Control and Cybernetics

vol. **31** (2002) No. 3

A sufficient condition for small-time local attainability of a set

by

Mikhail I. Krastanov

Institute of Mathematics & Informatics
Sofia, Bulgaria

Abstract: The notion of small-time local attainability (STLA) of a closed set with respect to a nonlinear control system is discussed and a new sufficient STLA condition is proved.

Keywords: small-time local attainability of a set, nonlinear control systems

1. Introduction

The problem of small-time local attainability of a closed set with respect to a control system is not reduced to the problem of small-time local attainability at every point of the set. So, it needs a specific study. This problem has been partially studied using mainly zero order and first order approach (see the papers Soravia, 1978, Bacciotti and Stefani, 1980, Veliov, 1994, Veliov, 1997, Clarke and Wolenski, 1996, etc.). We would like to mention also the paper Krastanov and Quincampoix (2001), which is closely related to the considered problem. A class of high-order variations to the attainable set is defined and a different sufficient STLA condition is proved there.

To state the problem of small-time local attainability, let us consider the following control system:

$$\dot{x}(t) \in F(x(t)), \tag{1}$$

where $F : R^n \Rightarrow R^n$ is a multifunction with compact and convex values. An absolutely continuous function $x(\cdot)$, satisfying (1) for almost every t from $[0,T]$, is called a trajectory of (1) defined on $[0,T]$. For a fixed point x and for $T > 0$, the attainable set $\mathcal{A}(x,T)$ of (1) from x at time $T > 0$ is defined as the set of all points that can be reached in time T from x by means of trajectories of (1).

This research was partially supported by the Ministry of Science and Higher Education -

DEFINITION 1.1 *Let S be a closed subset of R^n. It is said that S is small-time locally attainable (STLA) with respect to the control system (1) iff for any $T > 0$ there exists a neighbourhood Ω of S such that for every point $x \in \Omega$ there exists an admissible trajectory of the control system (1) starting from the point x and reaching the set S in time not greater than T, i.e. $\mathcal{A}(x,t) \cap S \neq \emptyset$ for some $t \in [0,T]$.*

To present a general sufficient STLA condition of zero-order of a control system with respect to a set, we follow the notations from the paper by Clarke and Wolenski (1996): Let S be a compact subset of R^n. We set

$$S_r := \{y \in R^n \mid \mathrm{d}_S(y,S) \leq r\},$$

where

$$\mathrm{d}_S(y) := \inf\{\|y - s\| \mid s \in S\}.$$

If x is an arbitrary point from $S_r \setminus S$, we set

$$\pi(x) := \{y \in S \mid \|y - x\| = \mathrm{d}_S(x)\},$$

i.e. $\pi(x)$ is the set of all metric projections of the point x on the set S.

Let us consider the control system (1) under the assumption that F is continuous of modulus ω near S, i.e.

$$\Delta(F(x), F(y)) \leq \omega(\|x - y\|), \quad \text{for all } x, y \text{ near } S,$$

where Δ denotes the Hausdorff metric and $\omega : [0,\infty) \to [0,\infty)$ is a nondecreasing continuous function with $\omega(0) = 0$. Let y belong to the boundary ∂S of the set S. A vector $\xi \in R^n$ is called a proximal normal to S at y provided there exists $r > 0$ such that the point $y + r\xi$ has y as the closest point. The set of all proximal normals at a point y is a cone. This cone is denoted by $N_S^p(y)$ (for a detailed treatment of proximal analysis and some of its applications, see for example the books by Clarke, 1983, and Clarke, Ledyaev, Stern and Wolenski, 1998). Using these notations, the results of the papers by Veliov (1994, 1997), and Clarke and Wolenski (1996) can be formulated as follows:

THEOREM 1.1 *Suppose that S is a nonempty and compact subset of R^n, and $F : R^n \rightrightarrows R^n$ is a continuous multifunction of modulus ω with compact convex values. Suppose that there exists $\delta > 0$ so that, whenever $y \in S$ and $\xi \in N_S^p(y)$, there exists $v \in F(y)$ for which*

$$\langle \xi, v \rangle \leq -\delta \|\xi\|. \tag{2}$$

Then S is STLA with respect to (1).

Unfortunately, if the inequality (2) is violated at some boundary point y of S (for example, when all admissible velocities are "tangent" to the closed set S at y), we can not apply Theorem 1.1. The following simple example demonstrates

EXAMPLE 1.1 *Let* $S_1 = \{(x,y) \in R^2 \mid 0 \le x \le 1,\ y \le 0\}$ *and let us consider the following control system*

$$\begin{aligned} \dot{x} &= u, & u &\in [-1,1], \\ \dot{y} &= y + 1 - x + v, & v &\in [-1,1]. \end{aligned} \tag{3}$$

The origin is a boundary point of S_1 and the vector $n = (0,1)^T$ is a proximal normal to S_1 at the origin. Since the scalar products of the vector n and all velocities admissible at the origin (see the control system (3)) are not negative, we can not apply Theorem 1.1. Applying some of the ideas related to small-time local controllability at a point (see, for example the papers by Hermes, 1978, Veliov and Krastanov, 1986), we can construct a high order variation of the attainable set at every point belonging to some neighbourhood of S. Using this variation, we can move towards the set S. For example, let $T \in [0,1]$, $t \in (0, T/2]$ and $z = (0,y)^T$ with $y > 0$. We set $v_t(s) = -1$ for every $s \in [0,t]$ and

$$u_t(s) = \begin{cases} 1, & \text{if } s \in [0,t]; \\ -1, & \text{if } s \in [t,2t]. \end{cases} \tag{4}$$

It can be directly checked that the trajectory $z_t(\cdot) = (x_t(\cdot), y_t(\cdot))$, starting from the point z and corresponding to the controls $u_t(\cdot)$ and $v_t(\cdot)$, is well defined on $[0,2t]$, $x_t(2t) = 0$ and

$$y_t(2t) = e^{2t}y + 2e^t - e^{2t} - 1. \tag{5}$$

Hence, we can represent $z_t(2t)$ as follows:

$$z_t(2t) = z + a(t,z) - t^2 A(z) + O(t^3, z), \tag{6}$$

where $A(z) = (0,-1)^T$, $a(t,z) = (0, (e^{2t}-1)y)^T$ and $O(t^3,z) = (0, 2e^t - e^{2t} - 1 + t^2)^T$. Taking into account that $d_{S_1}(z) = y$, we obtain that

$$|a(t,z)| \le M.t.d_{S_1}(z), \quad |O(t^3,z)| \le N.t^3 \tag{7}$$

for suitable chosen positive number M and N. The expansion of the solution z_t in the form (6) and the estimates (7) motivate our Definition 2.1 of a high order variation of the attainable set at a point (for this example, $A(\cdot)$ is a variation of second order of the attainable set with respect to S_1 at the point z).

It can be directly verified that the origin is a metric projection of the point z on the set S_1, the vector $n = (0,1)^T$ is a normal to S_1 at the origin and

$$\langle n, A(0) \rangle = -1.$$

So, a natural question is to ask whether a result analogous to Theorem 1.1 holds true, if we replace the velocities from the inequality (2) by high order variations. The answer is "yes", and this is the main result of the present paper. As a direct

EXAMPLE 1.2 *Let* $S_2 = \{(x,y) \in R^2 | \;\; y \le 0\}$ *and let us consider the following control system*

$$\begin{aligned} \dot{x} &= u, \\ \dot{y} &= x^3 + x^2. \end{aligned} \tag{8}$$

The set S_2 is not compact. Moreover, at the origin all trajectories are tangent to S_2. Hence, we can not apply Theorem 1.1. If we assume that there exists a constant $M > 0$ such that the values of the admissible controls belong to the interval $[-M, M]$, then it could be directly shown that the set S_2 is not STLA with respect to the control system (8). But, when there are no bounds on the values of the admissible controls, our main result implies that the set S_2 is STLA with respect to the control system (8) (see, also Jurdjevic and Kupka, 1985). The proof is based again on construction of suitable variations: Let $T \in [0,1]$, $(x,0)$ be an arbitrary boundary point of S_2, $z = (x,y)^T$, $y > 0$, $m > \max(1, |x|)$, $t > 0$, $2t^4/m^3 < T$, $M = m^4/t^5$. We define the following control function:

$$u_t(s) = \begin{cases} -M, & \text{if } s \in [0, t^4/m^3]; \\ M, & \text{if } s \in [t^4/m^3, 2t^4/m^3]. \end{cases} \tag{9}$$

It can be directly checked that the trajectory $z_t(\cdot) = (x_t(\cdot), y_t(\cdot))$ starting from the point z and corresponding to the control $u_t(\cdot)$ is well defined on $[0, 2t^4/m^3]$, $x_t(2t^4/m^3) = 0$ and

$$y_t(2t^4/m^3) = y - \frac{t}{2} + \left(\frac{2}{3m} + \frac{2x}{m}\right)t^2 - \left(\frac{2x}{m^2} + \frac{3x^2}{2m^2}\right)t^3$$
$$+ \left(\frac{2x^3}{m^3} + \frac{2x^2}{m^3}\right)t^4.$$

Thus, we may represent $z_t(2t^4/m^3)$ as follows:

$$z_t(2t^4/m^3) = z - tA(z) + O(t^2, z), \text{ where } A(z) = \left(0, -\frac{1}{2}\right)^T \text{ and}$$

$$O(t^2, z) = \left(0, \left(\frac{2}{3m} + \frac{2x}{m}\right)t^2 - \left(\frac{2x}{m^2} + \frac{3x^2}{2m^2}\right)t^3 + \left(\frac{2x^3}{m^3} + \frac{2x^2}{m^3}\right)t^4\right)^T \tag{10}$$

Taking into account our choice of m, we obtain that

$$|O(t^2, z)| \le 12t^2. \tag{11}$$

The expansion of the solution z_t in the form (10) and the estimate (11) show that $A(\cdot)$ is a variation of first order of the attainable set at the point z.

The vector $(0,1)^T$ is a normal to S_2 at the point $(x,0)$ and

$$\langle n, A(x,0)\rangle = -\frac{1}{2}.$$

Since $(x,0)$ is an arbitrary boundary point of S_2, our main result implies that

2. The main result

Our approach to the study of the STLA property is based on a suitable class of high-order variations:

DEFINITION 2.1 *Let S be a closed subset of R^n, $r > 0$, $x \in S_r \setminus S$ and $A : S_r \to R^n$ be a continuous function. It is said that A is a variation of order $\alpha > 0$ of the attainable set of the control system (1) at the point x iff there exist positive real numbers T, M, N, θ, $\beta > \alpha$, p_i, $i = 1, \ldots, k$ and $1 \leq q_1 < q_2 < \cdots < q_k$, such that for each $t \in [0, T]$ the following inclusion holds true*

$$x + t^\alpha A(x) + a(t, x) + O(t^\beta, x) \in \mathcal{A}(x, p(t)), \tag{12}$$

where $p(t) = \sum_{i=1}^k p_i t^{q_i}$, and the continuous functions $a(\cdot, \cdot) : [0, T] \times S_r \to R^n$ and $O(\cdot, \cdot) : [0, T] \times S_r \to R^n$ satisfy the following estimates

$$\|a(t, x)\| \leq M \; t^\theta \; d_S(x) \text{ and } O(t^\beta, x) \leq N \; t^\beta, \; t \in [0, T].$$

By $\mathcal{V}_x^\alpha$ we denote the set of all variations of order α of the attainable set at x.

REMARK 2.1 *An open question is how to construct elements of the set $\mathcal{V}_x^\alpha$. Partial answers can be found in the papers where the local properties of the attainable set are studied (for example, Agrachev and Gamkrelidze, 1993, Bianchini and Stefani, 1990, Brunovsky, 1974, Frankowska, 1989, Hermes, 1982, Kawski, 1988, Sussmann, 1987, Veliov, 1988, etc.).*

DEFINITION 2.2 *Let $r > 0$ and $T > 0$. It is said that $\mathcal{V}$ is a regular subset of the set $\cup_{x \in S_r \setminus S} \mathcal{V}_x^\alpha$, provided that there exist positive constants T, $\beta > \alpha$, θ, L, M, N, C and P such that for every $A \in \mathcal{V} \cap \mathcal{V}_x^\alpha$ with corresponding $p(t)$, $a(t, x)$ and $O(t^\beta, x)$ (according to Definition (2.1)), the following relations hold true:*

i) $\|A(x) - A(y)\| \leq L\|x - y\|$ for all $y \in \pi(x)$;
ii) $\|A(x)\| \leq C$;
iii) $\|a(t, x))\| \leq M.t^\theta.d_S(x)$ for all $t \in [0, T]$;
iv) $\|O(t^\beta, x))\| \leq N.t^\beta$ for $t \in [0, T]$;
v) $|p(t)| \leq P.t$ for all $t \in [0, T]$;
vi) $x + t^\alpha A(x) + a(t, x) + O(t^\beta, x) \in \mathcal{A}(x, p(t))$ for all $t \in [0, T]$.

REMARK 2.2 *The regularity of a set of variarions of the reachable set $\mathcal{V}$ means that: 1) all elements of $\mathcal{V}$ are Lipschitz continuous functions defined on a neighbourhood of the set S with one and the same Lipschitz constant; 2) all functions related to the elements of $\mathcal{V}$ (according to Definition (2.1)) are uniformly bounded on this neighbourhood. In our further consideration we shall use only regular subsets of the set $\cup_{x \in S_r \setminus S} \mathcal{V}_x^\alpha$. This assumption is technical and guarantees the existence of suitable bounded trajectories of the considered control system (1) well defined on some fixed interval $[0, T]$. This is especially important for*

Now we can formulate the main result:

THEOREM 2.1 *Suppose that S is a nonempty closed subset of R^n, $\mathcal{V}$ is a regular subset of $\cup_{x\in S_{r_0}\setminus S}\mathcal{V}_x^\alpha$ and $\delta > 0$. Let us assume that whenever $x \in S_{r_0} \setminus S$, $y \in \pi(x)$ and $\xi \in N_S^p(y)$ there exists $A \in \mathcal{V} \cap \mathcal{V}_x^\alpha$ for which*

$$\langle \xi, A(y)\rangle \leq -\delta.\|\xi\|. \tag{13}$$

Then the control system (1) is STLA with respect to S.

Let $\Theta(x)$ be the minimal time of steering to the set S from the point x by means of a trajectory of the control system (1), i.e. $\Theta(x) := \inf\{t \geq 0$, such that $z(0) = x$, $z(t) \in S$ for some trajectory $z(.)$ of the control system (1)$\}$. The map $\Theta(\cdot)$ is called time optimal map of reaching the set S. Theorem 2.1 implies directly the following corollary:

COROLLARY 2.1 *Suppose that the assumptions of Theorem 2.1. hold true. Then there exists a constant $C > 0$ such that*

$$\Theta(x) \leq C.d_S(x)^{1/\alpha} \tag{14}$$

for every x from some neighbourhood U of S.

PROPOSITION 2.1 *Suppose that there exist positive constants α, r, C, K and σ for which the following conditions hold true:*
i/ $\Theta(x) \leq C.d_S(x)^{1/\alpha}$ for every x from S_r.
ii/ if $z(.)$ is a trajectory of the control system (1) defined on $[0,T]$ such that $z(T) \in S$, and if y is a point from $S_r \setminus S$ such that $\|y - z(0)\| \leq \sigma$, then there exists a trajectory $z_y(.)$ of (1) such that

$$z_y(0) = y \quad and \quad \|z_y(t) - z(t)\| \leq e^{Kt}\|z_y(0) - z(0)\| \quad for\ every\ t \in [0,T].$$

Then Θ is $1/\alpha$- Hölder continuous in S_r.

REMARK 2.3 *Under the assumptions of Theorem 2.1, condition i/ holds always true. Condition ii/ is satisfied for control systems with Lipschitz continuous right-hand side (see, for example the paper by Bianchini and Stefani, 1990).*

3. Proofs

Proof of Theorem 2.1. The regularity of $\mathcal{V}$ implies (according to Definition (2.2)) the existence of positive constants T_0, $\beta > \alpha$, θ, L, M, N, C and P for which

Let us fix an arbitrary T from the interval $(0, T_0]$ and let r be a real number for which the following relations hold true:

$$0 < r < \min\left\{r_0, \frac{\delta}{2.L}, 2\frac{(1-p)^\alpha T^\alpha C^2}{P^\alpha \delta}, 2\frac{T_0^\alpha C^2}{\delta}\right\} \tag{15}$$

$$M^2\left(\frac{r.\delta}{2C^2}\right)^{2\theta/\alpha} + \frac{r.L.\delta}{C^2} + 2M\left(\frac{r.\delta}{2C^2}\right)^{\theta/\alpha} < \frac{\delta^2}{16C^2}, \tag{16}$$

$$MC\left(\frac{r.\delta}{2C^2}\right)^{\theta/\alpha} + MN\left(\frac{r.\delta}{2C^2}\right)^{(\beta+\theta-\alpha)/\alpha} + N\left(\frac{r.\delta}{2C^2}\right)^{(\beta-\alpha)/\alpha} < \frac{\delta}{2}, \tag{17}$$

$$\frac{N^2}{C^2}\left(\frac{r.\delta}{2C^2}\right)^{2(\beta-\alpha)/\alpha} + 2\frac{N}{C}\left(\frac{r.\delta}{2C^2}\right)^{(\beta-\alpha)/\alpha} < \frac{1}{4}, \tag{18}$$

where

$$p := \left(1 - \frac{\delta^2}{8C^2}\right)^{1/2\alpha}.$$

This choice of r is important for obtaining all the necessary estimates in the proof.

Let x be an arbitrary point from $S_r \setminus S$ and y be an arbitrary point from the set $\pi(x)$. Then, $\|y-x\| = \mathrm{d}_S(x)$, $0 \neq x-y \in N_S^p(y)$, and according to (13), there exists an element A from $\mathcal{V} \cap \mathcal{V}_x^\alpha$, for which

$$\|A(y)\| \geq \left\langle \frac{-\xi}{\|\xi\|}, A(y)\right\rangle \geq \delta.$$

Then, (15) implies that

$$C \geq \|A(x)\| \geq \|A(y)\| - L\|y-x\| \geq \delta - L.r \geq \delta - \frac{\delta}{2} = \frac{\delta}{2}. \tag{19}$$

According to Definition 2.1, the relation $A \in \mathcal{V}_x^\alpha$ implies the existence of $p_{x,y}(\cdot)$, $a(\cdot,\cdot)$ and $O(\cdot,\cdot)$ such that for all $t \in [0, T]$

$$z_{x,y}(x,t) := x + a(t,x) + t^\alpha A(x) + O(t^\beta, x) \in \mathcal{A}(x, p_{x,y}(t)). \tag{20}$$

Moreover, the regularity of $\mathcal{V}$ implies that for all $t \in [0, T]$

$$\begin{aligned} &\|A(x) - A(y)\| \leq L\|x-y\|, \quad \|A(x)\| \leq C, \\ &\|a(t,x)\| \leq M.t^\theta.\mathrm{d}_S(x), \\ &\|O(t^\beta, x)\| \leq N.t^\beta \text{ and } |p(t)| \leq P.t. \end{aligned} \tag{21}$$

Then for every $t \in [0, T]$ we have that

$$\mathrm{d}_S(z_{x,y}(x,t))^2 \leq \|z_{x,y}(x,t) - y\|^2 = \|(z_{x,y}(x,t) - x) + (x-y)\|^2$$

$$+2\left\langle a(t,x)+t^{\alpha}A(x)+O(t^{\beta},x),x-y\right\rangle+\|x-y\|^{2}$$
$$\leq\left(M.t^{\theta}\mathrm{d}_S(x)+t^{\alpha}C+t^{\beta}N\right)^{2}+2t^{\alpha}\langle A(y),x-y\rangle$$
$$+2t^{\alpha}\langle A(x)-A(y),x-y\rangle+2\left\langle a(t,x)+O(t^{\beta},x),x-y\right\rangle+\mathrm{d}_S(x)^{2}$$
$$\leq M^{2}t^{2\theta}\mathrm{d}_S^{2}(x)+t^{2\alpha}C^{2}+N^{2}t^{2\beta}+2MCt^{\theta+\alpha}\mathrm{d}_S(x)+2MNt^{\theta+\beta}\mathrm{d}_S(x)$$
$$+2NCt^{\alpha+\beta}-2\delta t^{\alpha}\mathrm{d}_S(x)+2t^{\alpha}L\mathrm{d}_S^{2}(x)+2Mt^{\theta}\mathrm{d}_S^{2}(x)+2t^{\beta}N\mathrm{d}_S(x)+\mathrm{d}_S^{2}(x)$$
$$=\mathrm{d}_S^{2}(x)\left[1+M^{2}t^{2\theta}+2t^{\alpha}L+2Mt^{\theta}\right]+2t^{\alpha}\mathrm{d}_S(x)\left[-\delta+MCt^{\theta}+\right.$$
$$\left.MNt^{\theta+\beta-\alpha}+Nt^{\beta-\alpha}\right]+t^{2\alpha}C^{2}\left[1+\frac{t^{2(\beta-\alpha)}N^{2}}{C^{2}}+\frac{2t^{(\beta-\alpha)}N}{C}\right].$$

We set

$$t_x:=\left(\frac{\delta\mathrm{d}_S(x)}{2C^{2}}\right)^{1/\alpha}.$$

According to (15) and (21),

$$0<t_x<T_0\ \text{ and }\ 0<p_{xy}(t_x)<T. \tag{22}$$

Then, by applying the inequalities (15)–(18) we obtain that

$$\mathrm{d}_S(z_{x,y}(x,t_x))^{2}\leq\mathrm{d}_S^{2}(x)\left[1+M^{2}t_x^{2\theta}+2t_x^{\alpha}L+2Mt_x^{\theta}\right]$$
$$+2t_x^{\alpha}\mathrm{d}_S(x)\left[-\delta+MCt_x^{\theta}+MNt_x^{\theta+\beta-\alpha}+Nt_x^{\beta-\alpha}\right]$$
$$+t_x^{2\alpha}C^{2}\left[1+\frac{t_x^{2(\beta-\alpha)}N^{2}}{C^{2}}+\frac{2t_x^{(\beta-\alpha)}N}{C}\right]$$
$$=\mathrm{d}_S^{2}(x)\left[1+M^{2}\left(\frac{\delta\mathrm{d}_S(x)}{2C^{2}}\right)^{2\theta/\alpha}+\frac{L\delta\mathrm{d}_S(x)}{C^{2}}+2M\left(\frac{\delta\mathrm{d}_S(x)}{2C^{2}}\right)^{\theta/\alpha}\right]$$
$$+2\frac{\delta\mathrm{d}_S^{2}(x)}{2C^{2}}\left[-\delta+MC\left(\frac{\delta\mathrm{d}_S(x)}{2C^{2}}\right)^{\theta/\alpha}+MN\left(\frac{\delta\mathrm{d}_S(x)}{2C^{2}}\right)^{(\theta+\beta-\alpha)/\alpha}\right.$$
$$\left.+N\left(\frac{\delta\mathrm{d}_S(x)}{2C^{2}}\right)^{(\beta-\alpha)/\alpha}\right]$$
$$+\frac{\delta^{2}\mathrm{d}_S^{2}(x)}{4C^{4}}C^{2}\left[1+\left(\frac{\delta\mathrm{d}_S(x)}{2C^{2}}\right)^{2(\beta-\alpha)/\alpha}\frac{N^{2}}{C^{2}}+2\left(\frac{\delta\mathrm{d}_S(x)}{2C^{2}}\right)^{(\beta-\alpha)/\alpha}\frac{N}{C}\right]$$
$$\leq\mathrm{d}_S^{2}(x)\left[1+M^{2}\left(\frac{\delta r}{2C^{2}}\right)^{2\theta/\alpha}+\frac{L\delta r}{C^{2}}+2M\left(\frac{\delta r}{2C^{2}}\right)^{\theta/\alpha}\right]$$
$$+\frac{\delta\mathrm{d}_S^{2}(x)}{C^{2}}\left[-\delta+MC\left(\frac{\delta r}{2C^{2}}\right)^{\theta/\alpha}+MN\left(\frac{\delta r}{2C^{2}}\right)^{(\theta+\beta-\alpha)/\alpha}\right.$$
$$\left.+N\left(\frac{\delta r}{\ldots}\right)^{(\beta-\alpha)/\alpha}\right]$$

$$+\frac{\delta^2 \mathrm{d}_S^2(x)}{4C^2}\left[1+\left(\frac{\delta r}{2C^2}\right)^{2(\beta-\alpha)/\alpha}\frac{N^2}{C^2}+2\left(\frac{\delta r}{2C^2}\right)^{(\beta-\alpha)/\alpha}\frac{N}{C}\right]$$
$$\leq \mathrm{d}_S^2(x)\left[1+\frac{\delta^2}{16C^2}-\frac{\delta^2}{2C^2}+\frac{5\delta^2}{16C^2}\right]=\mathrm{d}_S^2(x)\left(1-\frac{\delta^2}{8C^2}\right).$$

Thus, for every point x from $S_r \setminus S$ the following estimation holds true:

$$\mathrm{d}_S(z_{x,y}(x,t_x))^2 \leq \mathrm{d}_S(x)^2\left(1-\frac{\delta^2}{8C^2}\right), \text{ i.e.}$$
$$\mathrm{d}_S\left(z_{x,y}(x,t_x)\right) \leq q.\mathrm{d}_S(x), \quad \text{where } \ q := \sqrt{1-\frac{\delta^2}{8C^2}} \tag{23}$$

Taking into account (19), we have that $0 < q < 1$. Thus, $z_{x,y}(x,t_x) \in S_r$. If we assume that $z_{x,y}(x,t_x) \in S$, then (20) and (22) imply that the set S can be reached from the point x in time $p_{x,y}(t_x)$, which is less than T, and we are done. Let us assume that $z_{x,y}(x,t_x) \notin S$. Then the estimation (23) means that $z_{x,y}(x,t_x) \in S_r \setminus S$. We set $p := q^{1/\alpha}$ (clearly, $0 < p < 1$), $x_1 := z_{x,y}(x,t_x)$ and $x_2 := z_{x_1 y_1}(x_1, t_{x_1})$, where y_1 is an arbitrary element of $\pi(x_1)$, and

$$t_{x_1} := \left(\frac{\delta \mathrm{d}_S(x_1)}{2C^2}\right)^{1/\alpha} \leq \left(\frac{\delta q \mathrm{d}_S(x)}{2C^2}\right)^{1/\alpha} = \left(\frac{\delta \mathrm{d}_S(x)}{2C^2}\right)^{1/\alpha} .q^{1/\alpha} = p.t_x,$$

(according to (23)). Since the estimation (23) holds true for arbitrary point $x \in S_r \setminus S$, we apply it for the point x_1 and obtain that

$$\mathrm{d}_S\left(x_2\right) = \mathrm{d}_S\left(z_{x_1 y_1}(x_1, t_{x_1})\right) \leq q.\mathrm{d}_S(x_1) = q.\mathrm{d}_S\left(z_y(x,t_x)\right) \leq q^2.\mathrm{d}_S(x).$$

Moreover,

$$x_2 = z_{x_1 y_1}(x_1, t_{x_1}) \in \mathcal{A}\left(x_1, p_{x_1 y_1}\left(t_{x_1}\right)\right) \subset \mathcal{A}\left(x, p_{x,y}\left(t_x\right) + p_{x_1 y_1}\left(t_{x_1}\right)\right)$$
$$=: \mathcal{A}(x, T_2)$$

where $T_2 = p_{x,y}\left(t_x\right) + p_{x_1 y_1}\left(t_{x_1}\right)$. Let us assume that $x_2 \in S$. Since

$$0 < T_2 = p_{x,y}\left(t_x\right) + p_{x_1 y_1}\left(t_{x_1}\right) \leq P\left(t_x + t_{x_1}\right) \leq P(1+p)t_x < T,$$

we obtain that the set S can be reached from the point x in time T_2 which is less than T, i.e. we are done.

Assuming that $x_1, x_2, \ldots, x_k, k \geq 2$, does not belong to S, we continue in the same way by setting $x_{j+1} = z_{x_j y_j}(x_j, t_{x_j}), j = 2, \ldots k-1$, where y_j is an arbitrary element of $\pi(x_j)$ and

$$t_{x_{j-1}} := \left(\frac{\delta \mathrm{d}_S(x_{j-1})}{\cdots}\right)^{1/\alpha}$$

Using again (23), it can be proved inductively that

$$t_{x_j} := \leq p.t_{x_{j-1}} \leq p^j.t_x, \quad \mathrm{d}_S(x_j) \leq q.\mathrm{d}_S(x_{j-1}) \leq q^j.\mathrm{d}_S(x) \tag{24}$$

$$x_{j+1} \in \mathcal{A}\left(x_j, p_{x_j y_j}\left(t_{x_j}\right)\right) \subset \mathcal{A}(x, T_{j+1}), \quad \text{where} \tag{25}$$

$$T_{j+1} := p_{x,y}\left(t_x\right) + p_{x_1 y_1}\left(t_{x_1}\right) + \ldots + p_{x_j y_j}\left(t_{x_j}\right) \tag{26}$$

$$\leq P.(1 + p + \ldots p^j)t_x \leq \frac{P.t_x}{1-p} < T \tag{27}$$

(the last inequality holds true because of (15)).

According to (24), (25), (26) and (27), the existence of a positive integer k such that $x_j \notin S$ for $j = 1, \ldots, k-1$ and $x_k \in S$ means that the set S can be reached from the point x in a positive time T_k, which is less than T, i.e. we are done. Let us assume that $x_k \notin S$ for every positive integer k. According to (25), every point $x_k, k = 1, 2, 3, \ldots$ determines an admissible trajectory $z_k(\cdot)$ of (1) defined on $[0, T_k]$ and such that $z_k(0) = x$ and $z_k(T_k) = x_k$. Clearly, $0 < T_k < T$ and $\{T_k\}_{k=1}^{\infty}$ is a monotonically increasing sequence of real numbers. Let $\{T_k\}_{k=1}^{\infty} \to T^*$. Clearly, $0 < T^* \leq T$. According to Theorem 3.1.7 from Clarke (1983) , there exists a subsequence $\{z_{k_j}(.)\}_{k=1}^{\infty}$ and a trajectory $z(\cdot)$ of (1) defined on $[0, T^*]$ such that $z_{k_j}(t) \to z(t)$ uniformly over $[0, \min\{T_{k_j}, T^*\}]$. Applying (24), we obtain that

$$\mathrm{d}_S(z(T^*)) = \lim_{j\to\infty} \mathrm{d}_S(z(T_{k_j})) = \lim_{j\to\infty} \mathrm{d}_S(z_{k_j}(T_{k_j})) = 0.$$

This and the inequalities $0 < T^* \leq T$ imply that the set S can be reached from the point x in a positive time T^* which is not greater than T, i.e. we are done in this case too. ■

Proof of Proposition 2.1. Our proof follows the corresponding proof from the paper by Bianchini and Stefani (1990), considering the case when S is a point and the control system is determined by a differential equation.

We set

$$D := e^{KCr^{1/\alpha}}. \tag{28}$$

Let x belong to the interior of the set S_r and let U be a neighbourhood of x, such that

$$U \subset S_r \cap \left\{y \in R^n \mid \|y - x\| \leq \frac{r}{2D}\right\} \cap \{y \in R^n \mid \|y - x\| \leq \sigma\}.$$

Let y_1 and y_2 be arbitrary points from U. Suppose $\Theta(y_1) < \Theta(y_2)$. Fix an arbitrary ε from the interval $(0, \Theta(y_2) - \Theta(y_1))$. Since $0 \leq \Theta(y_1) < \Theta(y_1) + \varepsilon$, there exists a trajectory $z_1(\cdot)$ of (1) starting from y_1 that reaches S in some time τ with

According to assumption ii/ there exists a trajectory $z_2(\cdot)$ of (1) starting from y_2, defined on $[0,\tau]$ and such that for every $t \in [0,\tau]$

$$\|z_2(t) - z_1(t)\| \le e^{Kt}\|y_2 - y_1\|.$$

Because $y_2 \in S_r$, we obtain that

$$\tau \le \Theta(y_2) \le C.r^{1/\alpha}.$$

Our choice of U, (28) and the inclusion $z_1(\tau) \in S$ imply that

$$\|\mathrm{d}_S(z_2(\tau))\| \le \|z_2(\tau) - z_1(\tau)\| \le e^{K\tau}\|y_2 - y_1\| \le D\|y_2 - y_1\| \le r. \qquad (29)$$

So, $z_2(\tau) \in S_r$ and by assumption i/

$$\Theta(z_2(\tau)) \le C.d_S(z_2(\tau))^{1/\alpha}.$$

Thus, according to (29),

$$\Theta(y_2) \le \tau + \Theta(z_2(\tau)) \le \Theta(y_1) + \epsilon + C.d_S(z_2(\tau))^{1/\alpha}$$
$$\le \Theta(y_1) + C.D^{1/\alpha}.\|y_2 - y_1\|^{1/\alpha} + \epsilon.$$

Since $\Theta(y_1)$ and $\Theta(y_2)$ do not depend on ϵ, we obtain that

$$\|\Theta(y_2) - \Theta(y_1)\| \le \widehat{C}\|y_2 - y_1\|^{1/\alpha},$$

where

$$\widehat{C} := C.D^{1/\alpha}.$$ ■

Acknowledgements. The author would like to express his gratitude to the anonymous referees for the valuable comments and suggestions which contributed much to improve the presentation of the paper.

References

AGRACHEV, A. and GAMKRELIDZE, R. (1993) Local controllability and semigroups of diffeomorphisms. *Acta Applicandae Mathematicae*, **32**, 1–57.

BACCIOTTI, A. and STEFANI, G. (1980) Self-accessibility of a set with respect to a multivalued field. *JOTA*, **31**, 535–552.

BIANCHINI, R. and STEFANI, G. (1990a) Time optimal problem and time optimal map. *Rend. Sem. Mat. Univers. Politecn. Torino*, **48**, 401–429.

BIANCHINI, R. and STEFANI, G. (1990b) Graded approximation and controllability along a trajectory. *SIAM Journal on Control and Optimization*, **28**, 903–924.

BRUNOVSKY, P. (1974) Local controllability of odd systems. *Banach Center Publications*, Warsaw, **1**, 39–45.

CLARKE, F. (1983) *Optimization and nonsmooth analysis.* John Wiley & Sons, Clarke, F. and Wolenski, P. (1996) Control of systems to sets and their interiors. *JOTA*, **88**, 3–23.

Clarke, F., Ledyaev, Yu., Stern, R. and Wolenski, P. (1998) *Nonsmooth analysis and control theory.* Volume **178** of *Graduate Texts in Mathematics*. Springer-Verlag, New York.

Frankowska, H. (1989) Local controllability of control systems with feedback. *JOTA*, **60**, 277–296.

Hermes, H. (1978) Lie algebras of vector fields and local approximation of attainable sets. *SIAM Journal on Control and Optimization*, **16**, 715–727.

Hermes, H. (1982) Control systems with decomposable Lie algebras. *Journal of Differential Equations*, **44**, 166–187.

Jurdjevic, V. and Kupka, I. (1985) Polynomial Control Systems. *Mathematische Annalen*, **272**, 361–368.

Kawski, M. (1988) Control variations with an increasing number of switchings. *Bulletin of the American Mathematical Society*, **18**, 149–152.

Krastanov, M. and Quincampoix, M. (2001) Local small-time controllability and attainability of a set for nonlinear control system. *ESAIM: Control Optim. Calc. Var.*, **6**, 499-516

Soravia, P. (1978) Hölder continuity of the minimal-time function for C^1-manifold targets. *JOTA*, **75**, 790–802.

Sussmann, H. (1987) A general theorem on local controllability. *SIAM Journal on Control and Optimization*, **25**, 158–194.

Veliov, V. (1988) On the controllability of control constrained linear systems. *Mathematica Balkanica, New Series*, **2**, 147–155.

Veliov, V. (1994) *Attractiveness and invariance: The case of uncertain measurement.* Volume **18** of *Prog. Syst. Control Theory "Modeling Techniques for uncertain systems"*. Birkhäuser, Basel, 277–288.

Veliov, V. (1997) On the Lipschitz continuity of the value function in optimal control. *JOTA*, **94**, 335–361.

Veliov, V., and Krastanov, M. (1986) Controllability of piecewise linear systems. *Systems & Control Letters*, **7**, 335–341.