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Second-order sufficient conditions for an extremum in optimal control

by

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Abstract: Sufficient quadratic optimality conditions for a weak and a strong minimum are stated in an optimal control problem on a fixed time interval with mixed state-control constraints, under the assumption that the gradients of all active mixed constraints with respect to control are linearly independent. The conditions are stated for the cases of both continuous and discontinuous controls and guarantee in each case a lower bound of the cost function increase at the reference point. They are formulated in terms of an accessory problem with quadratic form, which must be positive-definite on the so-called critical cone. In the case of discontinuous control the quadratic form has some new terms related to the control discontinuity.

Keywords: maximum principle, broken extremal, weak minimum, strong minimum, control discontinuities, quadratic form, critical cone, accessory problem, sufficient optimality condition.

1. Introduction

The classical sufficient quadratic optimality conditions for a problem with constraints require that the second variation of Lagrangian is positive definite in all critical directions. In this paper, we give sufficient quadratic conditions for a strong minimum in an optimal control problem with mixed state-control constraints and endpoint constraints of both equality and inequality type, for an extremal with at most a finite number of control discontinuities.

In the case of continuous control these conditions have an almost classical form. We begin with this case in Section 2. However, even in this case, the set Λ_0 of normed collections of Lagrange multipliers of the problem does not necessarily consist of a single element. Therefore, second-order conditions are formulated not in terms of a quadratic form, but of a maximum of quadratic forms over the

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set Λ_0. This situation is characteristic for problems with inequality constraints, particularly for optimal control problems.

In Section 3, which is central to the paper, we formulate sufficient quadratic conditions for the general case of extremal with a finite number of control discontinuities. In this case the quadratic form has new terms related to control jumps.

In Section 4, we explain the methodology of the proofs, which are based on the abstract theory of higher order conditions derived by Levitin, Milyutin and Osmolovskii (1974, 1978), and then developed by Milyutin and Osmolovskii (1993).

Two important illustrative examples are presented in Section 5: a finite dimensional example, due to Milyutin, where the maximum of quadratic forms is positive on all critical directions without any single one being so, and an example of extremals with control jumps, which is investigated using the quadratic conditions formulated in Section 3.

The main results presented in this paper are due to the author and were published in Osmolovskii (1975, 1986, 1988A, 1988B, 1995, 2000), Milyutin and Osmolovskii (1998, Part 2).

Statement of the problem. The following optimal control problem on a fixed time interval $\Delta = [t_0, t_1]$ will be considered and referred to as the *canonical problem*:

$$\begin{aligned} &\mathcal{J}(x,u) = J(x(t_0), x(t_1)) \to \min, \\ &F(x(t_0), x(t_1)) \le 0, \quad K(x(t_0), x(t_1)) = 0, \\ &\dot{x} = f(t,x,u), \quad g(t,x,u) = 0, \quad \varphi(t,x,u) \le 0, \\ &(x(t_0), x(t_1)) \in \mathcal{P}, \quad (t,x,u) \in \mathcal{Q}, \end{aligned} \tag{1}$$

where $\mathcal{P}$ and $\mathcal{Q}$ are open sets, the functions J, F, and K are defined and twice continuously differentiable on $\mathcal{P}$, and the functions f, g, and φ are defined and twice continuously differentiable on $\mathcal{Q}$. We assume that the gradients with respect to control $g_{iu}(t,x,u)$, $i = 1, \ldots, d(g)$ and $\varphi_{ju}(t,x,u)$, $j \in I_\varphi(t,x,u)$ are jointly linearly independent at each point $(t,x,u) \in \mathcal{Q}$ such that $g(t,x,u) = 0$, $\varphi(t,x,u) \le 0$, where

$$I_\varphi(t,x,u) = \{j \in \{1, \ldots, d(\varphi)\} \mid \varphi_j(t,x,u) = 0\}$$

is the set of indices of the active constraints and $d(a)$ denotes the dimension of the vector a.

The minimum is taken over the pairs of functions $w(\cdot) = (x(\cdot), u(\cdot))$ such that $x(\cdot) \in W^{1,1}_{d(x)}$ and $u(\cdot) \in L^\infty_{d(u)}$, where $W^{1,1}_{d(x)}$ is the space of absolutely continuous functions from Δ into $\mathbb{R}^{d(x)}$, endowed with the norm

$$\|x\|_{1,1} = |x(t_0)| + \int |\dot{x}(t)|\, dt,$$

and $L^\infty_{d(u)}$ denotes the space of bounded measurable functions from Δ into $\mathbb{R}^{d(u)}$, endowed with the norm

$$\|u\|_\infty = \operatorname*{ess\,sup}_{\Delta} |u(t)|.$$

Set

$$\mathcal{W} = W^{1,1}_{d(x)} \times L^\infty_{d(u)}.$$

Thus, the problem is considered in $\mathcal{W}$.

For brevity, we put

$$x_0 = x(t_0), \quad x_1 = x(t_1), \quad p = (x_0, x_1), \quad w = (x, u).$$

Let $w^0 = (x^0, u^0) \in \mathcal{W}$ be an admissible point of the canonical problem (1) satisfying the following condition: there exists a compact set $\mathcal{C} \subset \mathcal{Q}$ such that $(t, x^0(t), u^0(t)) \in \mathcal{C}$ a.e. on Δ. It means that there exists an $\varepsilon > 0$ such that a.e., on Δ, the distance of the point $(t, x^0(t), u^0(t)) \in \mathcal{Q}$ to the boundary $\partial\mathcal{Q}$ is not less than ε.

Weak γ-sufficiency. Each sufficient second-order optimality condition is related to a local minimum of some specific shape, or some specific order. For example, the second-order sufficient condition

$$\Phi'(x^0) = 0, \qquad \Phi''(x^0) > 0$$

for a local minimum of a smooth function $\Phi(x)$ at the point x^0 is equivalent to the following condition: there exists $C > 0$ such that

$$\delta\Phi := \Phi(x^0 + \delta x) - \Phi(x^0) \geq C|\delta x|^2$$

for all sufficiently small δx. We refer to this condition as *γ-minimum* at a point x^0 for $\gamma = |\delta x|^2$, and to the function γ as the *order of the condition*, or simply the *order*.

What is a typical order in optimal control, particularly in problem (1), at the given admissible point $w^0 = (x^0, u^0) \in \mathcal{W}$? The answer is not unique. It depends on the type of minimum under consideration (weak, strong or some other), and on the properties of the control $u^0(t)$ (continuous, discontinuous, singular, nonsingular, etc.). Second-order sufficient conditions for a weak minimum, obtained in the recent three decades by many authors (see, e.g. Dontchev et al., 1995, Levitin, Milyutin, Osmolovskii, 1978, Maurer, 1981, 1992, Maurer, Pickenhain, 1995, Osmolovskii, 1975, 1988, 1995, Pickenhain, Tammer, 1991, Pickenhain, 1992, Zeidan, 1983, 1989, 1994), guarantee the following lower bound of the cost functional increase at the reference point:

where

$$\gamma(\delta w) = \max_{\Delta} |\delta x(t)|^2 + \int_{\Delta} |\delta u(t)|^2 \, dt. \tag{3}$$

Inequality (2) holds with some $\varepsilon > 0$ for all admissible variations $\delta w = (\delta x, \delta u)$ in some neighbourhood of zero in $\mathcal{W}$. (A variation $\delta w \in \mathcal{W}$ is called *admissible* if $w^0 + \delta w$ satisfies the constraints of the problem.) We call condition (2) a *weak minimum of the order* γ, or a *weak* γ*-minimum*. The standard order (3) evolved naturally out of the calculus of variations, where minimum of the order $\gamma = \int |\delta \dot{x}(t)|^2 \, dt$ is equivalent to the Jacobi sufficient condition.

However, it could be proved that the second-order conditions mentioned above guarantee not only a weak minimum of the order γ, but also some stronger condition, which we call a *weak* γ*-sufficiency* at the point w^0 and define as follows: there exists $\varepsilon > 0$ such that

$$\sigma(\delta w) \geq \varepsilon \gamma(\delta w),$$

in some neighbourhood of zero in $\mathcal{W}$, where

$$\sigma(\delta w) = \sum_{i=0}^{d(F)} F_i^+(p^0 + \delta p) + |K(p^0 + \delta p)| + \int_{\Delta} |\dot{x}^0 + \delta \dot{x} - f(t, w^0 + \delta w)| \, dt$$

$$+ \operatorname*{ess\,sup}_{\Delta} |g(t, w^0 + \delta w)| + \sum_{j=1}^{d(\varphi)} \operatorname*{ess\,sup}_{\Delta} \varphi_j^+(t, w^0 + \delta w).$$

In this definition, we used the following notations:

$$\delta w = (\delta x, \delta u) \in \mathcal{W}, \quad \delta p = (\delta x(t_0), \delta x(t_1)), \quad F_0(p) = J(p) - J(p^0),$$
$$p^0 = (x^0(t_0), x^0(t_1)), \quad a^+ = \max\{a, 0\}.$$

Obviously, $\sigma \geq 0$ and $\sigma(0) = 0$. We call σ the *violation function* of the canonical problem.

It is easy to show that, at the point w^0, *a weak* γ*-sufficiency* $\Rightarrow$ *a weak* γ*-minimum* $\Rightarrow$ *a strict weak minimum.*

Strong minimum. The commonly used notion of a strong minimum in the calculus of variations corresponds only to the proximity of the state components of trajectories. We shall give a new definition of a strong minimum, which is even stronger than the one just mentioned. To this end, let us define a notion of an unessential component of a vector x.

DEFINITION. The state variable x_i (or the ith component x_i of vector x) is said to be *unessential* if the functions f, g, φ do not depend on it and J, F, K are affine in x_{i0} and x_{i1}, where $x_{i0} = x_i(t_0)$, $x_{i1} = x_i(t_1)$. A state variable x_i, which

Accordingly, we will speak of essential and unessential components of a vector x.

For example, consider some optimization problem with the integral functional

$$\mathcal{J} = \int_{t_0}^{t_1} G(t, x, u)\, dt.$$

We can rewrite it to the canonical form by representing $\mathcal{J}$ as

$$y(t_1) - y(t_0),$$

where y is a new state variable satisfying the equation

$$\dot{y} = G(t, x, u).$$

Clearly, y is the unessential component in the new problem, and it must not be taken into consideration in the definition of a strong minimum.

We denote by $\underline{x}$ a vector composed of all essential components of vector x.

DEFINITION. We say that w^0 is a point of a *strict strong minimum* if there exists an $\varepsilon > 0$ such that $\delta\mathcal{J} > 0$ for all admissible nonzero variations $\delta w = (\delta x, \delta u)$ satisfying the conditions

$$\max_{\Delta} |\delta\underline{x}(t)| < \varepsilon, \quad |\delta x(t_0)| < \varepsilon. \tag{4}$$

DEFINITION. We say that w^0 is a point of a *strong γ-sufficiency* if there exists an $\varepsilon > 0$ such that $\sigma(\delta w) \geq \varepsilon\gamma(\delta w)$ for all variations $\delta w = (\delta x, \delta u)$ satisfying conditions (4) and the inequality $\sigma(\delta w) < \varepsilon$.

If δw is an admissible variation such that $\delta\mathcal{J} \leq 0$, then $\sigma(\delta w) = 0$. Therefore, *a strong γ-sufficiency implies a strict strong minimum* at the point w^0.

DEFINITION. We say that w^0 is a point of a *strict bounded-strong minimum* if for any compact set $\mathcal{C} \subset \mathcal{Q}$, there exists an $\varepsilon > 0$ such that $\delta\mathcal{J} > 0$ for all admissible nonzero variations $\delta w = (\delta x, \delta u)$ satisfying the conditions

$$\begin{aligned} &(t, x^0(t), u^0(t) + \delta u(t)) \in \mathcal{C} \text{ a.e. on } \Delta, \\ &\max_{\Delta} |\delta\underline{x}(t)| < \varepsilon, \quad |\delta x(t_0)| < \varepsilon. \end{aligned} \tag{5}$$

DEFINITION. We say that w^0 is a point of a *bounded-strong γ-sufficiency* if there exists an $\eta > 0$ such that, for any compact set $\mathcal{C} \subset \mathcal{Q}$, there exists an $\varepsilon > 0$ such that $\sigma(\delta w) \geq \eta\gamma(\delta w)$ for all variations $\delta w = (\delta x, \delta u)$ satisfying conditions (5) and the inequality $\sigma(\delta w) < \varepsilon$.

Obviously, *a bounded-strong γ-sufficiency implies a strict bounded-strong minimum* at the point w^0.

Also, it is clear that the notions of the bounded-strong γ-sufficiency and the bounded-strong minimum are weaker than the notions of the strong γ-sufficiency and the strong minimum, respectively. It is easy to give a condition which guarantees the equivalence of these notions. Set

$$U(t,x) = \{u \in \mathbb{R}^{d(u)} \mid (t,x,u) \in \mathcal{Q},\ g(t,x,u) = 0,\ \varphi(t,x,u) \le 0\}.$$

PROPOSITION 1 *Assume that there exists a compact set $\mathcal{C} \subset \mathcal{Q}$ such that the conditions $t \in \Delta$, $u \in U(t, x^0(t))$ imply that $(t, x^0(t), u) \in \mathcal{C}$. Then, the bounded-strong γ-sufficiency and the bounded-strong minimum are equivalent to the strong γ-sufficiency and the strong minimum, respectively.*

The proposition follows from the definitions.

We shall formulate sufficient conditions for a weak minimum, which are equivalent to a weak γ-sufficiency, and their strengthening to sufficient conditions for a bounded-strong minimum, which are equivalent to a bounded-strong γ-sufficiency. We shall do it first for the case of a continuous control u^0 and the order γ defined by formula (3), and then for the case of a discontinuous control u^0 and a higher order.

2. Sufficient optimality conditions in the case of continuous control

First order necessary conditions. In this section we assume that *the control u^0 is continuous*, in order to achieve a better understanding of more general results of Section 3 for discontinuous controls.

Let us state the well-known first-order necessary conditions for a weak minimum and for the so-called Pontryagin minimum introduced by Milyutin (see Milyutin, Osmolovskii, 1998). These conditions are often referred to as the local and the integral maximum principle, respectively. The local maximum principle, or Euler–Lagrange equation, which we give first, is conveniently identified with the nonemptiness of the set Λ_0 defined below. Let

$$l = \alpha_0 J + \langle \alpha, F \rangle + \langle \beta, K \rangle, \quad H = \langle \psi, f \rangle, \quad \bar{H} = H - \langle \nu, g \rangle - \langle \mu, \varphi \rangle,$$

where α_0 is a scalar, α, β, ψ, ν, and μ are vectors of the same dimension as F, K, f, g, and φ, respectively, and $\langle \cdot, \cdot \rangle$ is the inner product. The functions l, H, and $\bar{H}$ depend on the following variables:

$$l = l(p, \alpha_0, \alpha, \beta), \quad H = H(t, w, \psi), \quad \bar{H} = \bar{H}(t, w, \psi, \nu, \mu).$$

Let λ denote an arbitrary tuple

$$(\alpha_0, \alpha, \beta, \psi(\cdot), \nu(\cdot), \mu(\cdot))$$

with

$$\alpha_0 \in \mathbb{R}, \quad \alpha \in \mathbb{R}^{d(F)}, \quad \beta \in \mathbb{R}^{d(K)}, \quad \psi(\cdot) \in W^{1,\infty}, \quad \nu(\cdot) \in L^{\infty}, \quad \mu(\cdot) \in L^{\infty}$$

Here, $W^{1,\infty}_{d(x)}$ is the space of Lipschitz continuous functions from Δ into $\mathbb{R}^{d(x)}$.

Denote by Λ_0 the set of all tuples λ satisfying the conditions

$$\begin{aligned}
&\alpha_0 \geq 0, \quad \alpha \geq 0, \quad \langle \alpha, F(p^0)\rangle = 0,\\
&\sum_{i=0}^{d(F)} \alpha_i + \sum_{j=1}^{d(K)} |\beta_j| = 1,\\
&\mu(t) \geq 0, \quad \langle \mu(t), \varphi(t, w^0(t))\rangle = 0, \qquad (6)\\
&\dot{\psi} = -\bar{H}_x, \quad \psi(t_0) = l_{x_0}, \quad \psi(t_1) = -l_{x_1},\\
&\bar{H}_u = 0,
\end{aligned}$$

where α_i and β_j are components of the vectors α and β, respectively. The gradients l_{x_0} and l_{x_1} are taken at the point $(p^0, \alpha_0, \alpha, \beta)$ with $p^0 = (x^0(t_0), x^0(t_1))$, and the gradients $\bar{H}_x$ and $\bar{H}_u$ are evaluated for the point $(t, w^0(t), \psi(t), \nu(t), \mu(t))$ with $t \in \Delta$.

It is well-known that *if w^0 is a weak minimum, then Λ_0 is nonempty* (see, e.g., Dubovitskii and Milyutin, 1965, 1971, 1981). The latter condition is just the *local maximum principle*. Note that Λ_0 can consist of more than one element. The following result pertains to this possibility (see, e.g., Osmolovskii, 1975, 1986, 1995).

PROPOSITION 2 *The set Λ_0 is a finite-dimensional compact set and the projection $\lambda = (\alpha_0, \alpha, \beta, \psi, \nu, \mu) \to (\alpha_0, \alpha, \beta)$ is injective on Λ_0.*

Note also the following property of Lagrange multipliers, which is specific to the case of continuous control u^0.

PROPOSITION 3 *For any $\lambda = (\alpha_0, \alpha, \beta, \psi, \nu, \mu) \in \Lambda_0$, the functions ν and μ are continuous.*

Similarly, the integral maximum principle, which is a first-order necessary condition for the so-called Pontryagin minimum at w^0 (see Section 4 for the definition), can be stated in terms of nonemptiness of the set M_0 defined below. Denote by M_0 the set of tuples $\lambda \in \Lambda_0$ such that for all $t \in \Delta$, the inclusion $u \in U(t, x^0(t))$ implies the inequality

$$H(t, x^0(t), u, \psi(t)) \leq H(t, x^0(t), u^0(t), \psi(t)). \qquad (7)$$

The requirement that M_0 be non-empty is the *integral* (or *Pontryagin*) *maximum principle* (see, e.g., Dubovitskii and Milyutin, 1981).

Note that, just like Λ_0, the set M_0 can contain more than one element. Since this set is closed, it follows from Proposition 2 that M_0 is also a finite-dimensional compact set.

In order to formulate sufficient conditions for a weak minimum at the point w^0 we must also define the notions of the critical cone and the corresponding

Critical cone. Denote by $W^{1,2}_{d(x)}$ the space of absolutely continuous functions

$$\bar{x}(\cdot) : \Delta \to \mathbb{R}^{d(x)}$$

with square integrable derivative. Denote by $\mathcal{W}_2$ the space of pairs $\bar{w} = (\bar{x}, \bar{u})$ such that

$$\bar{x} \in W^{1,2}_{d(x)}, \quad \bar{u} \in L^2_{d(u)},$$

where $L_2^{d(u)}$ is the space of square integrable functions

$$\bar{u}(\cdot) : \Delta \to \mathbb{R}^{d(u)}.$$

Let

$$I_F(p^0) = \{i \in \{1, \ldots, d(F)\} \mid F_i(p^0) = 0\}$$

be the set of indices of active inequality constraints $F_i(p) \leq 0$ at the point p^0, where F_i are the components of a vector function F. Let $\mathcal{K}$ denote the set of $\bar{w} = (\bar{x}, \bar{u}) \in \mathcal{W}_2$ such that

$$\begin{aligned} &\langle J_p(p^0), \bar{p}\rangle \leq 0, \quad \langle F_{ip}(p^0), \bar{p}\rangle \leq 0 \quad \forall i \in I_F(p^0), \quad K_p(p^0)\bar{p} = 0, \\ &\dot{\bar{x}} = f_w(t, w^0)\bar{w}, \\ &g_w(t, w^0)\bar{w} = 0, \quad \langle \varphi_{jw}(w^0, t), \bar{w}\rangle \leq 0 \quad \forall j \in I_\varphi(t, w^0(t)), \end{aligned} \tag{8}$$

where $\bar{p} = (\bar{x}(t_0), \bar{x}(t_1))$, $\bar{w} = (\bar{x}, \bar{u})$. Obviously, $\mathcal{K}$ is a closed convex cone in the space $\mathcal{W}_2$.

The following question is of interest: which inequalities in the definition of $\mathcal{K}$ can be replaced by equalities without affecting $\mathcal{K}$? This question is answered below.

PROPOSITION 4 *For any $\lambda = (\alpha_0, \alpha, \beta, \psi, \nu, \mu) \in \Lambda_0$ and $\bar{w} \in \mathcal{K}$, we have*

$$\begin{aligned} &\alpha_0\langle J_p(p^0), \bar{p}\rangle = 0, \quad \alpha_i\langle F_{ip}(p^0), \bar{p}\rangle = 0 \quad \forall i \in I_F(p^0), \\ &\mu_j\langle \varphi_{jw}(t, w^0), \bar{w}\rangle = 0 \quad \forall j = 1, \ldots, d(\varphi), \end{aligned}$$

where α_i and μ_j are the components of the vectors α and μ, respectively.

Hence, each inequality in the definition of $\mathcal{K}$ can be replaced by equality if the corresponding Lagrange multiplier is positive for some $\lambda \in \Lambda_0$.

The following question is also of interest: under what conditions can one of the terminal inequalities in the definition of $\mathcal{K}$ be omitted without affecting $\mathcal{K}$?

PROPOSITION 5 *Suppose that there exists a $\lambda \in \Lambda_0$ such that $\alpha_0 > 0$. Then the relations*

$$\begin{aligned}
&\langle F_{ip}(p^0), \bar{p}\rangle \le 0, \quad \alpha_i \langle F_{ip}(p^0), \bar{p}\rangle = 0 \ \forall i \in I_F(p^0); \ K_p(p^0)\bar{p} = 0;\\
&\dot{\bar{x}} = f_w(t, w^0)\bar{w}, \quad g_w(t, w^0)\bar{w} = 0,\\
&\langle \varphi_{jw}(t, w^0), \bar{w}\rangle \le 0 \quad \mu_j \langle \varphi_{jw}(t, w^0), \bar{w}\rangle = 0 \quad \forall j \in I_\varphi(t, w^0(t))
\end{aligned}$$

represent an equivalent definition of $\mathcal{K}$.

A similar assertion holds for any other endpoint inequality

$$\langle F_{ip}(p^0), \bar{p}\rangle \le 0, \quad i \in I_F(p^0).$$

Obviously, if there exists a $\lambda \in \Lambda_0$ such that $\alpha_i > 0$ for all $i \in I_F(p^0)$ and, for any $t \in \Delta$, we have $\mu_j(t) > 0$ for all $j \in I_\varphi(t, w^0(t))$, then $\mathcal{K}$ is a subspace. It means that all Lagrange multipliers of active inequality constraints are positive. Certainly, this is a rather strong assumption. To avoid it, most of the authors extend $\mathcal{K}$ to a subspace by omitting inequalities in the definition of $\mathcal{K}$ for Lagrange multipliers, which are either zero or even small enough. It clears away most of the difficulties in the proofs, but leads to much stronger sufficient conditions for a local minimum than that, which we present in this paper.

Quadratic form. We are now ready to introduce the quadratic form. For any $\lambda \in \Lambda_0$ and $\bar{w} = (\bar{x}, \bar{u}) \in \mathcal{W}_2$, we set

$$\Omega^\lambda(\bar{w}) = \frac{1}{2}\langle l_{pp}\bar{p}, \bar{p}\rangle - \frac{1}{2}\int_\Delta \langle \bar{H}_{ww}\bar{w}, \bar{w}\rangle\, dt,$$

where

$$\begin{aligned}
&l_{pp} = l_{pp}(p^0, \alpha_0, \alpha, \beta), \ \bar{H}_{ww} = \bar{H}_{ww}(t, w^0(t), \psi(t), \nu(t), \mu(t)),\\
&\bar{p} = (\bar{x}(t_0), \bar{x}(t_1)), \ \bar{w} = (\bar{x}, \bar{u}).
\end{aligned}$$

Obviously, $\Omega^\lambda(\bar{w})$ is quadratic in $\bar{w}$ and linear in λ.

The set $\mathrm{Leg}_+(\Lambda_0)$. An element $\lambda = (\alpha_0, \alpha, \beta, \psi, \nu, \mu) \in \Lambda_0$ is said to be *strictly Legendrian* if the following condition is satisfied: for any $t \in \Delta$, the quadratic form

$$-\langle \bar{H}_{uu}(t, w^0(t), \psi(t), \nu(t), \mu(t))\bar{u}, \bar{u}\rangle \tag{9}$$

of the variable $\bar{u}$ is positive definite on the cone formed by the vectors $\bar{u} \in \mathbb{R}^{d(u)}$ such that

$$\begin{aligned}
&g_u(t, w^0(t))\bar{u} = 0,\\
&\langle \varphi_{ju}(t, w^0(t)), \bar{u}\rangle \le 0 \quad \forall j \in I_\varphi(t, w^0(t)),
\end{aligned} \tag{10}$$

Denote by $\text{Leg}_+(\Lambda_0)$ the set of all strictly Legendrian elements $\lambda \in \Lambda_0$.

Basic sufficient condition for a weak minimum. Let

$$\bar{\gamma}(\bar{w}) = \langle \bar{x}(t_0), \bar{x}(t_0) \rangle + \int_{\Delta} \langle \bar{u}(t), \bar{u}(t) \rangle \, dt.$$

DEFINITION. We say that a point w^0 *satisfies condition* $\mathcal{B}_0$ if there exist a nonempty compact set $M \subset \text{Leg}_+(\Lambda_0)$ and a constant $\varepsilon > 0$ such that

$$\max_{\lambda \in M} \Omega^\lambda(\bar{w}) \geq \varepsilon \bar{\gamma}(\bar{w}) \quad \forall \bar{w} \in \mathcal{K}.$$

THEOREM 1 *Condition $\mathcal{B}_0$ is equivalent to a weak γ-sufficiency at the point w^0.*

Condition $\mathcal{B}_0$ obviously holds if the set $\text{Leg}_+(\Lambda_0)$ is nonempty and the cone $\mathcal{K}$ consists only of zero.

Most of the authors use only one quadratic form in sufficient conditions. In condition $\mathcal{B}_0$, it corresponds to the case when M is a singleton.

How far is condition $\mathcal{B}_0$ from sufficient conditions for a strong minimum? Remarkably, in the case of continuous control u^0, which we are considering now, condition $\mathcal{B}_0$ is very close to such conditions. In order to obtain them, we do not need to change the quadratic form or the critical cone. We only must strengthen the maximum principle.

Basic sufficient condition for a strong minimum. Let M_0^+ denote the set of $\lambda \in M_0$ such that

$$H(t, x^0(t), u, \psi(t)) < H(t, x^0(t), u^0(t), \psi(t)) \tag{11}$$

if $t \in \Delta$, $u \in U(t, x^0(t))$, $u \neq u^0(t)$. For a given $\lambda \in M_0$, we call this condition the *strict maximum principle*.

Denote by $\text{Leg}_+(M_0^+)$ the set of all strictly Legendrian elements $\lambda \in M_0^+$.

DEFINITION. We say that a point w^0 *satisfies condition* $\mathcal{B}_0^S$ if there exist a nonempty compact set $M \subset \text{Leg}_+(M_0)$ and a constant $\varepsilon > 0$ such that

$$\max_{\lambda \in M} \Omega^\lambda(\bar{w}) \geq \varepsilon \bar{\gamma}(\bar{w}) \quad \forall \bar{w} \in \mathcal{K}.$$

THEOREM 2 *Condition $\mathcal{B}_0^S$ is equivalent to a bounded-strong γ-sufficiency at the point w^0.*

Thus, condition $\mathcal{B}_0^S$ is sufficient for a strict bounded-strong minimum (see Supplement S2 to Chapter 4 due to Osmolovskii in Levitin, Milyutin and Osmolovskii, 1978), which is equivalent to a strict strong minimum if the hypothesis of Proposition 1 holds.

Sufficient conditions for a strong minimum which do not require the hypoth-

3. Sufficient optimality conditions in the case of discontinuous control

The results in this Section are central to the paper and more general than those of Section 2. Assume now that the control u^0 is piecewise continuous, with nonempty set $\theta = \{t^1, \dots, t^s\}$ of all discontinuity points, where $t_0 < t^1 < \dots < t^s < t_1$. Also assume that u^0 is Lipschitz continuous on each interval of the set $\Delta \setminus \theta$.

We shall indicate the changes in the theory of sufficient conditions in this case, using the same notations for the new quadratic form, critical cone and some other objects as in the case of continuous control.

Sufficient conditions for a weak minimum of the order (3) do not change. But now these conditions could not be strengthened to sufficient conditions of a strong minimum. In a sense their local role has now decreased. To restore this role, we have to change the concepts of critical cone, quadratic form, and even the very concept of a weak minimum. We shall extend the class of L^∞-small variations of the control – corresponding to a weak minimum – to a broader class which defines the so called θ-weak minimum.

Denote by $\overline{u^0}$ the closure in $\mathbb{R}^{1+d(u)}$ of the set

$$\{(t, u) \in \mathbb{R}^{1+d(u)} \mid t \in \Delta \backslash \theta,\ u = u^0(t)\}.$$

Definition. We say that $w^0 = (x^0, u^0)$ is a strict *θ-weak minimum* if there exist an $\varepsilon > 0$ and a neighbourhood V of the compact set $\overline{u^0}$ such that, for each admissible pair $w = (x, u) \in \mathcal{W}$ satisfying the conditions

$$(t, u(t)) \in V \quad \text{a.e. on } \Delta, \quad \max_{\Delta} |x(t) - x^0(t)| < \varepsilon, \quad w(\cdot) \neq w^0(\cdot)$$

we have

$$\mathcal{J}(w) > \mathcal{J}(w^0).$$

What are the sufficient conditions for a θ-weak minimum? What is the order of these conditions which corresponds to a typical θ-weak minimum? These questions become quite nontrivial. The notion of an order, which will be presented now may seem strange. It is not homogeneous and it is not defined via analytic expressions. However, it suits well the fact that a minimum in optimal control, is, 'as a rule', quadratic with respect to the variations of the control that retain the jump points unchanged and it is of the first order with respect to the shifts of the jump points.

Denote by

the closure in $\mathbb{R}^{1+d(u)}$ of the graph of the restriction of $u^0(\cdot)$ to the interval (t^{k-1}, t^k), $k = 1, \ldots, s+1$, where $t^0 = t_0$, $t^{s+1} = t_1$. Put

$$u^{0k-} = u^0(t^k - 0), \quad u^{0k+} = u^0(t^k + 0), \quad [u^0]^k = u^{0k+} - u^{0k-},$$
$$k = 1, \ldots, s.$$

Let $\mathcal{V}$ be a fixed neighborhood of the compact set $\overline{u^0}$ with $\mathcal{V}$ being the union of disjoint neighborhoods $\mathcal{V}_k$ of the compact sets $\overline{u^0(t^{k-1}, t^k)}$, $k = 1, \ldots, s+1$. Define the function

$$\Gamma(t, u) : \mathcal{V} \mapsto \mathbb{R}^1$$

by the following three conditions:

(a) $\Gamma(t, u) = |u - u^0(t)|^2$ if $(t, u) \in \mathcal{V}_k$, $t \in (t^{k-1}, t^k)$, $k = 1, \ldots, s+1$;

(b) $\Gamma(t, u) = |u - u^{0k-}|^2 + 2|t - t^k|$ if $(u, t) \in \mathcal{V}_k$, $t > t^k$, $k = 1, \ldots, s$;

(c) $\Gamma(t, u) = |u - u^{0k+}|^2 + 2|t - t^k|$ if $(t, u) \in \mathcal{V}_{k+1}$, $t < t^k$, $k = 1, \ldots, s$.

We call $\Gamma(t, u)$ the *order function*.

Put

$$\gamma(\delta w) = \max_{\Delta} |\delta x(t)|^2 + \int_{\Delta} \Gamma(t, u^0(t) + \delta u(t))\, dt. \tag{12}$$

DEFINITION. We say that θ-weak γ-sufficiency holds at the point $w^0 = (x^0, u^0)$ if there exist $\varepsilon > 0$ and a neighbourhood $V \subset \mathcal{V}$ of the compact set $\overline{u^0}$ such that, for each admissible variation $\delta w = (\delta x, \delta u) \in \mathcal{W}$ satisfying the conditions

$$(t, u^0(t) + \delta u(t)) \in V \quad \text{a.e. on } \Delta, \quad \max_{\Delta} |\delta x(t)| < \varepsilon,$$

we have

$$\sigma(\delta w) \geq \varepsilon \gamma(\delta w).$$

We shall formulate the results of Osmolovskii (1988A, 1988B, 1995), which concern 'decoding' of this sufficient condition for a strict θ-weak minimum. Although the order γ under consideration is not quadratic (and, moreover, not even homogeneous of any degree), the results are again obtained in terms of an accessory problem, but with a specific quadratic form Ω which must be positive definite on a new "critical cone" $\mathcal{K}$.

Critical cone. Denote by $P_\theta W^{1,2}_{d(x)}$ the space of piecewise continuous functions

$$\bar{x}(\cdot) : [t_0, t_1] \to \mathbf{R}^{d(x)}$$

which are absolutely continuous on each interval of the set $(t_0, t_1) \setminus \theta$ and have functions are contained in θ. In what follows, given $t^k \in \theta$ and $\bar{x}(\cdot) \in P_\theta W^{1,2}_{d(x)}$, we use the notations

$$\bar{x}^{k-} = \bar{x}(t^k - 0), \quad \bar{x}^{k+} = \bar{x}(t^k + 0), \quad [\bar{x}]^k = \bar{x}^{k+} - \bar{x}^{k-}.$$

Denote by $Z_2(\theta)$ the space of triples $\bar{z} = (\bar{\xi}, \bar{x}, \bar{u})$ such that

$$\bar{\xi} = (\bar{\xi}_1, \dots, \bar{\xi}_s) \in \mathbf{R}^s, \quad \bar{x} \in P_\theta W^{1,2}_{d(x)}, \quad \bar{u} \in L^2_{d(u)}.$$

Let $\mathcal{X}_j(t)$ be the characteristic function of the set

$$\{t \in \Delta \mid \varphi_j(t, w^0(t)) = 0\}$$

for $j = 1, \dots, d(\varphi)$.

Let $\mathcal{K}$ denote the set of $\bar{z} = (\bar{\xi}, \bar{x}, \bar{u}) \in Z_2(\theta)$ such that

$$\begin{aligned} &\langle J_p(p^0), \bar{p}\rangle \le 0, \quad \langle F_{ip}(p^0), \bar{p}\rangle \le 0 \quad \forall i \in I_F(p^0), \quad K_p(p^0)\bar{p} = 0, \\ &\dot{\bar{x}} = f_w(t, w^0)\bar{w}, \quad [\bar{x}]^k = [\dot{x}^0]^k \bar{\xi}_k \quad \forall t^k \in \theta, \\ &g_w(t, w^0)\bar{w} = 0, \quad \langle \varphi_{jw}(t, w^0), \bar{w}\rangle \mathcal{X}_j \le 0 \quad \forall j = 1, \dots, d(\varphi), \end{aligned} \tag{13}$$

where $\bar{p} = (\bar{x}(t_0), \bar{x}(t_1))$, $\bar{w} = (\bar{x}, \bar{u})$, and $[\dot{x}^0]^k$ denotes the jump of the function $\dot{x}^0(t)$ at $t^k \in \theta$, i.e.,

$$[\dot{x}^0]^k = \dot{x}^{0k+} - \dot{x}^{0k-} = \dot{x}^0(t^k + 0) - \dot{x}^0(t^k - 0).$$

Let Λ_0 be defined by the same conditions (6) as before, and let M_0 be the set of tuples $\lambda \in \Lambda_0$ such that, for all $t \in \Delta \setminus \theta$, the inclusion $u \in U(t, x^0(t))$ implies the inequality (7). Given any $\lambda = (\alpha_0, \alpha, \beta, \psi, \nu, \mu) \in \Lambda_0$, we denote by $[H]^k$ the jump of the function $H(t, x^0(t), u^0(t), \psi(t))$ at $t^k \in \theta$. Let Λ_0^θ be the set of all $\lambda \in \Lambda_0$ such that $[H]^k = 0 \;\; \forall t^k \in \theta$.

PROPOSITION 6 *Λ_0^θ is a finite-dimensional compact set such that*

$$M_0 \subset \Lambda_0^\theta \subset \Lambda_0.$$

If w^0 is a point of θ-weak minimum, then Λ_0^θ is not empty. In the new definition of critical cone $\mathcal{K}$, Proposition 4 holds if Λ_0 in it is replaced by Λ_0^θ.

Proposition 5 has the following analogue:

PROPOSITION 7 *Suppose that there exists an $\lambda \in \Lambda_0^\theta$ such that $\alpha_0 > 0$. Then, $\mathcal{K}$ is characterised by the following relations*

$$\begin{aligned} &\langle F_{ip}(p^0), \bar{p}\rangle \le 0, \quad \alpha_i\langle F_{ip}(p^0), \bar{p}\rangle = 0 \;\; \forall i \in I_F(p^0); \quad K_p(p^0)\bar{p} = 0; \\ &\dot{\bar{x}} = f_w(t, w^0)\bar{w}, \quad [\bar{x}]^k = [\dot{x}^0]\bar{\xi}_k \;\; \forall t^k \in \theta; \quad g_w(t, w^0)\bar{w} = 0, \\ &\langle \varphi_{jw}(t, w^0), \bar{w}\rangle \mathcal{X}_j \le 0, \quad \mu_j\langle \varphi_{jw}(t, w^0), \bar{w}\rangle = 0 \quad \forall j = 1, \dots, d(\varphi). \end{aligned}$$

Quadratic form. The following property of elements of Λ_0 is specific for

PROPOSITION 8 *For any $\lambda = (\alpha_0, \alpha, \beta, \psi, \nu, \mu) \in \Lambda_0$ the functions ν and μ are piecewise-continuous, and all their discontinuity points are contained in θ.*

Thus, the quantities

$$\mu^{k-} = \mu(t^k - 0), \quad \mu^{k+} = \mu(t^k + 0), \quad \nu^{k-} = \nu(t^k - 0), \quad \nu^{k+} = \nu(t^k + 0)$$

are well defined for any $\lambda \in \Lambda_0$ and $t^k \in \theta$.

Given any $\lambda \in \Lambda_0$ and $t^k \in \theta$, we define

$$\Delta^k(\bar{H})(t) = \bar{H}(t, x^0(t), u^{0k+}, \psi(t), \nu^{k+}, \mu^{k+}) \\ - \bar{H}(t, x^0(t), u^{0k-}, \psi(t), \nu^{k-}, \mu^{k-}),$$

where $u^{0k-} = u^0(t^k - 0)$ and $u^{0k+} = u^0(t^k + 0)$.

PROPOSITION 9 *For any $\lambda \in \Lambda_0$ and $t^k \in \theta$, the derivative of the function $\Delta^k(\bar{H})(t)$ exists at t^k. Denoting this derivative by $D^k(\bar{H})$, we have*

$$D^k(\bar{H}) = \langle \bar{H}_x^{k+}, \bar{H}_\psi^{k-} \rangle - \langle \bar{H}_\psi^{k+}, \bar{H}_x^{k-} \rangle + [\bar{H}_t]^k,$$

where $\bar{H}_x^{k+}$, $\bar{H}_\psi^{k+}$, $\bar{H}_x^{k-}$, $\bar{H}_\psi^{k-}$ are, respectively, the right and left limits of the functions $\bar{H}_x(t, w^0(t), \psi(t), \nu(t), \mu(t))$ and $\bar{H}_\psi = f(t, w^0(t))$ at t^k, whereas $[\bar{H}_t^\lambda]^k$ is the jump of the function $\bar{H}_t(t, w^0(t), \psi(t), \nu(t), \mu(t))$ at t^k.

Note that $D^k(\bar{H})$ is linear in λ.

We have also the following result.

PROPOSITION 10 *Let $\lambda \in M_0$. Then $D^k(\bar{H}) \geq 0 \quad \forall t^k \in \theta$.*

Furthermore, given any $\lambda \in \Lambda_0$ and $t^k \in \theta$, we define

$$[\bar{H}_x]^k = \bar{H}_x^{k+} - \bar{H}_x^{k-},$$

which is the jump of $\bar{H}_x(t, w^0(t), \psi(t), \nu(t), \mu(t))$ at t^k.

We are now ready to define the quadratic form we seek. For any $\lambda \in \Lambda_0$ and $\bar{z} \in Z_2(\theta)$, we set

$$\Omega^\lambda(\bar{z}) = \frac{1}{2} \sum_{k=1}^{s} (D^k(\bar{H}) \bar{\xi}_k^2 - 2\langle [\bar{H}_x]^k, \bar{x}_{\text{av}}^k \rangle \bar{\xi}_k) \\ + \frac{1}{2}\Big(\langle l_{pp}\bar{p}, \bar{p} \rangle - \int_{t_0}^{t_1} \langle \bar{H}_{ww}\bar{w}, \bar{w} \rangle \, dt \Big), \tag{14}$$

where

$$l_{pp} = l_{pp}(p^0, \alpha_0, \alpha, \beta), \;\; \bar{H}_{ww} = \bar{H}_{ww}(t, w^0(t), \psi(t), \nu(t), \mu(t)),$$
$$\bar{p} = (\bar{x}(t_0), \bar{x}(t_1)), \;\; \bar{w} = (\bar{x}, \bar{u}),$$
$$\bar{x}_{\text{av}}^k = \tfrac{1}{2}(\bar{x}^{k-} + \bar{x}^{k+}).$$

The set $\operatorname{Leg}_+(\Lambda_0)$. In the case of a discontinuous control u^0 under consideration, the definition of a strictly Legendrian element includes some additional conditions. Put $w^{0k-} = w^0(t^k - 0)$, $w^{0k+} = w^0(t^k + 0)$.

DEFINITION. An element $\lambda = (\alpha_0, \alpha, \beta, \psi, \nu, \mu) \in \Lambda_0$ is said to be *strictly Legendrian* if the following conditions are satisfied:

(i) $[H]^k = 0$, $D^k(\bar{H}) > 0$ $\forall t^k \in \theta$;

(ii) for any $t \in \Delta \setminus \theta$, the quadratic form (9) is positive definite on the cone formed by the vectors $\bar{u} \in \mathbb{R}^{d(u)}$ satisfying conditions (10);

(iii) for any $t^k \in \theta$, the quadratic form

$$-\langle \bar{H}_{uu}(t^k, w^{0k-}, \psi(t^k), \nu^{k-}, \mu^{k-})\bar{u}, \bar{u}\rangle \tag{15}$$

of the variable $\bar{u}$ is positive definite on the cone formed by the vectors $\bar{u} \in \mathbb{R}^{d(u)}$ such that

$$\begin{aligned} &g_u(t^k, w^{0k-})\bar{u} = 0, \\ &\langle \varphi_{ju}(t^k, w^{0k-}), \bar{u}\rangle \le 0 \quad \forall j \in I_\varphi(t^k, w^{0k-}), \\ &\mu_j^{k-}\langle \varphi_{ju}(t^k, w^{0k-}), \bar{u}\rangle = 0 \quad \forall j \in I_\varphi(t^k, w^{0k-}); \end{aligned} \tag{16}$$

(iv) for any $t^k \in \theta$, the quadratic form

$$-\langle \bar{H}_{uu}(t^k, w^{0k+}\psi(t^k), \nu^{k+}, \mu^{k+})\bar{u}, \bar{u}\rangle \tag{17}$$

of the variable $\bar{u}$ is positive definite on the cone formed by the vectors $\bar{u} \in \mathbb{R}^{d(u)}$ such that

$$\begin{aligned} &g_u(t^k, w^{0k+})\bar{u} = 0, \\ &\langle \varphi_{ju}(t^k, w^{0k+}), \bar{u}\rangle \le 0 \quad \forall j \in I_\varphi(t^k, w^{0k+}), \\ &\mu_j^{k+}\langle \varphi_{ju}(t^k, w^{0k+}), \bar{u}\rangle = 0 \quad \forall j \in I_\varphi(t^k, w^{0k+}). \end{aligned} \tag{18}$$

Note that every $\lambda \in M_0$ is a *nonstrictly Legendrian* element in the sense that

$$[H^\lambda]^k = 0, \quad D^k(\bar{H}^\lambda) \ge 0 \quad \forall t^k \in \theta$$

and that the quadratic forms (9), (15), and (17) in conditions (ii), (iii), and (iv) are positive semidefinite on the cones (10), (16), and (18), respectively. In other words, the following result is true.

PROPOSITION 11 $\operatorname{Leg}(M_0) = M_0$, *where* $\operatorname{Leg}(M)$ *denotes the subset of all non-strictly Legendrian elements of a set* $M \subset \Lambda_0$.

Denote by $\operatorname{Leg}_+(\Lambda_0)$ the set of all strictly Legendrian elements $\lambda \in \Lambda_0$.

Basic sufficient condition for a θ-weak minimum

DEFINITION. We say that the point w^0 *satisfies condition* $\mathcal{B}_0^\theta$ if there exist a nonempty compact set $M \subset \operatorname{Leg}_+(\Lambda_0)$ and a constant $\varepsilon > 0$ such that

$$\max \Omega^\lambda(\bar{z}) \ge \varepsilon \bar{\gamma}(\bar{z}) \quad \forall \bar{z} \in K$$

where

$$\bar{\gamma}(\bar{z}) = \langle \bar{\xi}, \bar{\xi} \rangle + \langle \bar{x}(t_0), \bar{x}(t_0) \rangle + \int_{\Delta} \langle \bar{u}(t), \bar{u}(t) \rangle \, dt. \tag{19}$$

THEOREM 3 *Condition $\mathcal{B}_0^\theta$ is equivalent to a weak γ-sufficiency at the point w^0.*

Remark on necessary conditions. Each sufficient quadratic condition presented in this paper is a natural strengthening of the corresponding necessary one, and the gap between them is minimal (see Osmolovskii, 1986, 1988A, 1988B, 1995). For example, the necessary quadratic condition for a θ-weak minimum can be formulated as follows.

For an admissible point w^0 satisfying the assumptions of this Section, we set $\mathcal{L}_0 = \operatorname{Leg} \Lambda_0$, where $\operatorname{Leg} \Lambda_0$ denotes the subset of all nonstrictly Legendrian elements $\lambda \in \Lambda_0$. Since $\mathcal{L}_0$ is a closed set and Λ_0 is a finite-dimensional compact set, $\mathcal{L}_0$ is also a finite dimensional compact set.

THEOREM 4 *If w^0 is a θ-weak minimum, then $\mathcal{L}_0$ is nonempty and*

$$\max_{\lambda \in \mathcal{L}_0} \Omega^\lambda(\bar{z}) \geq 0 \quad \forall \bar{z} \in \mathcal{K}. \tag{20}$$

(see Osmolovskii, 1995, Theorem 6.1).

Sufficient condition for a strong minimum. Now we need to define the order γ in the entire space $\mathcal{W}$. To this end we assume that there exists a neighborhood $\mathcal{V}$ of the compact set $\overline{u^0}$ and a continuous function

$$\Gamma(t, u) : \mathbb{R}^{1+d(u)} \mapsto \mathbb{R}^1$$

such that the restriction of Γ to $\mathcal{V}$ is an order function satisfying, in addition to conditions (a),(b),(c) of its definition, the following two conditions:

(d) $\Gamma(t, u) > 0$ on $\mathbb{R}^{1+d(u)} \setminus \mathcal{V}$;

(e) $\Gamma(t, u)$ is Lipschitz continous in u on each compact set $\mathcal{F} \subset \mathbb{R}^{1+d(u)} \setminus \mathcal{V}$.

An extension of such an order function to $\mathbb{R}^{1+d(u)}$ satisfying these two conditions will again be called the *order function*.

The following lower bound for the order function Γ is of interest in applications (see Milyutin and Osmolovskii, 1998, Part 2, Proposition 9.3, p. 273).

LEMMA 1 *Let $\mathcal{C} \subset \mathcal{Q}$ be a compact set and let $\delta u \in L^\infty(\Delta, \mathbb{R}^{d(u)})$ be a variation such that $(t, x^0(t), u^0(t) + \delta u(t)) \in \mathcal{C}$ a.e on Δ. Then*

$$\int_{\Delta} \Gamma(t, u^0(t) + \delta u(t)) \, dt \geq \operatorname{const}(\|\delta u\|_1)^2,$$

where $\|\delta u\|_1 = \int |\delta u(t)| \, dt$, and the constant depends only on $\mathcal{C}$ and Γ.

Now we can easily obtain a sufficient condition for a strong minimum by strengthening Condition $\mathcal{B}_0^\theta$. As in the case of continuous control, we need only to strengthen the maximum principle.

Let M_0^+ denote the set of $\lambda \in M_0$ such that

(a) $H(t, x^0(t), u, \psi(t), t) < H(t, x^0(t), u^0(t), \psi(t))$

$$\text{if} \quad t \in \Delta \setminus \theta, \quad u \in U(t, x^0(t)), \quad u \neq u^0(t);$$

(b) $H(t^k, x^0(t^k), u, \psi(t^k)) < H^{k-} = H^{k+}$

$$\text{if} \quad t^k \in \theta, \quad u \in U(t^k, x^0(t^k)), \quad u \notin \{u^{0k-}, u^{0k+}\},$$

where $H^{k-} = H(t^k, x^0(t^k), u^{0k-}, \psi(t^k))$, $H^{k+} = H(t^k, x^0(t^k), u^{0k+}, \psi(t^k))$.

Denote by $\mathrm{Leg}_+(M_0^+)$ the set of all strictly Legendrian elements $\lambda \in M_0^+$.

Definition. We say that an admissible point w^0 *satisfies condition* $\mathcal{B}$ if there exist a nonempty compact set $M \subset \mathrm{Leg}_+(M_0^+)$ and a constant $\varepsilon > 0$ such that

$$\max_{\lambda \in M} \Omega^\lambda(\bar{z}) \geq \varepsilon \bar{\gamma}(\bar{z}) \quad \forall \bar{z} \in \mathcal{K}.$$

Theorem 5 *Condition $\mathcal{B}$ is equivalent to the existence of an order function $\Gamma(t,u) : \mathbb{R}^{1+d(u)} \mapsto \mathbb{R}^1$ such that the bounded-strong γ-sufficiency holds for the corresponding order γ defined by (12).*

If the hypothesis of Proposition 1 holds, then Condition $\mathcal{B}_0^S$ is sufficient for a strict strong minimum.

Sufficient conditions for a strong minimum which do not require the hypothesis of Proposition 1 are presented in Osmolovskii (1995).

Condition $\mathcal{B}$ obviously holds if the set $\mathrm{Leg}_+(M_0^+)$ is nonempty and the cone $\mathcal{K}$ consists only of zero. Therefore, Theorem 5 implies the following result.

Corollary 1 *If the set $\mathrm{Leg}_+(M_0^+)$ is nonempty and $\mathcal{K} = \{0\}$, then w^0 is a strict bounded-strong minimum.*

The hypotheses of Corollary 1 are first-order sufficient conditions for a bounded-strong minimum.

Remarks. Sufficient conditions for a *weak* minimum, as well as necessary conditions for a weak minimum, do not require any new quadratic forms, they are formulated by means of the traditional one. A fairly complete second order theory for a weak minimum was developed by a number of researchers (see, e.g., Dontchev, Hager, Poor and Yang, 1995, Malanowski, 1994, Maurer, 1981, Maurer and Pickenhain, 1995, Pickenhain, 1992, Osmolovskii, 1975, Zeidan, 1983, 1989, 1994). The sufficient second order conditions for a weak minimum can minimum only by strengthening the maximum principle to the "strict maximum principle", but this strengthening automatically implies that the control is *continuous*. Such a strengthening was formulated in Section 2.

The new quadratic form (with additional terms) appears when we consider the second-order conditions for some type of the "local" minimum which is *stronger* than the weak minimum and related to variations of the type of "shifts" of the control discontinuity points. These variations are no longer small in the L^∞-norm, but small in the L^1-norm (or in any other integral norm) and have the character of "needle-shaped" variations, concentrated near the control discontinuity points. The notion of θ-weak minimum, and hence the notions of Pontryagin (see Section 5 for the definition), bounded-strong and strong minima include such type of variations.

The problem of deriving the second-order conditions, which take into account the variations of this type appears already for the broken extremals in the simplest problem of calculus of variations. The complete solution of this problem was given in Milyutin and Osmolovskii (1998, Part 2, Chapters 1 and 2).

In optimal control, the same results (formulated in Section 3) lead, in particular, to the finite-dimensional second-order sufficient conditions for bang-bang control in the problem which is linear in control. This was shown in Milyutin and Osmolovskii (1998, Part 2, Chapter 3).

A number of reseachers developed the theory of sufficient conditions for broken extremals based on the related notions of field of extremals, Hamilton-Jacobi theory and geometrical methods in optimal control (see, e.g., Agrachev, Stefani, Zezza, 2002, Noble and Shattler, 1999, 2002, Nowakowski, 1988, Sarychev, 1997), but this is beyond the scope of our article.

4. The methodology of the proofs

Abstract scheme. The concept of second-order conditions for extremum problems with constraints has a long history, which deserves a special paper. Here, we shall point out only some important facts concerning investigations of higher order conditions in Milyutin's school.

In 1965, there appeared two papers by Dubovitskii and Milyutin, where the concept of the critical variations for problems with constraints was introduced, and a theory of second-order conditions, based on this concept, was developed for a fairy general abstract model. Later it became clear that, in the infinite-dimensional case, this theory concerns mainly necessary conditions, but it is inadequate for the sufficient ones. This situation stimulated the interest of Milyutin and his colleagues in the problem of higher order conditions, both necessary and sufficient, for an acceptably general abstract model, since it was not clear what could replace the concept of critical variation.

Milyutin, Levitin and Osmolovskii (1974, 1978) developed an abstract the-of an arbitrary order. The very concept of an order acquired a new meaning in this theory, namely that of a nonnegative functional in the space of variations. The choice of the order in a given class of problems and for a given type of trajectory is a key point in this approach. As soon as the order is chosen, the scheme of obtaining conditions of this order becomes strictly defined, and the difficulties of implementing it depend on both the class of the problems and the type of the trajectory under consideration. In applications to calculus of variations and to finite-dimensional problems, the theory led to all the classical results and also to some new ones, as e.g., in the mathematical programming (see Levitin, Milyutin, Osmolovskii, 1974, 1978). In optimal control, the theory made it possible to obtain new quadratic conditions for different type of minima, both necessary and sufficient for trajectories with nonsingular discontinuous controls (Osmolovskii, 1988, 1995) and for trajectories with singular controls (Dmitruk, 1987, 1999).

About the proofs. The complete proofs of the sufficient quadratic conditions for discontinuous controls, formulated in Section 3 for optimal control problem (1), are presented (together with the proofs of the corresponding necessary quadratic conditions) in Osmolovskii (1996, 1988A). For wider availability, the author plans to publish them in a separate book. For the case of broken externals in the problem of calculus of variations the proofs are given in the monograph by Milyutin and Osmolovskii (1998, Part 2, Chapter 2).

Here, for general problem (1), we shall give only some explanations of the methodology of the proofs of the results stated in Section 3 (which are more general than those in Section 2).

The central and most nontrivial part of the proofs concerns the conditions for the so-called Pontryagin minimum, or minimum on the set of Pontryagin's sequences. Below we shall give the definition of Pontryagin minimum, introduced by Milyutin in Levitin, Milyutin, and Osmolovskii (1978).

Let Π be the set of sequences of variations $\delta w_n = (\delta x_n, \delta u_n)$ in the space $\mathcal{W}$, which *converge to zero in Pontryagin's sence*, i.e. such that the following conditions hold:

(a) $\max\limits_{t\in\Delta} |\delta x_n(t)| \to 0$;

(b) there exists a compact set $\mathcal{C} \in \mathcal{Q}$ (depending on the sequence) such that $(t, x^0(t) + \delta x_n(t), u^0(t) + \delta u_n(t)) \in \mathcal{C}$ a.e. on Δ;

(c) $\int\limits_{\Delta} |\delta u_n(t)|\, dt \to 0$.

Note that this "convergence" is stronger than L^1-convergence of variations of the control defined in item (c), since there is an important condition (b) which guarantees a uniform boundedness in the L^∞-norm of all terms of the sequence $\{\delta w_n\}$. This type of "convergence" is not related to any norm or topology (the second closure of a set in the sense of this convergence is not identical with the a local minimum in the usual sence. But, in spite of this "strange property", there is a rich theory of both necessary and sufficient conditions of the first and higher orders for the Pontryagin minimum introduced below.

DEFINITION. We say that w^0 is a *minimum on* Π, or a *Pontryagin minimum*, if there is no sequence $\{\delta w_n\} \in \Pi$ such that all its terms satisfy the conditions

$$\begin{aligned} &J(p^0+\delta p_n) < J(p^0), \quad F(p^0+\delta p_n) \le 0, \quad K(p^0+\delta p_n)=0, \\ &\dot{x}+\delta\dot{x}_n = f(t, w^0+\delta w_n), \quad g(t, w^0+\delta w_n)=0, \quad \varphi(t, w^0+\delta w_n) \le 0, \\ &(p^0+\delta p_n) \in \mathcal{P}, \quad w^0+\delta w_n \in \mathcal{Q}, \end{aligned}$$

where $\delta p_n = (\delta x_n(t_0), \delta x_n(t_1))$, $\delta w_n = (\delta x_n, \delta u_n)$.

Let $\Gamma(t,u)$ be an order function and $\gamma(\delta w)$ be the corresponding order (defined by (12)).

DEFINITION. We say that w^0 is a point γ-sufficiency on Π, or Pontryagin γ-sufficiency, if there exists an $\varepsilon > 0$ such that, for any sequence $\{\delta w_n\} \in \Pi$, we have $\sigma(\delta w_n) \ge \varepsilon\gamma(\delta w_n)$ for all sufficiently large n.

Obviously, γ-sufficiency on Π implies minimum on Π.

Similarly, we define minimum and γ-sufficiency on an arbitrary set of sequences S, which is closed in the operation of taking subsequences. For example, the weak minimum is a minimum on the set Π_0 of all sequences $\{\delta w_n\}$ in $\mathcal{W}$ such that $\|\delta w_n\|_\infty \to 0$. Obviously, $\Pi_0 \subset \Pi$, hence every Pontryagin minimum is a weak minimum.

The derivations of both necessary and sufficient quadratic conditions for Pontryagin minimum are based on the abstract theory of higher order conditions for nonsmooth problems with constraints initiated in Levitin, Milyutin, and Osmolovskii (1974, 1978), and then developed in Milyutin and Osmolovskii (1993). For higher order γ, the general theory yields the existence of a constant C_γ such that $C_\gamma \ge 0$ and $C_\gamma > 0$ are a necessary and a sufficient condition, respectively, for a minimum in the class of sequences at hand. We will demonstrate the form of the constant C_γ in the canonical optimal control problem on the set of Pontryagin's sequences.

We divide the system of constraints of the canonical problem into two subsystems

(a) $F(p) \le 0, \quad K(p)=0, \quad \dot{x}=f(w,t)$;

(b) $g(w,t)=0, \quad \varphi(w,t) \le 0$.

In the sequel, constraints of subsystem (b) are satisfied by the sequences of variations, while the functions F and K are included into the Lagrange function together with the cost function $J(p)$. For this reason, subsystems (a) and (b) are called *free* and *nonfree*, respectively. All functions and sets asssociated with the two subsystems will be identified with the superscript "sb" (for "subsystem"). Set where $\delta_n g = g(t, w^0 + \delta w_n) - g(t, w^0)$, $\delta_n \varphi = \varphi(t, w^0 + \delta w_n) - \varphi(t, w^0)$, $\varphi^0 = \varphi(t, w^0)$. Define the Lagrange function for the free subsystem of constraints

$$\Psi^{sb,\lambda}(\delta w) = \alpha_0 \delta J + \alpha \delta F + \beta \delta K + \int \psi(\delta \dot{x} - \delta f)\, dt$$
$$= \delta l^\lambda + \int \psi(\delta \dot{x} - \delta f) \delta t,$$

where $\delta J = J(p^0 + \delta p) - J(p^0)$, $\delta f = f(t, w^0 + \delta w) - f(t, w^0)$, etc. Put

$$\Psi_0^{sb}(\delta w) = \max_{\lambda \in \Lambda_0} \Psi^{sb,\lambda}(\delta w).$$

Define the set of sequences

$$\Pi_{\sigma\gamma}^{sb} = \{\{\delta w_n\} \in \Pi^{sb} \mid \sigma(\delta w_n) \leq O(\gamma(\delta w_n)\}.$$

Let

$$C_\gamma^{sb} = \inf_{\{\delta w_n\} \in \Pi_{\sigma\gamma}^{sb}} \liminf_n \frac{\Psi_0^{sb}(\delta w_n)}{\gamma(\delta w_n)}.$$

The abstract theory, developed in Milyutin and Osmolovskii (1993), can be applied to the subsystem of constraints in the problem of minimization on a set of the above sequences. It yields the following theorem:

THEOREM 6 (i) *Condition $C_\gamma^{sb} \geq 0$ is necessary for a Pontryagin minimum at the point w^0.*
(ii) *Condition $C_\gamma^{sb} > 0$ is equivalent to Pontryagin's γ-sufficiency at the point w^0.*

Then, we are faced with the problem of "decoding" the constant C_γ^{sb}, which turns out to be a very difficult one. (One can get some ideas about this from the investigation of the problem of calculus of variations in Milyutin and Osmolovskii, 1998.) As a result of decoding C_γ^{sb} for the problem (1), the following theorem is established.

THEOREM 7 *Condition $\mathcal{B}$ of Section 3 is equivalent to the existence of an order function $\Gamma(t, u)$ such that, for the corresponding order γ defined by formula (12) the inequality $C_\gamma^{sb} > 0$ holds.*

The proof of this theorem utilizes, in particular, a generalization of Hoffman's lemma to the case where the system of linear inequalities is considered on a convex cone (see Osmolovskii, 1977).

Theorems 6 and 7 imply that Condition $\mathcal{B}$ of Section 3 is equivalent to the existence of an order function $\Gamma(t, u)$ such that, for the corresponding order γ defined by formula (12), a Pontryagin's γ-sufficiency holds. Since, under Condition $\mathcal{B}$, Pontryagin's γ-sufficiency is equivalent to bounded-strong γ-sufficiency

Moreover, it is easy to see that a point w^0 is a θ-weak minimum in the canonical problem (1) if and only if there exists an open set $V \in \mathbb{R}^{1+d(u)}$ such that w^0 is a Pontryagin minimum for the same problem under the additional constraint $(t, u) \in V$. Therefore, conditions for a θ-weak minimum can be derived from the conditions for a Pontryagin minimum.

5. Two examples

An example of the maximum of quadratic forms being positive without any single one being so. The following important example belongs to A.A. Milyutin (see Levitin, Milyutin, and Osmolovskii, 1985). Define four functions in $\mathbb{R}^3$

$$\varphi_1(x) = x_1^2 + 2x_2x_3,\ \varphi_2(x) = x_2^2 + 2x_1x_3,\ \varphi_3(x) = x_3^2 + 2x_1x_2,$$
$$\psi(x) = x_1 + x_2 + x_3,$$

where $x = (x_1, x_2, x_3)$. Consider the problem

$$\varphi_1(x) \to \min, \quad \varphi_2(x) \le 3, \quad \varphi_3(x) \le 3, \quad \psi(x) = 1.$$

This problem can be viewed as a special case of problem (1). The Lagrange function here takes the form

$$L(x, \lambda) = \sum_{i=1}^{3} \alpha_i \varphi_i(x) + \beta\psi(x),$$

where $\lambda = (\alpha, \beta)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$, $\beta \in \mathbb{R}$.

The point $x^0 = (1, 1, 1)$ is admissible in this problem. Let us show that, at this point, the set Λ_0 of normed collections of Lagrange multipliers is nonempty, the critical cone $\mathcal{K}$ is a subspace, and the maximum of quadratic forms

$$\frac{1}{2}\langle L_{xx}(x^0, \alpha, \beta)\bar{x}, \bar{x}\rangle = \sum_{i=1}^{3} \alpha_i \varphi_i(\bar{x})$$

over the compact set Λ_0 is positive for each nonzero element $\bar{x} \in \mathcal{K}$, i.e. that the second-order sufficient condition is satisfied. At the same time, none of the quadratic forms which correspond to λ in Λ_0 is positive-definite on $\mathcal{K}$. Moreover, there exists one zero form on $\mathcal{K}$ and the rest of the forms have alternating signs on $\mathcal{K}$.

Define the set Λ_0 and the critical cone $\mathcal{K}$ at the point x^0. Since $\varphi_i(x^0) = 3$, $\varphi_i'(x^0) = (2, 2, 2)$, $i = 1, 2, 3$ and $\psi'(x^0) = (1, 1, 1)$, then

$$\Lambda_0 = \{\lambda = (\alpha, \beta) \in \mathbb{R}^4 \mid \alpha_1 \ge 0,\ \alpha_2 \ge 0,\ \alpha_3 \ge 0,$$

and $\mathcal{K}$ is a subspace defined by

$$\mathcal{K} = \{\bar{x} \in \mathbb{R}^3 \mid \bar{x}_1 + \bar{x}_2 + \bar{x}_3 = 0\}.$$

The projection of Λ_0 under the mapping $(\alpha, \beta) \mapsto \alpha$ is the standard simplex in $\mathbb{R}^3$. Consequently,

$$\Omega(\bar{x}) = \max_{\lambda \in \Lambda_0} \sum_{i=1}^{3} \alpha_i \varphi_i(\bar{x}) = \max_{1 \le i \le 3} \varphi_i(\bar{x}).$$

Let us show that $\Omega(\cdot)$ is positive on $\mathcal{K} \setminus \{0\}$. Indeed, if $\bar{x} \in \mathcal{K}$, then, obviously,

$$\sum_{i=1}^{3} \varphi_i(\bar{x}) = \Big(\sum_{i=1}^{3} \bar{x}_i\Big)^2 = 0.$$

Consequently, if $\bar{x} \in \mathcal{K}$ and $\sum_{i=1}^{3} |\varphi_i(\bar{x})| > 0$, then $\Omega(\bar{x}) = \max_{1 \le i \le 3} \varphi_i(\bar{x}) > 0$. Hence, it is sufficient to show that, if $\bar{x} \in \mathcal{K} \setminus \{0\}$, then $\sum_{i=1}^{3} |\varphi_i(\bar{x})| > 0$. Thus, we must show that $x_1 = x_2 = x_3 = 0$ is the unique solution of the system

$$x_1^2 + 2x_2x_3 = 0, \quad x_2^2 + 2x_1x_3 = 0, \quad x_3^2 + 2x_1x_2 = 0, \quad x_1 + x_2 + x_3 = 0.$$

In fact, by substracting the second equation from the first and taking the last into account, we obtain $x_1^2 - x_2^2 = 0$. Similarly, $x_2^2 - x_3^2 = 0$, $x_1^2 - x_3^2 = 0$. Thus, $|x_1| = |x_2| = |x_3|$. But, if x satisfies these conditions and $x_1 + x_2 + x_3 = 0$, then, obviously, $x = 0$. Hence, $\Omega(\cdot)$ is positive on $\mathcal{K} \setminus \{0\}$.

Now let us show that none of the elements $\lambda \in \Lambda_0$ generates a positive-definite quadratic form on $\mathcal{K}$, and, moreover, that each element, except for $\widehat{\lambda} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -2)$, generates the quadratic form with alternating signs on $\mathcal{K}$, while $\widehat{\lambda}$ generates the zero form. Indeed, if $\lambda \in \Lambda_0$, $\bar{x} \in \mathcal{K}$, then $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and $\bar{x}_3 = -(\bar{x}_1 + \bar{x}_2)$, whence it easily follows that

$$\sum_{i=1}^{3} \alpha_i \varphi_i(\bar{x}) = p\bar{x}_1^2 + q\bar{x}_2^2 - 2r\bar{x}_1\bar{x}_2,$$

where $p = 1 - 3\alpha_1$, $q = 1 - 3\alpha_2$, $r = 1 - 3\alpha_3$, and therefore $p + q + r = 0$. The determinant of the matrix of this quadratic form is equal to $\Delta = pq - r^2 = pq - (p+q)^2 = -(p^2 + q^2 + pq)$, so that $\Delta < 0$ if $|p| + |q| > 0$. And if $|p| + |q| = 0$, then $p = q = r = 0$, i.e. $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}$. In the last case we get zero form on $\mathcal{K}$.

Thus, at the point x^0, no element $\lambda \in \Lambda_0$ generates a positive-definite quadratic form on $\mathcal{K}$, while the maximum over Λ_0 of the forms is positive on $\mathcal{K} \setminus \{0\}$. Hence γ-sufficiency holds with $\gamma = |\delta x|^2$.

A simple illustrative example of analyzing extremals with control discontinuities. Consider the problem

$$\mathcal{J}(x, u) = \int^{1} x^2 \, dt \to \max, \quad \dot{x} = u, \; x(0) = x(1) = 0, \; |u| = 1. \tag{21}$$

This is a slightly modified problem from the book by Milyutin and Osmolovskii (1998, see part 2, Section 15, p.341), where the constraint $|u| \le 1$ is replaced by $|u| = 1$.

The canonical form of this problem is as follows:

$$y_1 \to \min, \quad y_0 = 0, \quad x_0 = 0, \quad x_1 = 0,$$
$$\dot{y} = -\frac{1}{2}x^2, \quad \dot{x} = u, \quad \frac{1}{2}(u^2 - 1) = 0, \tag{22}$$

where $y_0 = y(0),\ y_1 = y(1),\ x_0 = x(0),\ x_1 = x(1)$.

The functions l, H and $\bar{H}$ are

$$l = \alpha_0 y_1 + \beta_{y_0} y_0 + \beta_{x_0} x_0 + \beta_{x_1} x_1, \quad H = -\frac{1}{2}\psi_y x^2 + \psi_x u,$$
$$\bar{H} = H - \frac{\nu}{2}(u^2 - 1),$$

where $\alpha_0 \ge 0$, and the adjoint system is

$$\dot{\psi_x} = \psi_y x, \quad \dot{\psi_y} = 0.$$

It follows from the maximum principle that $u = \operatorname{sign} \psi_x$, and from the transversality condition we obtain $\psi_y(1) = -\alpha_0$. Clearly $\alpha_0 > 0$, so we can put $\alpha_0 = 1$. Thus, the extremality conditions become

$$\dot{\psi_x} = -x, \quad \dot{x} = u, \quad u = -\operatorname{sign} \psi_x.$$

It follows that ψ_x is continuously differentiable function, whose graph consists of parabolas of the form

$$\psi_x = -\frac{1}{2}t^2 + bt + c, \quad \psi_x = \frac{1}{2}t^2 + bt + c,$$

which have a common tangent line lying on the t-axis; these points correspond to the control switching times. There are countably many such extremals. For each of them, $\Lambda_0 = \Lambda_0^\theta = M_0$ is a singleton (see Sections 2 and 3 for the definitions).

Let us fix an arbitrary extremal $(x, y, u, \psi_x, \psi_y)$ and write down for it the conditions, which determine the critical cone $\mathcal{K}$. By Proposition 7,

$$\mathcal{K} = \{\bar{z} = (\bar{\xi}, \bar{x}, \bar{y}, \bar{u}) \mid \bar{u} = 0, \quad \bar{x}(0) = \bar{x}(1) = 0, \quad \dot{\bar{y}} = -x\bar{x},$$
$$\bar{y}(0) = 0,\ [\bar{y}]^k = 0\ \forall k,\ \dot{\bar{x}} = 0,\ [\bar{x}]^k = [u]^k \bar{\xi}_k\ \forall k\}.$$

Now we write down the quadratic form Ω for this extremal. In this case $\bar{H}_x = x$ and $\bar{H}_y = 0$, hence $[\bar{H}_x]^k = [\bar{H}_y]^k = 0$ for all k. Moreover, $l_{pp} = 0$, and, for the elements of the critical cone, we have $\langle \bar{H}_{ww}\bar{w}, \bar{w}\rangle = H_{xx}\bar{x}^2 = \bar{x}^2$. Let us calculate $D^k(\bar{H})$. By the definitions, $\Delta^k \bar{H}(t) = \psi_x(t)[u]^k$. Hence $D^k(\bar{H}) = \dot{\psi_x}(t^k)[u]^k = -x(t^k)[u]^k$. For an extremal with s switchings the

Also, we have $|[u]^k| = 2$. Consequently, $D^k(\bar{H}) = 1/s$. Therefore, according to (14),

$$\Omega = \frac{1}{s}\sum_{k=1}^{s}\overline{\xi}_k^2 - \int_0^1 \bar{x}^2\,dt.$$

Since this expression does not involve $\bar{y}$, we are to determine the sign of Ω on the subspace

$$L = \{(\overline{\xi}, \bar{x}) \mid \dot{\bar{x}} = 0, \quad [\bar{x}]^k = [u]^k\overline{\xi}_k \ \forall k, \quad \bar{x}(0) = \bar{x}(1) = 0\}.$$

Put $\eta_k = [\bar{x}]^k \ \forall k$. Then $\eta_k^2 = ([\bar{x}]^k)^2 = 4\overline{\xi}_k^2 \ \forall k$. Moreover, since $t^{k+1} - t^k = 1/s,\ k = 1, \dots, s-1$, we have for elements of the subspace L

$$\int_0^1 \bar{x}^2\,dt = \frac{1}{s}(\eta_1^2 + (\eta_1 + \eta_2)^2 + \cdots + (\eta_1 + \cdots + \eta_{s-1})^2).$$

Here L is specified by the conditions

$$\eta_1 + \cdots + \eta_s = 0, \tag{23}$$

and Ω on L has the form

$$\Omega = \frac{1}{4s}\sum_{k=1}^{s}\eta_k^2 - \frac{1}{s}\sum_{k=1}^{s-1}(\eta_1 + \cdots + \eta_k)^2. \tag{24}$$

We must determine the sign of the quadratic form (24) on the subspace (23). If $s \geq 2$, then by putting $\eta_1 = -\eta_2 = \zeta$ and $\eta_i = 0$ for $i \geq 2$, we obtain $\Omega = -2\zeta^2/(4s)$, i.e., the quadratic form (24) is not nonnegative on the subspace (23). Then, according to Theorem 4 an extremal with $s \geq 2$ switchings does not yield a θ-weak minimum in the problem.

For $s = 1$, the cone $\mathcal{K}$ consists only of the origin and the single element of the set Λ_0 is strictly Legendrian, hence the set $\mathrm{Leg}_+(M_0)$ is not empty. (Although $\bar{H}_{uu}$ is identically zero, the strengthened Legendre condition is trivially satisfied, because, at each point $t \in [0,1] \setminus \{t^1\}$ the "cone of critical directions" $\{\bar{u} \in \mathbb{R} \mid u(t)\bar{u} = 0\}$ is equal to $\{0\}$, since $u(t) \neq 0$. For $t = t^1$, the same is true for the "cones" $\{\bar{u} \in \mathbb{R} \mid u(t^1 - 0)\bar{u} = 0\}$ and $\{u(t^1 + 0)\bar{u} = 0\}$, since $u(t^1 - 0) \neq 0$ and $u(t^1 + 0) \neq 0$). Therefore, according to Corollary 1, the extremal with a single switching yields a strict strong minimum in problem (22). Hence, it yields a strict strong maximum in problem (21) (since the component y is unessential, see Section 1 for the definition). Actually, this extremal yields the absolute maximum in the problem.

Moreover, according to Theorem 5, we have for some $\varepsilon > 0$ the following growth estimate for the cost functional in problem (21): $-\delta\mathcal{J} \geq \varepsilon\gamma(\delta w)$ for all

$\gamma(\delta w) \geq \text{const}(\|\delta u\|_1)^2$, we also have the estimate $-\delta \mathcal{J} \geq c(\|\delta u\|_1)^2$ with some $c > 0$ for all admissible variations $\delta w = (\delta x, \delta u)$ with $\max|\delta x(t)| < \varepsilon$.

Some interesting examples of applying the quadratic conditions to the analysis of extremals with jumps of the control are presented in Dmitruk and Osmolovskii (1992), Osmolovskii (1994, 1998), Milyutin and Osmolovskii (1998, Part 2, Chapter 4).

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