Goncharov (1999), Ülger (1991). The strong compactness of the set-valued Pettis integral $\int_0^1 \Gamma(t)\,dt$ can be proved by using Banach-Dieudonné's theorem or a typical convergence result for Young measures (Castaing et al., 2002, Theorem 6.3.6).

2) The lower semicontinuity for functional integrals can be found in Balder (1986, 1995, 2000a, 2000b), Castaing and Clauzure (1982), Castaing and Valadier (1977), Castaing et al. (2002), Jalby (1992), Valadier (1990a).

3) Theorem 6 provides a new type of second order differential inclusion dealing with unusual $W_{P,E}^{2,1}$ solutions. When Γ is a convex compact valued, mesurable and integrably bounded multifunction, it is obvious that $W_{P,E}^{2,1}$ solutions coincide with $W_{B,E}^{2,1}$ solutions, and so Theorem 6 is reduced to Theorem 5. These results extend to infinite dimensional spaces the ones obtained in Gomaa (2000), Gupta (1992), Ibrahim and Gomaa (2000), Marano (1992, 1994), Ricceri and Ricceri (1990). In this context, we refer to Castaing and Valadier (1969), Maruyama (2001) dealing with first order differential inclusions in locally convex spaces.

4) By assuming that the multifunction Γ in the above results are weakly compact valued and the multifunction F is upper semicontinuous on $E_\sigma \times E_\sigma$, we get the existence of *weak solutions*, see e.g. Castaing and Valadier (1969), Maruyama (2001). Details are left to the readers.

References

AMRANI, A. and CASTAING, C. (1997) Weak compactness in Pettis integration. *Bull. Pol. Acad. Sci. Math.*, **45** (2), 139–150.

AMRANI, A., CASTAING, C., and VALADIER, M. (1992) Méthodes de troncature appliquées à des problèmes de convergence faible ou forte dans L^1. *Arch. Rational Mech. Anal.*, **117**, 167–191.

AMRANI, A., CASTAING, C., and VALADIER, M. (1998) Convergence in Pettis norm under extreme point condition. *Vietnam J. of Math.*, **26** (4), 323–335.

ATTOUCH, H., CABOT, A., and REDONT, P. (2000–2001) Shock solutions via epigraphical regularizations of a second order in time differential inclusion. Preprint, Université de Montpellier II.

AZZAM, D. L. and BOUNKHEL, M. (2001) Existence of the second order nonconvex sweeping process with perturbations. Technical report, Publicaciones Tecnicas, Centro de Modelamiento Matemático, Universidad de Chile, UMR CNRS.

BALDER, E. J. (1986) On seminormality of integral functionals and their integrands. *SIAM J. Control and Optimization*, **24**, 95–121.

BALDER, E. J. (1995) Lectures on Young measures. *Cahiers de mathématiques de la décision*, 9517, CEREMADE, Université Paris-Dauphine.

BALDER, E. J. (2000a) Lectures on Young measure theory and its applications in economics. *Rend. Istit. Mat. Univ. Trieste*, 31, suppl., 1–69. Workshop di Teoria della Misura et Analisi Reale, Grado, 1997.

BALDER, E. J. (2000b) New fundamentals of Young measure convergence. In *Calculus of Variations and Optimal Control (Haifa 1998)*, Boca Raton, FL. Chapman & Hall, 24–48.

BENABDELLAH, H. (2000) Existence of solutions to the nonconvex sweeping process. *J. Differential Equations*, **164** (2), 286–295.

BENABDELLAH, H., CASTAING, C., SALVADORI, A., and SYAM, A. (1996) Nonconvex sweeping process. *J. Appl. Anal.*, **2** (2), 217–240.

CASTAING, C. (1969) Quelques applications du théorème de Banach–Dieudonné à l'intégration. Preprint 67, Université de Montpellier II.

CASTAING, C. (1972) Quelques résultats de compacité liés à l'intégration. In: *Actes du Colloque d'Analyse Fonctionnelle de Bordeaux (Univ. Bordeaux, 1971), Bull. Soc. Math. France, Mém.* 31–32. Soc. Math. France, Paris, 73–81.

CASTAING, C. (1980) Topologie de la convergence uniforme sur les parties uniformément intégrables de L^1 et théorèmes de compacité dans certains espaces du type Köthe–Orlicz. Séminaire d'Analyse Convexe 10, Université de Montpellier II.

CASTAING, C. (1984) Représentation intégrale des multifonctions séparément σ–additives et séparément continues. Séminaire d'Analyse Convexe 18, Université de Montpellier II.

CASTAING, C. (1996) Weak compactness and convergence in Bochner and Pettis integration. *Vietnam J. Math.*, **24** (3), 241–286.

CASTAING, C. and CLAUZURE, P. (1982) Semicontinuité des fonctionnelles intégrales. *Acta Math. Vietnam.*, **7**, 139–170 (first published in Séminaire d'Analyse Convexe **11**, 1981, 15.1–15.45, Montpellier, France).

CASTAING, C., RAYNAUD DE FITTE, P., and VALADIER, M. (2002) Young measures on topological spaces. With applications in control theory and probability theory. Preprint.

CASTAING, C. and SAADOUNE, M. (2000) Dunford–Pettis–type theorems and convergence in set valued integration. *J. Nonlinear and Convex Analysis*, **1** (1), 37–71.

CASTAING, C. and VALADIER, M. (1969) Équations différentielles multivoques dans les espaces localement convexes. *Revue Française de Recherche Opérationnelle*, **3**, 3–16.

CASTAING, C. and VALADIER, M. (1977) *Convex Analysis and Measurable Multifunctions.* LNM 580, Springer Verlag, Berlin.

CELLINA, A. (1980) On the differential inclusion $x' \in [-1, +1]$. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)*, **69** (1–2), 1–6 (1981).

COLOMBO, G. and GONCHAROV, V. V. (1999) The sweeping processes without convexity. *Set-Valued Anal.*, **7** (4), 357–374.

EL AMRI, K. and HESS, C. (2000) On the Pettis integral of closed valued multifunctions. *Set-Valued Anal.*, **8** (4), 329–360.

GEITZ, R. (1981) Pettis integration. *Proc. Amer. Math. Soc.*, **82**, 81–86.

GOMAA, A. M. M. (2000) *Set-Valued Functions and Set-Valued Differential Equations*. PhD thesis, Faculty of Sciences, Cairo University.

GROTHENDIECK, A. (1964) *Espaces Vectoriels Topologiques*. Publ. Soc. Mat. Saõ Paulo, São Paulo. 3rd ed.

GUPTA, C. P. (1992) Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equation. *J. Math. Anal. Appl.*, **168** (2), 540–551.

HARTMAN, P. (1964) *Ordinary differential equations*. John Wiley & Sons Inc., New York.

HUFF, R. (1986) Remarks on Pettis integration. *Proc. Amer. Math. Soc.*, **96**, 402–404.

IBRAHIM, A. G. AND GOMAA, A. M. (2000) Existence theorems for a functional multivalued three-point boundary value problem of second order. *J. Egyptian Math. Soc.*, **8** (2), 155–168.

JALBY, V. (1992) Mesures de Young sur un espace souslinien complètement régulier. Applications aux suites de fonctions de L^1_E où E n'est pas nécessairement réflexif.

KOMLÓS, J. (1967) A generalization of a problem of Steinhaus. *Acta Math. Acad. Sci. Hungar.*, **18**, 217–229.

MARANO, S. A. (1992) Existence theorems for a multivalued boundary value problem. *Bull. Austral. Math. Soc.*, **45** (2), 249–260.

MARANO, S. A. (1994) A remark on a second-order three-point boundary value problem. *J. Math. Anal. Appl.*, **183** (3), 518–522.

MARUYAMA, T. (2001) A generalization of the weak convergence theorem in Sobolev spaces with application to differential inclusions in a Banach space. *Proc. Japan Acad. Ser. A Math. Sci.*, **77** (1), 5–10.

MONTEIRO MARQUES, M. D. P. (1993) *Differential Inclusions in Nonsmooth Mechanical Problems, Shocks and Dry Friction*. Progress in Nonlinear Differential Equations and their Applications, 9. Birkhäuser, Basel.

MOREAU, J.-J. (1977) Evolution problem associated with moving convex set in a Hilbert space. *J. Differential Equations*, **26**, 347–374.

MUSIAL, K. (1987) Vitali and Lebesgue theorems for Pettis integral in locally convex spaces. *Atti Sem. Mat Fis. Modena*, **25**, 159–166.

MUSIAL, K. (1991) Topics in the theory of Pettis integration. *Rendiconti dell'istituto di matematica dell'Università di Trieste*, 23, 176–262. School on Measure Theory and Real Analysis, Grado.

PEDREGAL, P. (1997) *Parametrized measures and variational principles.* Progress in Nonlinear Differential Equations and their Applications, 30. Birkhäuser, Basel.

PIANIGIANI, G. (1990) Differential inclusions. The Baire category method. In: *Methods of nonconvex analysis (Varenna, 1989).* Springer, Berlin, 104–136.

PICCININI, L. and VALADIER, M. (1995) Uniform integrability and Young measures. *J. Math. Anal. Appl.*, **195** (2), 428–439.

POLIQUIN, R. A., ROCKAFELLAR, R. T., and THIBAULT, L. (2000) Local differentiability of distance functions. *Trans. Amer. Math. Soc.*, **352** (11), 5231–5249.

RICCERI, O. N. and RICCERI, B. (1990) An existence theorem for inclusions of the type $\psi(u)(t) \in F(t, \phi(u)(t))$ and application to a multivalued boundary value problem. *Appl. Anal.*, **38** (4), 259–270.

ROUBÍČEK, T. (1997) *Relaxation in Optimization Theory and Variational Calculus.* De Gruyter Series in Nonlinear Analysis and Applications, 4. Walter de Gruyter, Berlin.

THIBAULT, L. (1999) Sweeping process with regular and nonregular sets. Technical report, Université Montpellier II.

TOLSTONOGOV, A. A. (2002) Properties of the set of extreme solutions for a class of nonlinear second-order evolution inclusions. *Set-Valued Anal.*, **10** (1), 53–77.

ÜLGER, A. (1991) Weak compactness in $L^1(\mu, X)$. *Proc. Amer. Math. Soc.*, **103**, 143–149.

VALADIER, M. (1990a) Application des mesures de Young aux suites uniformément intégrables dans un Banach séparable. Séminaire d'Analyse Convexe, Université de Montpellier II.

VALADIER, M. (1990b) Young measures. In: Cellina, A., ed., *Methods of Nonconvex Analysis*, LNM 1446, Berlin. Springer Verlag, 152–158.

VALADIER, M. (1994) A course on Young measures. *Rendiconti dell'istituto di matematica dell'Università di Trieste*, 26, suppl., 349–394. Workshop di Teoria della Misura et Analisi Reale, Grado, 1993.

VRABIE, I. I. (1987) *Compactness Methods for Nonlinear Evolutions.* Longman Scientific & Technical, Harlow. With a foreword by A. Pazy.