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Boundary optimal control problem for the phase-field transition system using fractional steps method

by

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Abstract: In this paper we prove the convergence of an iterative scheme of fractional steps type for boundary optimal control problem which is governed by the phase–field transition system. The existence of an optimal control and necessary optimality conditions are given for approximating problem. A gradient type algorithm and numerical implementation of this algorithm is discussed.

Keywords. phase-field transition system, free boundary problem, optimal control, fractional steps method, optimality conditions, finite element method, gradient method.

1. Introduction

Consider the following problem:

(P_0) Minimize $\int_Q \chi_0(\varphi(t,x) - 1)^2 dxdt + \int_0^T w^2(t)dt$, on all $(u, \varphi, w, v) \in (C(0,T; H^1(\Omega)))^2 \times W^{1,\infty}(0,T) \times \mathcal{U}$ subject to

$$\begin{aligned} & u_t + \frac{\ell}{2}\varphi_t = k\Delta u && \text{in } Q = [0,T] \times \Omega, \\ & \frac{\partial u}{\partial \nu} + hu = w(t)g(x) && \text{in } \Sigma = [0,T] \times \partial\Omega, \\ & u(0,x) = u_0(x) && x \in \Omega, \end{aligned} \tag{1.1}$$

$$\begin{aligned} & w'(t) = \beta w(t) + v(t) && t \in [0,T], \\ & w(0) = 0, \end{aligned} \tag{1.2}$$

$$\begin{aligned} & \tau\varphi_t = \xi^2\Delta\varphi + \frac{1}{2a}(\varphi - \varphi^3) + 2u && \text{in } Q \\ & \varphi = 1 && \text{in } \Sigma, \\ & \varphi(0,x) = \varphi_0(x) && x \in \Omega. \end{aligned} \tag{1.3}$$

Here Ω is a bounded domain in R^n $(n = 1, 2, 3)$ with a sufficiently smooth boundary $\partial\Omega$, u is the reduced temperature, φ is the phase function used to of the surrounding at $\partial\Omega$ and it is manipulated by a heating (cooling) system according to the equation (1.2), $v \in \mathcal{U}$ (the boundary control), with

$$\mathcal{U} = \{v \in L^\infty(0,T), \quad 0 \le v(t) \le R, \quad \text{a.e. } t \in [0,T]\},$$

$g \in H^1(\partial\Omega)$ is a given function, χ_0 is the characteristic function of Ω_0, $\Omega_0 \subset \Omega$.

At the moment t the material is liquid if φ is close to $+1$ and $u(t,x) \ge \delta_1$ and it is solid if φ is close to -1 and $u(t,x) \le -\delta_1$, with $\delta_1 \ge 0$. We define as *interface at the moment* t (or simply *interface*) the set $\{x \in \Omega,\ |u(t,x)| \le \delta_1,\ |\varphi(t,x)| \le 1-\delta_2,\ \delta_2 \ge 0\}$.

This model, introduced by Caginalp (1986), has been established in the literature as an extension of the classical two phase Stefan problem to capture the effects of surface tension, supercooling, and superheating. For detailed discussions of the phase field transition system we refer to Caginalp (1986) and Fix (1982). The positive parameters $\tau,\ \xi,\ \ell,\ k,\ h$, are constants (see Caginalp, 1986); a depends on ξ.

The distributed optimal control problem governed by the phase field equation has been analyzed in Chen and Hoffmann (1991), Hoffmann and Jiang (1992), Heinkenschloss and Sachs (1994) and Heinkenschloss and Tröltzsch (1995). The numerical approach of the optimal control problem stated here, associated with an inverse problem, has been investigated in Moroşanu (1993). The method stated in the present paper can also be applied to the case of distributed optimal control problem (see the phase-field system considered in Moroşanu, 1997).

It is more convenient to reformulate problem (P_0) as

(P) Minimize $\int_Q \chi_0 \varphi^2(t,x)\,dx\,dt + \int_0^T w^2(t)\,dt$, subject to (1.1)–(1.2) and

$$\begin{aligned} &\tau\varphi_t - \xi^2\Delta\varphi = 2u + \frac{1}{2a}\varphi - \frac{1}{2a}(\varphi+1)^3 + \frac{1}{2a} && \text{in } Q \\ &\varphi = 0 && \text{in } \Sigma, \\ &\varphi(0,x) = \varphi_0(x) - 1 && x \in \Omega. \end{aligned} \tag{1.4}$$

For every $\varepsilon > 0$, we associate with the system (1.1), (1.2), (1.4) the following approximating scheme:

$$\begin{aligned} &u_t^\varepsilon + \frac{\ell}{2}\varphi_t^\varepsilon - k\Delta u^\varepsilon = 0 \quad \text{in } Q_i^\varepsilon = (i\varepsilon,(i+1)\varepsilon)\times\Omega, \\ &\frac{\partial u^\varepsilon}{\partial\nu} + hu^\varepsilon = w(t)g(x) \quad \text{on } \Sigma_i^\varepsilon = (i\varepsilon,(i+1)\varepsilon)\times\partial\Omega, \\ &u^\varepsilon(0,x) = u_0(x) \qquad x \in \Omega, \end{aligned} \tag{1.5}$$

$$\begin{aligned} &w'(t) = \beta w(t) + v(t) \quad t \in [0,T], \\ &w(0) = 0, \end{aligned} \tag{1.6}$$

$$\begin{aligned} &\tau\varphi_t^\varepsilon - \xi^2\Delta\varphi^\varepsilon = \frac{1}{2a}\varphi^\varepsilon + 2u^\varepsilon + \frac{1}{2a} \quad \text{in } Q_i^\varepsilon, \\ &\varphi^\varepsilon = 0 \qquad \text{on } \Sigma_i^\varepsilon, \end{aligned} \tag{1.7}$$

where $z(\cdot,\varphi_-^\varepsilon(i\varepsilon,x))$ is the solution of

$$\begin{aligned} &z'(s)+\frac{1}{2a}(z(s)+1)^3=0 \quad s\in[0,T],\\ &z(0)=\varphi_-^\varepsilon(i\varepsilon,x), \quad \varphi_-^\varepsilon(0,x)=\varphi_0(x)-1, \end{aligned} \tag{1.8}$$

computed at $s=\varepsilon$, for $i=\overline{0,M_\varepsilon-1}$, with $M_\varepsilon=[\frac{T}{\varepsilon}]$ and $Q^\varepsilon_{M_\varepsilon-1}=((M_\varepsilon-1)\varepsilon,T)\times\Omega$. Here $\varphi_+^\varepsilon(i\varepsilon)=\lim_{t\downarrow i\varepsilon}\varphi^\varepsilon(t)$, $\varphi_-^\varepsilon(i\varepsilon)=\lim_{t\uparrow i\varepsilon}\varphi^\varepsilon(t)$.

Due to the form of boundary condition (1.1_2), we cannot set the phase-field system (1.1)–(1.3) into the abstract framework and so, we cannot treat the convergence of this numerical scheme on the basis of the abstract approximation results known in mathematical literature. On the other hand, this particular form is essentially used to obtain the estimates (2.4), (3.22) and (3.27).

Corresponding to the approximating scheme (1.5)–(1.8), we consider the approximating optimal control problem:

(P^ε) Minimize $\int_Q \chi_0(\varphi^\varepsilon(t,x))^2dxdt+\int_0^T w^2(t)dt$, on all $(u^\varepsilon,\varphi^\varepsilon,w,v)$ subject to (1.5)–(1.8).

The main result of this paper amounts to saying that problem (P) can be approximated for $\varepsilon\to 0$ by the sequence of problems (P^ε). The convergence of the approximating process leads to an idea of numerical approximation of the optimal control of problem (P), namely (see algorithm **CPHT-2D**, step **P2**, Section 5), at every iteration *iter*, the computation of the approximate solution corresponding to the nonlinear phase-field transition system is substituted by computation of the approximate solution for an ordinary equation and a linear system. Hence a large amount of time is saved concerning computations.

Such a convergence scheme was studied for an optimal control problem governed by nonlinear parabilic variational inequalities by Barbu (1988). For other works in this context see Aniţa (1988), Barbu (1984), Popa (1995), for example.

The plan of this work is the following: In Section 2 we shall prove the existence of an optimal control in problem (P). The convergence of the optimal solution of problem (P^ε) to the optimal solution of problem (P), as $\varepsilon\to 0$, is derived in Section 3. Besides the existence of an optimal control in problem (P^ε), the necessary optimality conditions for this problem will be proved in Section 4. A conceptual algorithm of gradient type for the calculation of the approximating optimal control of problem (P^ε) and a numerical result are presented in Section 5.

We shall use the standard notation for the Sobolev spaces on Ω and Q.

2. The optimal control problem (P)

In this Section we will give an existence result for problem (P). First of all we recall the notion of weak solution for (1.1)–(1.2) and (1.4).

DEFINITION 2.1 *By weak solution (u,φ) to (1.1)–(1.2) and (1.4) we mean a*

$$\leq \left[\int_Q (\varphi_n - \varphi^*)^6 dxdt\right]^{1/3} \left[\int_Q |\varphi_n^2 + \varphi_n\varphi^* + (\varphi^*)^2|^2 dxdt\right]^{2/3}$$
$$\leq C\int_0^T \|\varphi_n - \varphi^*\|^2_{L^6(\Omega)} dt.$$

This, combined with (2.8) and Sobolev's imbedding theorem indicates that

$$\varphi_n^3 \to (\varphi^*)^3 \quad \text{strongly in } L^2(0,T;L^2(\Omega)).$$

So, letting n tend to $+\infty$ in (2.7), we get

$$\begin{cases} \dfrac{\partial u^*}{\partial t} + \dfrac{\ell}{2}\dfrac{\partial \varphi^*}{\partial t} = k\Delta u^* & \text{in } Q, \\ \tau\dfrac{\partial \varphi^*}{\partial t} = \xi^2\Delta\varphi^* + \dfrac{1}{2a}\varphi^* - \dfrac{1}{2a}(\varphi^*+1)^3 + 2u^* + \dfrac{1}{2a} & \text{in } Q, \\ \dfrac{\partial u^*}{\partial \nu} + hu^* = w^*(t)g(x), \quad \varphi^* = 0 & \text{in } \Sigma, \\ u^*(0,x) = u_0(x), \quad \varphi^*(0,x) = \varphi_0(x) - 1 & \text{in } \Omega, \\ (w^*)'(t) = \beta w^*(t) + v^*(t) \quad t\in[0,T], & \\ w^*(0) = 0. & \end{cases} \tag{2.9}$$

Thus, the uniqueness of solution for (2.9) implies that (u^*, φ^*) is the solution of problem (1.1)–(1.2) and (1.4), corresponding to $v^* \in \mathcal{U}$.

Since j is continuous, then by (2.6), we see that $d = j(v^*)$ and the proof is completed.

3. A convergence result

The aim of this section is to establish a convergence result for the sequence of optimal solutions for problems (P^ε), when ε converges to 0. We set

$$j^\varepsilon(v) = \int_Q \chi_0(\varphi_\varepsilon^v(t,x))^2 dxdt + \int_0^T w^2(t)dt, \tag{3.1}$$

where φ_ε^v is the solution to (1.5)–(1.8) corresponding to $v \in \mathcal{U}$. Then (see (2.5)), we may rewrite (P) and (P^ε) as

$(P) \quad \min\{j(v),\ v\in\mathcal{U}\}$,

$(P^\varepsilon) \quad \min\{j^\varepsilon(v),\ v\in\mathcal{U}\}$.

Now we come back to the iterative scheme (1.5)–(1.8) and note that if $u_0 \in H^1(\Omega)$, $\varphi_0 \in H_0^1(\Omega)$ satisfy the compatibility conditions, then for every $\varepsilon > 0$ this problem has a unique solution $(u^\varepsilon, \varphi^\varepsilon) \in (W_p^{2,1}(Q_i^\varepsilon) \cap L^\infty(Q_i^\varepsilon))^2$ on every interval $[i\varepsilon, (i+1)\varepsilon]$, $i = 0, 1, \dots, M_\varepsilon - 1$, $p > \frac{n+2}{2}$ (see Moroşanu, 1997). As in

PROPOSITION 3.1 *For $u_0 \in H^1(\Omega)$, $\varphi_0 \in H_0^1(\Omega)$ (as in Proposition 2.1) problem (P^ε) has at least one solution $(u_\varepsilon^*, \varphi_\varepsilon^*, w_\varepsilon^*, v_\varepsilon^*)$.*

The main result of this paper is

THEOREM 3.1 *Let $\{v_\varepsilon^*\}$ be a sequence of optimal controllers for problems (P^ε). Then*

$$\lim_{\varepsilon\to 0}(\inf j^\varepsilon(v)) = \inf\{j(v); v \in \mathcal{U}\} \text{ and } \tag{3.2}$$

$$\lim_{\varepsilon\to 0} j(v_\varepsilon^*) = \inf\{j(v); v \in \mathcal{U}\}. \tag{3.3}$$

Moreover, every weak limit point of $\{v_\varepsilon^*\}$ is an optimal controller for problem (P).

Theorem 3.1 amounts to saying that (P^ε) approximates problem (P) and an optimal controller $\{v_\varepsilon^*\}$ of (P^ε) is a suboptimal controller for problem (P).

The main ingredient of the proof of Theorem 3.1 is the following lemma:

LEMMA 3.1 *If $\{v_\varepsilon^*\}$ is a sequence of optimal controllers for problems (P^ε) then there exists $\{\varepsilon_n\} \to 0$ such that*

$$v_{\varepsilon_n}^* \to v^* \quad \text{weak star in } L^\infty(0,T), \tag{3.4}$$

$$w_{\varepsilon_n}^* \to w^* \quad \text{strongly in } C[0,T], \tag{3.5}$$

$$(w_{\varepsilon_n}^*)' \to (w^*)' \quad \text{weak star in } L^\infty(0,T), \tag{3.6}$$

$$\varphi_{\varepsilon_n}^*(t) \to \varphi^*(t) \quad \text{strongly in } L^2(\Omega), \text{ for any } t \in [0,T], \tag{3.7}$$

$$u_{\varepsilon_n}^* \to u^* \quad \text{strongly in } L^2(0,T,H^1(\Omega)), \tag{3.8}$$

where $(u_{\varepsilon_n}^*, \varphi_{\varepsilon_n}^*, w_{\varepsilon_n}^*) = (u_{\varepsilon_n}^{v_{\varepsilon_n}^*}, \varphi_{\varepsilon_n}^{v_{\varepsilon_n}^*}, w_{\varepsilon_n}^{v_{\varepsilon_n}^*})$ is the solution to (1.5)–(1.8) corresponding to $v = v_{\varepsilon_n}^*$ and $(u^*, \varphi^*, w^*) = (u^{v^*}, \varphi^{v^*}, w^{v^*})$ is the solution to (1.1)–(1.2) and (1.4) corresponding to $v = v^*$.

Proof. For $\{v_\varepsilon^*\}$ independent of ε this lemma was proved in Moroşanu (1997). Here we shall adapt the arguments of Moroşanu (1997) to this case (see also Barbu, 1988). Let $\{v_\varepsilon^*\}$ be an optimal controller for problem (P^ε) and let $(u_\varepsilon^*, \varphi_\varepsilon^*, w_\varepsilon^*)$ be the corresponding solution of (1.5)–(1.8) with $v = v_\varepsilon^*$. Since $\{v_\varepsilon^*\}$ is bounded in $L^\infty(0,T)$, there exist $v^* \in L^\infty(0,T)$ and $\{\varepsilon_n\}$ such that

We set $v^*_{\varepsilon_n} = v^*_n$, $w_{\varepsilon_n} = w_n$, $u^{\varepsilon_n} = u^n$ and $\varphi^{\varepsilon_n} = \varphi^n$. Then, (1.5)–(1.8) becomes (setting $\varepsilon_n = \varepsilon$):

$$\begin{aligned} &u^n_t + \frac{\ell}{2}\varphi^n_t - k\Delta u^n = 0 && \text{in } Q^\varepsilon_i,\\ &\frac{\partial u^n}{\partial \nu} + hu^n = w_n(t)g(x) && \text{in } \Sigma^\varepsilon_i,\\ &u^n(0,x) = u_0(x), && x \in \Omega, \end{aligned} \tag{3.9}$$

$$\begin{aligned} &w'_n(t) = \beta w_n(t) + v^*_n(t) \qquad t \in [0,T],\\ &w_n(0) = 0, \end{aligned} \tag{3.10}$$

$$\begin{aligned} &\tau\varphi^n_t - \xi^2\Delta\varphi^n = \frac{1}{2a}\varphi^n + 2u^n + \frac{1}{2a} && \text{in } Q^\varepsilon_i,\\ &\varphi^n = 0 && \text{in } \Sigma^\varepsilon_i,\\ &\varphi^n_+(i\varepsilon,x) = z(\varepsilon,\varphi^n_-(i\varepsilon,x)), \end{aligned} \tag{3.11}$$

where $z(\cdot,\varphi^n_-(i\varepsilon,x))$ is the solution of

$$\begin{aligned} &z'(s) + \frac{1}{2a}(z(s)+1)^3 = 0 \quad s \in [0,T],\\ &z(0) = \varphi^n_-(i\varepsilon,x), \quad \varphi^n_-(0,x) = \varphi_0(x) - 1, \end{aligned} \tag{3.12}$$

computed at $s = \varepsilon$, for $i = \overline{0, M_\varepsilon - 1}$. We see that if $\varphi^n_-(i\varepsilon,x) \in L^\infty(\Omega)$, then $z(\cdot,\varphi^n_-(i\varepsilon,x)) \in L^\infty(\Omega)$ and the following estimates hold

$$\|\nabla\varphi^n_+(i\varepsilon,x)\|_{L^2(\Omega)} \le \|\nabla\varphi^n_-(i\varepsilon,x)\|_{L^2(\Omega)}, \tag{3.13}$$

$$\|\varphi^n_+(i\varepsilon,x) - \varphi^n_-(i\varepsilon,x)\|_{L^2(\Omega)} \le L\varepsilon, \tag{3.14}$$

for $i = 0,1,\dots,M_\varepsilon - 1$, where $L > 0$ is a constant depending on Ω, $\|\varphi^n_-(i\varepsilon,x)\|_{L^\infty(\Omega)}$ and a (see Moroşanu, 1997, Lemmas 3.2 and 3.3).

Multiplying (3.9_1) by $\frac{4a}{\ell}u^n$ and (3.11_1) by $a\varphi^n_t$, using integration by parts and Green's formula, yields

$$\begin{aligned} &\frac{2a}{\ell}\frac{\partial}{\partial t}\int_\Omega (u^n)^2dx + 2a\int_\Omega u^n\varphi^n_t dx + \frac{4ak}{\ell}\int_\Omega |\nabla u^n|^2dx\\ &+ \frac{4akh}{\ell}\int_{\partial\Omega}(u^n)^2dx = \frac{4ak}{\ell}\int_{\partial\Omega}u^n w_n(t)g(x)dx, \end{aligned} \tag{3.15}$$

$$a\tau\int_\Omega(\varphi^n_t)^2dx + \frac{a\xi^2}{2}\frac{\partial}{\partial t}\int_\Omega|\nabla\varphi^n|^2dx = 2a\int_\Omega u^n\varphi^n_t dx$$

(3.16)

If we now multiply (3.11_1) by $\dfrac{1}{\tau}\varphi^n$ and then we integrate over Ω, by Green's formula we get

$$\int_\Omega \varphi_t^n \varphi^n dx + \frac{\xi^2}{\tau}\int_\Omega |\nabla\varphi^n|^2 dx$$
$$= \frac{2}{\tau}\int_\Omega u^n\varphi^n dx + \frac{1}{2a\tau}\int_\Omega (\varphi^n)^2 dx + \frac{1}{2a\tau}\int_\Omega \varphi^n dx. \tag{3.17}$$

Adding (3.15)–(3.17), performing some computation involving Cauchy's and Hölder's inequality, we find

$$\frac{2a}{\ell}\frac{\partial}{\partial t}\int_\Omega (u^n)^2 dx + \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega (\varphi^n)^2 dx$$
$$+\frac{4ak}{\ell}\int_\Omega |\nabla u^n|^2 dx + \frac{\xi^2}{\tau}\int_\Omega |\nabla\varphi^n|^2 dx$$
$$+\frac{a\tau}{2}\int_\Omega (\varphi_t^n)^2 dx + \frac{a\xi^2}{2}\frac{\partial}{\partial t}\int_\Omega |\nabla\varphi^n|^2 dx + \frac{2akh}{\ell}\int_{\partial\Omega}(u^n)^2 dx$$
$$\le C\Big(\int_\Omega (u^n)^2 dx + \int_\Omega (\varphi^n)^2 dx\Big) + \frac{1}{2h}\|w_n(t)\|^2_{L^3(\partial\Omega)}\|g(x)\|^2_{L^6(\partial\Omega)}$$
$$+|\Omega|\frac{1}{4a\tau}. \tag{3.18}$$

Integration over $(0,\varepsilon)$ and by parts gives now

$$\frac{2a}{\ell}\|u^n(\varepsilon)\|^2_{L^2(\Omega)} + \frac{1}{4}\|\varphi_-^n(\varepsilon)\|^2_{L^2(\Omega)} + \frac{4ak}{\ell}\int_0^\varepsilon \|\nabla u^n\|^2_{L^2(\Omega)} ds$$
$$+\frac{\xi^2}{\tau}\int_0^\varepsilon \|\nabla\varphi^n\|^2_{L^2(\Omega)} ds + \frac{a\tau}{2}\int_0^\varepsilon \|\varphi_t^n\|^2_{L^2(\Omega)} ds$$
$$+\frac{a\xi^2}{2}\|\nabla\varphi_-^n(\varepsilon)\|^2_{L^2(\Omega)} + \frac{2akh}{\ell}\int_0^\varepsilon\int_{\partial\Omega}(u^n)^2 ds dx$$
$$\le \frac{2a}{\ell}\|u_0\|^2_{L^2(\Omega)} + \frac{1}{4}\|\varphi_0\|^2_{L^2(\Omega)} + \frac{a\xi^2}{2}\|\nabla\varphi_0\|^2_{L^2(\Omega)}$$
$$+C\int_0^\varepsilon (\|u^n(s)\|^2_{L^2(\Omega)} + \|\varphi^n(s)\|^2_{L^2(\Omega)}) ds$$
$$+\frac{1}{2h}\int_0^\varepsilon \|w_n(t)\|^2_{L^3(\partial\Omega)}\|g(x)\|^2_{L^6(\partial\Omega)} ds + |\Omega|\frac{\varepsilon}{4a\tau}. \tag{3.19}$$

Similarly, for any Q_i^ε, $i = 1, 2, \ldots, M_\varepsilon - 2$, we obtain

$$\frac{2a}{\ell}\|u^n((i+1)\varepsilon)\|^2_{L^2(\Omega)} + \frac{1}{4}\|\varphi_-^n((i+1)\varepsilon)\|^2_{L^2(\Omega)}$$
$$+\frac{4ak}{\ell}\int^{(i+1)\varepsilon} \|\nabla u^n\|^2 \, ds + \frac{\xi^2}{\tau}\int^{(i+1)\varepsilon} \|\nabla\varphi^n\|^2 \, ds$$

$$
\begin{aligned}
&+\frac{a\tau}{2}\int_{i\varepsilon}^{(i+1)\varepsilon}\|\varphi_t^n\|^2_{L^2(\Omega)}ds+\frac{a\xi^2}{2}\|\nabla\varphi_-^n((i+1)\varepsilon)\|^2_{L^2(\Omega)}\\
&+\frac{2akh}{\ell}\int_{i\varepsilon}^{(i+1)\varepsilon}\int_{\partial\Omega}(u^n)^2dsdx\le\frac{2a}{\ell}\|u^n(i\varepsilon)\|^2_{L^2(\Omega)}+\frac{1}{4}\|\varphi_+^n(i\varepsilon)\|^2_{L^2(\Omega)}\\
&+\frac{a\xi^2}{2}\|\nabla\varphi_+^n(i\varepsilon)\|^2_{L^2(\Omega)}+C\int_{i\varepsilon}^{(i+1)\varepsilon}(\|u^n(s)\|^2_{L^2(\Omega)}+\|\varphi^n(s)\|^2_{L^2(\Omega)})ds\\
&+\frac{1}{2h}\int_{i\varepsilon}^{(i+1)\varepsilon}\|w_n(t)\|^2_{L^3(\partial\Omega)}\|g(x)\|^2_{L^6(\partial\Omega)}ds+|\Omega|\frac{\varepsilon}{4a\tau}. \qquad (3.20)
\end{aligned}
$$

On $Q^{\varepsilon}_{M_\varepsilon-1}$ we have the estimates

$$
\begin{aligned}
&\frac{2a}{\ell}\|u^n(T)\|^2_{L^2(\Omega)}+\frac{1}{4}\|\varphi_-^n(T)\|^2_{L^2(\Omega)}\\
&+\frac{4ak}{\ell}\int_{(M_\varepsilon-1)\varepsilon}^{T}\|\nabla u^n\|^2_{L^2(\Omega)}ds+\frac{a\xi^2}{\tau}\int_{(M_\varepsilon-1)\varepsilon}^{T}\|\nabla\varphi^n\|^2_{L^2(\Omega)}ds\\
&+\frac{a\tau}{2}\int_{(M_\varepsilon-1)\varepsilon}^{T}\|\varphi_t^n\|^2_{L^2(\Omega)}ds+\frac{a\xi^2}{2}\|\nabla\varphi_-^n(T)\|^2_{L^2(\Omega)}\\
&+\frac{2akh}{\ell}\int_{(M_\varepsilon-1)\varepsilon}^{T}\int_{\partial\Omega}(u^n)^2dsdx\le\frac{2a}{\ell}\|u^n((M_\varepsilon-1)\varepsilon)\|^2_{L^2(\Omega)}\\
&+\frac{1}{4}\|\varphi_+^n((M_\varepsilon-1)\varepsilon)\|^2_{L^2(\Omega)}+\frac{a\xi^2}{2}\|\nabla\varphi_+^n((M_\varepsilon-1)\varepsilon)\|^2_{L^2(\Omega)}\\
&+C\int_{(M_\varepsilon-1)\varepsilon}^{T}(\|u^n(s)\|^2_{L^2(\Omega)}+\|\varphi^n(s)\|^2_{L^2(\Omega)})ds\\
&+\frac{1}{2h}\int_{(M_\varepsilon-1)\varepsilon}^{T}\|w_n(t)\|^2_{L^3(\partial\Omega)}\|g(x)\|^2_{L^6(\partial\Omega)}ds+|\Omega|\frac{T-(M_\varepsilon-1)\varepsilon}{4a\tau}. \qquad (3.21)
\end{aligned}
$$

From (3.19)–(3.21), taking into account (3.13), we get

$$
\begin{aligned}
&\frac{2a}{\ell}\|u^n(T)\|^2_{L^2(\Omega)}+\frac{1}{4}\|\varphi_-^n(T)\|^2_{L^2(\Omega)}\\
&+\frac{4ak}{\ell}\int_0^T\|\nabla u^n\|^2_{L^2(\Omega)}ds+\frac{a\xi^2}{\tau}\int_0^T\|\nabla\varphi^n\|^2_{L^2(\Omega)}ds\\
&+\frac{a\tau}{2}\sum_{i=0}^{M_\varepsilon-1}\int_{i\varepsilon}^{(i+1)\varepsilon}\|\varphi_t^n\|^2_{L^2(\Omega)}ds+\frac{a\tau}{2}\int_{(M_\varepsilon-1)\varepsilon}^{T}\|\varphi_t^n\|^2_{L^2(\Omega)}ds\\
&+\frac{a\xi^2}{2}\|\nabla\varphi_-^n(T)\|^2_{L^2(\Omega)}+\frac{2akh}{\ell}\int_0^T\int_{\partial\Omega}(u^n)^2dsdx\\
&\le\frac{2a}{\ell}\|u_0\|^2_{L^2(\Omega)}+\frac{1}{4}\|\varphi_0\|^2_{L^2(\Omega)}+\frac{a\xi^2}{2}\|\nabla\varphi_0\|^2_{L^2(\Omega)}\\
&+C\int_0^T(\|u^n(s)\|^2+\|\varphi^n(s)\|^2)ds
\end{aligned}
$$

$$+\frac{1}{2h}\int_0^T \|w_n(t)\|^2_{L^3(\partial\Omega)}\|g(x)\|^2_{L^6(\partial\Omega)}ds+|\Omega|\frac{T}{4a\tau}.$$

By Gronwall inequality, we may derive finally the inequality

$$\sum_{i=0}^{M_\varepsilon-1}\int_{i\varepsilon}^{(i+1)\varepsilon}\|\varphi_t^n\|^2_{L^2(\Omega)}ds+\int_{(M_\varepsilon-1)\varepsilon}^{T}\|\varphi_t^n\|^2_{L^2(\Omega)}ds$$
$$\int_0^T(\|\nabla u^n\|^2_{L^2(\Omega)}+\|\nabla\varphi^n\|^2_{L^2(\Omega)})ds+\int_\Sigma(u^n)^2dxds\le C_1\quad\forall\varepsilon>0,\ (3.22)$$

where $C_1>0$ does not depend on M_ε and ε (C_1 depends on $\tau,\ k,\ \ell,\ \xi,\ a,\ h,\ T,\ \Omega,\ \|u_0\|_{L^2(\Omega)},\ \|\nabla\varphi_0\|_{L^2(\Omega)},\ \|\varphi_0\|_{L^2(\Omega)},\ \|w_n(t)\|_{L^3(\partial\Omega)},\ \|g\|_{L^6(\partial\Omega)}$).

Multiplying now (3.9_1) by u_t^n, integrating over $(i\varepsilon,(i+1)\varepsilon)$, $i=0,1,\ldots,M_\varepsilon-1$, and using Green's formula, Cauchy–Schwarz's inequality, and Young's inequality, we obtain

$$\frac{1}{2}\int_0^\varepsilon\int_\Omega(u_t^n)^2dsdx+\frac{k}{2}\int_\Omega|\nabla u^n(\varepsilon)|^2dx+\frac{kh}{2}\int_{\partial\Omega}(u^n(\varepsilon))^2dx$$
$$\le\frac{\ell^2}{8}\int_0^\varepsilon\|\varphi_t^n\|^2_{L^2(\Omega)}ds+\frac{k}{2}\int_\Omega|\nabla u_0|^2dx+\frac{kh}{2}\int_{\partial\Omega}u_0^2dx$$
$$+k\int_0^\varepsilon\int_{\partial\Omega}u_t^nw_n(t)g(x)dsdx,\qquad(3.23)$$
$$\frac{1}{2}\int_{i\varepsilon}^{(i+1)\varepsilon}\int_\Omega(u_t^n)^2dsdx+\frac{k}{2}\int_\Omega|\nabla u^n((i+1)\varepsilon)|^2dx+\int_{\partial\Omega}(u^n((i+1)\varepsilon))^2dx$$
$$\le\frac{\ell^2}{8}\int_{i\varepsilon}^{(i+1)\varepsilon}\|\varphi_t^n\|^2_{L^2(\Omega)}ds+\frac{k}{2}\int_\Omega|\nabla u^n(i\varepsilon)|^2dx+\frac{kh}{2}\int_{\partial\Omega}(u^n(i\varepsilon))^2dx$$
$$+k\int_{i\varepsilon}^{(i+1)\varepsilon}\int_{\partial\Omega}u_t^nw_n(t)g(x)dsdx,\ i=1,\ldots,M_\varepsilon-2,\qquad(3.24)$$
$$\frac{1}{2}\int_{(M_\varepsilon-1)\varepsilon}^{T}\int_\Omega(u_t^n)^2dsdx+\frac{k}{2}\int_\Omega|\nabla u^n(T)|^2dx+\int_{\partial\Omega}(u^n(T))^2dx$$
$$\le\frac{\ell^2}{8}\int_{(M_\varepsilon-1)\varepsilon}^{T}\|\varphi_t^n\|^2_{L^2(\Omega)}ds+\frac{k}{2}\int_\Omega|\nabla u^n((M_\varepsilon-1)\varepsilon)|^2dx$$
$$+\frac{kh}{2}\int_{\partial\Omega}(u^n((M_\varepsilon-1)\varepsilon))^2dx+k\int_{(M_\varepsilon-1)\varepsilon}^{T}\int_{\partial\Omega}u_t^nw_n(t)g(x)dsdx.\qquad(3.25)$$

From (3.23)–(3.25), taking into account (3.22), we obtain

$$\frac{1}{2}\int_0^T\int_\Omega(u_t^n)^2dsdx+\frac{k}{2}\int_\Omega|\nabla u^n(T)|^2dx+\frac{kh}{2}\int_{\partial\Omega}(u^n(T))^2dx$$
$$\le\frac{\ell^2}{8}\sum_{i=0}^{M_\varepsilon-1}\int_{i\varepsilon}^{(i+1)\varepsilon}\|\varphi_t^n\|^2_{L^2(\Omega)}ds+\frac{\ell^2}{8}\int_{(M_\varepsilon-1)\varepsilon}^{T}\|\varphi_t^n\|^2_{L^2(\Omega)}ds$$

$$+\frac{k}{2}\|\nabla u_0\|^2_{L^2(\Omega)}+\frac{kh}{2}\|u_0\|^2_{L^2(\Omega)}+k\int_0^T\int_{\partial\Omega}u_t^n w_n(t)g(x)dsdx$$
$$\leq\frac{\ell^2}{8}\bar{C}_1+k\int_0^T\int_{\partial\Omega}u_t^n w_n(t)g(x)dsdx, \tag{3.26}$$

where $\bar{C}_1>0$ depends on C_1 and on $\|\nabla u_0\|_{L^2(\Omega)}$. But

$$k\int_0^T\int_{\partial\Omega}u_t^n w_n(t)g(x)dxds=k\int_{\partial\Omega}g(x)dx\int_0^T\frac{\partial}{\partial t}(u^n(t,x)w_n(t))ds$$
$$-k\int_{\partial\Omega}g(x)dx\int_0^T u^n(t,x)w_n'(t)ds$$
$$=k\int_{\partial\Omega}u^n(t,x)w_n(t)g(x)dx-k\int_0^T\int_{\partial\Omega}u^n w_n'(t)g(x)dxds$$

and (using Cauchy–Schwarz's inequality, Hölder's inequality, and Young's inequality)

$$k\int_{\partial\Omega}u^n(t,x)w_n(t)g(x)dxds$$
$$\leq\frac{kh}{4}\int_{\partial\Omega}(u^n)^2(t,x)dx+\frac{k}{h}\|w_n\|^2_{L^3(\partial\Omega)}\|g\|^2_{L^6(\partial\Omega)},$$
$$k\int_0^T\int_{\partial\Omega}u^n w_n'(t)g(x)dxds$$
$$\leq\frac{k}{2}\int_0^T\int_{\partial\Omega}(u^n)^2dxds+\frac{k}{2}\|w_n'\|^2_{L^3(\Sigma)}\|g\|^2_{L^6(\Sigma)}.$$

So, (3.26) becomes

$$\frac{1}{2}\int_0^T\int_\Omega(u_t^n)^2dsdx+\frac{k}{2}\int_\Omega|\nabla u^n(T)|^2dx+\frac{kh}{4}\int_{\partial\Omega}(u^n(T))^2dx$$
$$\leq\bar{C}_2+\frac{k}{2}\int_0^T\|u^n(s)\|^2_{L^2(\partial\Omega)}ds, \tag{3.27}$$

where $\bar{C}_2>0$ depends on $\bar{C}_1$, T, $\|g\|_{L^6(\partial\Omega)}$, $\|w_n(t)\|_{L^3(\partial\Omega)}$ and $\|w_n'(t)\|_{L^3(\partial\Omega)}$. Using now the Gronwall inequality, we obtain

$$\int_0^T\|u_t^n\|^2_{L^2(\Omega)}ds\leq C_2\quad\forall\,\varepsilon>0, \tag{3.28}$$

where $C_2>0$ does not depend on M_ε and ε (C_2 depends on $\bar{C}_1$ and $\bar{C}_2$).

By virtue of estimate (3.14) we get

$$\sum^{M_\varepsilon-1}\|\varphi_+^n(i\varepsilon,x)-\varphi_-^n(i\varepsilon,x)\|_{L^2(\Omega)}\leq LT=C \tag{3.29}$$

Combining (3.22), (3.28) and (3.29), we obtain

$$\overset{T}{\underset{0}{V}}\varphi^n + \int_0^T \|u_t^n(t)\|^2_{L^2(\Omega)}dt$$
$$+ \sum_{i=0}^{M_\varepsilon -1} \int_{i\varepsilon}^{(i+1)\varepsilon} \|\varphi_t^n(t)\|^2_{L^2(\Omega)}dt + \int_{(M_\varepsilon-1)\varepsilon}^T \|\varphi_t^n(t)\|^2_{L^2(\Omega)}dt$$
$$+ \int_0^T \|\nabla u^n\|^2_{L^2(\Omega)}dt + \int_0^T \|\nabla \varphi^n\|^2_{L^2(\Omega)}dt \le C \quad \forall \varepsilon > 0, \forall t \in [0,T], \tag{3.30}$$

where $\overset{T}{\underset{0}{V}}\varphi^n$ stands for the variation of $\varphi^n : [0,T] \to L_2(\Omega)$. Since the injection of $L_2(\Omega)$ into $H^{-1}(\Omega)$ is compact and the set $\{\varphi_t^n(t)\}$ is bounded in $L_2(\Omega)$ for every $t \in [0,T]$, by an infinite dimensional version of Helly-Foiaş theorem (see, for instance, Barbu and Precupanu, 1986, Remark 3.2, pp. 60), we conclude that there exists a bounded variation function $\varphi^*(t) \in BV([0,T];H^{-1}(\Omega))$ such that, on a subsequence also denoted $\varphi^n(t)$, we have

$$\varphi^n(t) \to \varphi^*(t) \text{ strongly in } H^{-1}(\Omega) \text{ for every } t \in [0,T]. \tag{3.31}$$

By (3.30) we may assume that

$$\varphi^n \to \varphi^* \ \text{ weakly in } L^2(0,T;H_0^1(\Omega)). \tag{3.32}$$

Now, since the inclusion of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact (see Brézis, 1983, Theorem IX.16, pp. 169), for every $\lambda > 0$ there exists $C(\lambda) > 0$ such that (see Lions, 1969, Chapter 1, Lemma 5.1)

$$\|\varphi^n(t)-\varphi^*(t)\|_{L^2(\Omega)} \le \lambda\|\varphi^n(t)-\varphi^*(t)\|_{H_0^1(\Omega)} + C(\lambda)\|\varphi^n(t)-\varphi^*(t)\|_{H^{-1}(\Omega)}$$
$$\forall \varepsilon > 0, \ \ \forall t \in [0,T],$$

where $C(\lambda) \to 0$ as $\lambda \to 0$.

Together with (3.31) and (3.32) this yields

$$\varphi^n \to \varphi^* \ \text{ strongly in } L^2(\Omega) \text{ for any } t \in [0,T]. \tag{3.33}$$

Taking into account (3.28), (3.30), we may obtain by (3.9_1) the estimates

$$\int_0^t \int_\Omega (\Delta u^n(s,x))^2 dxds \le C \quad \forall t \in (0,T]. \tag{3.34}$$

By the elliptic boundary regularity, (3.34) implies

$$\|u^n\|_{L^2(0,T;H^2(\Omega))} \le C. \tag{3.35}$$

Similarly, by (3.11_1), using (3.30), we obtain

$$\int_0^t \int_\Omega (\Delta \varphi^n(s,x))^2 dxds \le C \quad \forall t \in (0,T] \tag{3.36}$$

and, by elliptic regularity,

$$\|\varphi^n\|_{L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega))} \leq C. \tag{3.37}$$

By (3.22), (3.35), it follows that the sequence $\{u^n\}$ is compact in $L^2(0,T;H^1(\Omega))$. Therefore, on a subsequence, again denoted $\{u^n\}$, we have

$$u^n \to u^* \text{ strongly in } L^2(0,T;H^1(\Omega)), \quad \text{weakly in } L^2(0,T;H^2(\Omega)), \tag{3.38}$$

$$u_t^n \to u_t^* \text{ weakly in } L^2(0,T;L^2(\Omega)),$$

and, by the Ascoli–Arzelà theorem

$$u^n \to u^* \text{ strongly in } C([0,T];L^2(\Omega)).$$

From (3.30) we also have

$$\begin{aligned} &\nabla u^n \to \nabla u^* \quad \text{weakly in } L^2(0,T;H^1(\Omega)),\\ &\nabla \varphi^n \to \nabla \varphi^* \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)). \end{aligned} \tag{3.39}$$

(3.34)–(3.36) clearly also imply that

$$\begin{aligned} &\Delta u^n \to \Delta u^* \quad \text{weakly in } L^2(0,T;L^2(\Omega)),\\ &\Delta \varphi^n \to \Delta \varphi^* \quad \text{weakly in } L^2(0,T;L^2(\Omega)). \end{aligned} \tag{3.40}$$

From (3.33) and (3.38) we may conclude that (3.7) and (3.8) holds.

Let $s < t$ be two arbitrary points of $[0,T]$ such that $i\varepsilon \leq s \leq (i+1)\varepsilon < \ldots < j\varepsilon \leq t$. Consider the problem

$$\begin{aligned} &\tau\varphi_t^n - \xi^2\Delta\varphi^n - \frac{1}{2a}\varphi^n = 2u^n + \frac{1}{2a} && \text{in } Q_k^\varepsilon,\\ &\varphi_+^n(k\varepsilon,x) = z(\varepsilon,\varphi_-^n(k\varepsilon,x)) && \text{on } \Sigma_k^\varepsilon,\\ &\varphi^n = 0 && x \in \Omega. \end{aligned} \tag{3.41}$$

In the usual way, from (3.41) we obtain

$$\begin{aligned} &\int_\Omega |\varphi_-^n((k+1)\varepsilon) - \varphi_+^n(k\varepsilon)|^2 dx + \frac{\xi^2}{2}\int_{k\varepsilon}^{(k+1)\varepsilon}\int_\Omega |\nabla\varphi^n|^2 dxdt\\ &\leq \frac{\xi^2}{2}\int_{k\varepsilon}^{(k+1)\varepsilon}\int_\Omega |\nabla\varphi_+^n|^2 dxdt + C\int_{k\varepsilon}^{(k+1)\varepsilon}\int_\Omega (\varphi_+^n(k\varepsilon))^2 dxdt\\ &+ C\int_{k\varepsilon}^{(k+1)\varepsilon}\int_\Omega [(u^n)^2 + (\varphi^n)^2] dxdt. \end{aligned}$$

Taking into account (3.12) the last inequality becomes

$$\begin{aligned} &\sum_{k=i}^{j-1} \|\varphi_-^n((k+1)\varepsilon) - \varphi_+^n(k\varepsilon)\|_{L^2(\Omega)}^2\\ &< C\sum_{k=i}^{j-1}\int_{k\varepsilon}^{(k+1)\varepsilon}\int_\Omega [(u^n)^2 + (\varphi^n)^2] dxdt. \end{aligned} \tag{3.42}$$

On the other hand, using (3.14), we get the estimate

$$\sum_{k=i}^{j-1} \|\varphi_+^n(k\varepsilon) - \varphi_-^n(k\varepsilon)\|_{L^2(\Omega)} \leq L(j-i)\varepsilon. \tag{3.43}$$

Hence

$$\|\varphi^n(t) - \varphi^n(s)\|_{L^2(\Omega)} \leq \|\varphi^n(s) - \varphi_-^n(i\varepsilon)\|_{L^2(\Omega)} + \|\varphi_-^n(j\varepsilon) - \varphi^n(t)\|_{L^2(\Omega)}$$
$$+ \sum_{k=i}^{j-1} \|\varphi_+^n(k\varepsilon) - \varphi_-^n(k\varepsilon)\|_{L^2(\Omega)} + \sum_{k=i}^{j-1} \|\varphi_-^n((k+1)\varepsilon) - \varphi_+^n(k\varepsilon)\|_{L^2(\Omega)}.$$

Along with (3.42) and (3.43) the last inequality implies

$$\|\varphi^n(t) - \varphi^n(s)\|_{L^2(\Omega)}$$
$$\leq C\left(|t-s| + |t-s|^{1/2}\left(\int_s^t \int_\Omega [(u^n)^2 + (\varphi^n)^2] dx d\tau\right)^{1/2}\right) \tag{3.44}$$

and therefore $\varphi^* : [0,T] \to L^2(\Omega)$ (the limit point of φ^n) is absolutely continuous and consequently almost everywhere differentiable on $[0,T]$. Hence $\varphi_t^*(t)$ exists a.e. on $(0,T)$.

Let $\nu \in L^6(\Omega)$ be an element arbitrary but fixed. By (3.41) we have

$$\tau(\varphi^n(t) - \varphi^n(s), \varphi^n(s) - \nu) + \tau \sum_{k=i+1}^{j} (\varphi_-^n(k\varepsilon) - \varphi_+^n(k\varepsilon), \varphi_+^n(k\varepsilon) - \nu)$$
$$\leq \xi^2 \int_s^t (\Delta\varphi^n, \varphi^n - \nu) d\tau + \frac{1}{2a} \int_s^t (\varphi^n, \varphi^n - \nu) d\tau$$
$$+ \int_s^t (2u^n + \frac{1}{2a}, \varphi^n - \nu) d\tau \quad \forall \nu \in L^6(\Omega), \tag{3.45}$$

where $(\cdot,\cdot)$ stands for the inner product of $L^2(\Omega)$ and also for the duality between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

We denote by $F : D(F) = L^6(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ the operator $z \to -\frac{1}{2a}(z+1)^3$. Clearly, F is m-dissipative and denote by e^{-Ft} the semigroup generated by F. Then (see (3.12))

$$z(t) = e^{-Ft}\varphi_-^n(i\varepsilon) \text{ for } t \in (0,T), \quad i = 0, 1, \ldots, M_\varepsilon - 1.$$

Thus

$$(\varphi_-^n(k\varepsilon) - \varphi_+^n(k\varepsilon), \varphi_+^n(k\varepsilon) - \nu)$$
$$= (\varphi_-^n(k\varepsilon) - e^{-F\varepsilon}\varphi_-^n(k\varepsilon), \varphi_-^n(k\varepsilon) - \nu)$$

We set $S_\varepsilon y = y - e^{-F\varepsilon}y$ and then (3.45) becomes, taking into account (3.46) and the monotonicity of S_ε,

$$\tau(\varphi^n(t)-\varphi^n(s),\varphi^n(s)-\nu)+\tau\sum_{k=i+1}^{j}(S_\varepsilon,\varphi^n_-(k\varepsilon)-\nu)$$

$$\le \tau\sum_{k=i+1}^{j}\|S_\varepsilon\varphi^n_-(k\varepsilon)\|^2_{L^2(\Omega)}+\xi^2\int_s^t(\Delta\varphi^n,\varphi^n-\nu)d\tau$$

$$+\frac{1}{2a}\int_s^t(\varphi^n,\varphi^n-\nu)d\tau+\int_s^t(2u^n+\frac{1}{2a},\varphi^n-\nu)d\tau\quad\forall\,\nu\in L^6(\Omega).\quad(3.47)$$

By (3.14) we have $\|S_\varepsilon\varphi^n_-(k\varepsilon)\|^2_{L^2(\Omega)}\le L^2\varepsilon^2$ and therefore

$$\sum_{k=i+1}^{j}\|S_\varepsilon\varphi^n_-(k\varepsilon)\|^2_{L^2(\Omega)}\le L^2(j-i)\varepsilon^2=\delta_\varepsilon,$$

where $\delta_\varepsilon\to 0$ for $\varepsilon\to 0$.

Now, we define $\tilde\varphi^n(\tau)=\varphi^n_-(k\varepsilon)$, for $\tau\in(k\varepsilon,(k+1)\varepsilon)$. Then

$$\varepsilon\sum_{k=i+1}^{j}\left(\frac{S_\varepsilon\nu}{\varepsilon},\varphi^n_-(k\varepsilon)-\nu\right)=\int_s^t\left(\frac{S_\varepsilon\nu}{\varepsilon},\tilde\varphi^n(\tau)-\nu\right)d\tau.$$

Since $\|\varphi^n_-(k\varepsilon)\|_{L^2(\Omega)}\le C$, then $\|\tilde\varphi^n(\tau)\|_{L^2(\Omega)}\le C$ and therefore the above integral is well defined. Using (3.7) we obtain

$$\lim_{\varepsilon\to 0}\varepsilon\sum_{k=i+1}^{j}\left(\frac{S_\varepsilon\nu}{\varepsilon},\varphi^n_-(k\varepsilon)-\nu\right)$$

$$=\int_s^t(-F\nu,\varphi^*(\tau)-\nu)d\tau\quad\forall\,\nu\in L^6(\Omega).\quad(3.48)$$

By (3.33) and (3.40) we have

$$\begin{aligned}&\int_0^t(-\Delta\varphi^n,\varphi^n-\nu)d\tau\to\int_0^t(-\Delta\varphi^*(\tau),\varphi^*(\tau)-\nu)d\tau,\\&\int_0^t(\varphi^n,\varphi^n-\nu)d\tau\to\int_0^t(\varphi^*,\varphi^*-\nu)d\tau.\end{aligned}\quad(3.49)$$

Taking into account (3.48), (3.49) and passing to the limit for $n\to\infty$ in (3.47) we obtain

$$\tau(\varphi^*(t)-\varphi^*(s),\varphi^*(s)-\nu)+\tau\int_s^t(-F\nu,\varphi^*(\tau)-\nu)d\tau$$

$$-\xi^2\int_s^t(\Delta\varphi^*(\tau),\varphi^*(\tau)-\nu)d\tau-\frac{1}{2a}\int_s^t(\varphi^*(\tau),\varphi^*(\tau)-\nu)d\tau$$

$$\le\int_s^t(2u^*(\tau)+\frac{1}{2a},\varphi^*(\tau)-\nu)d\tau\quad\forall\,\nu\in L^6(\Omega).\quad(3.50)$$

Dividing (3.50) by $t-s$ and letting s tend to t we see that

$$\left(\tau\frac{\partial\varphi^*}{\partial t}(t)-\xi^2\Delta\varphi^*(t)-\frac{1}{2a}\varphi^*(t)-F\nu,\varphi^*(t)-\nu\right)$$
$$\leq(2u^*(t)+\frac{1}{2a},\varphi^*(t)-\nu),\ \text{a.e. } t\in[0,T]\quad\forall\nu\in L^6(\Omega).\tag{3.51}$$

Using now the maximal monotonicity of F, we infer from (3.51) that

$$\tau\frac{\partial\varphi^*}{\partial t}(t)-\xi^2\Delta\varphi^*(t)$$
$$=\frac{1}{2a}\Big(\varphi^*(t)+1\Big)-\frac{1}{2a}\Big(\varphi^*(t)+1\Big)^3+2u^*(t),\ \text{a.e. } t\in[0,T].$$

Hence $\varphi^*(t,x)$ satisfies (1.4), a.e. $t\in[0,T]$.

By (3.9_1) we get

$$\frac{\ell}{2}\int_Q(\psi(t),d\varphi^n(t))dx+\int_Q(u_t^n-k\Delta u^n)\psi dxdt=0,\ \forall\psi\in L^2(0,T;H^1(\Omega)).$$

Taking into account (3.32), (3.38) and (3.40) we may pass now to the limit for $n\to\infty$, and, by Helly's theorem (see Barbu and Precupanu, 1986, Theorem 3.5, pp. 58)

$$\frac{\ell}{2}\int_Q(\psi(t),d\varphi^*(t))dx+\int_Q(u_t^*-k\Delta u^*)\psi dxdt=0$$
$$\forall\psi\in L^2(0,T;H^1(\Omega)).\tag{3.52}$$

Since φ^* is absolutely continuous from $[0,T]$ to $L^2(\Omega)$ (see (3.44)) the first Stieltjes integral can by written as $\int_Q(\psi(t),\varphi_t^*(t))dxdt$ and so (3.52) yields

$$u_t^*(t)+\frac{\ell}{2}\varphi_t^*(t)-k\Delta u^*(t)=0,\ \text{a.e. } t\in[0,T].\tag{3.53}$$

By trace theorem (the map $u^n\to u^n|_{\partial\Omega}$ is continuous from $H^1(\Omega)$ into $H^{1/2}(\partial\Omega)\subset L^2(\partial\Omega)$) and taking into account (3.4)–(3.6), we may conclude from (3.53) that (u^*,φ^*) satisfies (1.1) a.e. $t\in[0,T]$. Therefore (u^*,φ^*) is a strong solution to (1.1)–(1.2) and (1.4) corresponding to $v=v^*$.

Proof of Theorem 3.1. The idea is the same as in Barbu (1988). Let $\{v_\varepsilon^*\}$ be an optimal controller for problem (P^ε) and let $(u_\varepsilon^*,\varphi_\varepsilon^*,w_\varepsilon^*)$ be the corresponding solution of (1.5)–(1.8) with $v=v_\varepsilon^*$. By virtue of Lemma 3.1 it results that there exist $v^*\in L^\infty(0,T)$ and $\{\varepsilon_n\}$ such that

$$\begin{aligned}&v_{\varepsilon_n}^*\to v^* &&\text{weak star in} && L^\infty(0,T),\\ &w_{\varepsilon_n}^{v_{\varepsilon_n}^*}\to w^{v^*} &&\text{strongly in} && C[0,T],\end{aligned}$$

where $(u_{\varepsilon_n}^{v_{\varepsilon_n}^*}, \varphi_{\varepsilon_n}^{v_{\varepsilon_n}^*}, w_{\varepsilon_n}^{v_{\varepsilon_n}^*})$ is the solution to (1.5)–(1.8) corresponding to $v = v_{\varepsilon_n}^*$ and $(u^{v^*}, \varphi^{v^*}, w^{v^*})$ is the solution to (1.1)–(1.2) and (1.4) corresponding to $v = v^*$.

Since $\varphi \to \int_Q \chi_0(\varphi(t,x))^2 dxdt$ and $w \to \int_0^T w^2(t)dt$ are convex continuous functions it follows that these are weakly lower semicontinuous functions (from $L^2(Q) \to \mathrm{R}$ and from $L^2(0,T) \to \mathrm{R}$, respectively). Hence

$$j(v^*) \le \liminf_{n\to\infty} \int_Q \chi_0(\varphi_{\varepsilon_n}^{v_{\varepsilon_n}^*}(t,x))^2 dxdt + \int_0^T (w_{\varepsilon_n}^{v_{\varepsilon_n}^*}(t))^2 dt. \tag{3.54}$$

Let $\tilde{v}^*$ be an optimal controller for problem (P). Since $v_{\varepsilon_n}^*$ is an optimal controller for problem (P^{ε_n}) it follows that

$$\begin{aligned}
&\int_Q \chi_0(\varphi_{\varepsilon_n}^{v_{\varepsilon_n}^*}(t,x))^2 dxdt + \int_0^T (w_{\varepsilon_n}^{v_{\varepsilon_n}^*}(t))^2 dt \\
&\le \int_Q \chi_0(\varphi_{\varepsilon_n}^{\tilde{v}^*}(t,x))^2 dxdt + \int_0^T (w_{\varepsilon_n}^{\tilde{v}^*}(t))^2 dt.
\end{aligned}$$

But $\varphi_{\varepsilon_n}^{\tilde{v}^*}(t) \to \varphi^{\tilde{v}^*}(t)$ strongly in $L^2(\Omega)$, $\forall\, t \in [0,T]$, and so the latter implies

$$\begin{aligned}
&\lim_{n\to\infty} \int_Q \chi_0(\varphi_{\varepsilon_n}^{\tilde{v}^*}(t,x))^2 dxdt + \int_0^T (w_{\varepsilon_n}^{\tilde{v}^*}(t))^2 dt \\
&\le \int_Q \chi_0(\varphi^{\tilde{v}^*}(t,x))^2 dxdt + \int_0^T (w^{\tilde{v}^*}(t))^2 dt.
\end{aligned} \tag{3.55}$$

From (3.54) and (3.55), we obtain

$$\begin{aligned}
j(v^*) &= \int_Q \chi_0(\varphi^{v^*}(t,x))^2 dxdt + \int_0^T (w^{v^*}(t))^2 dt \\
&\le \liminf_{n\to\infty} \left[\int_Q \chi_0(\varphi_{\varepsilon_n}^{v_{\varepsilon_n}^*}(t,x))^2 dxdt + \int_0^T (w_{\varepsilon_n}^{v_{\varepsilon_n}^*}(t))^2 dt \right] \\
&\le \limsup_{n\to\infty} \left[\int_Q \chi_0(\varphi_{\varepsilon_n}^{v_{\varepsilon_n}^*}(t,x))^2 dxdt + \int_0^T (w_{\varepsilon_n}^{v_{\varepsilon_n}^*}(t))^2 dt \right] \\
&\le \int_Q \chi_0(\varphi^{\tilde{v}^*}(t,x))^2 dxdt + \int_0^T (w^{\tilde{v}^*}(t))^2 dt.
\end{aligned}$$

Hence

$$\lim_{\varepsilon_n\to 0} \inf j^{\varepsilon_n}(v_{\varepsilon_n}^*) = j(\tilde{v}^*) = \inf \quad \{j(v), \ \ v \in \mathcal{U}\}$$

To prove (3.3) we set $\tilde{\varphi}_\varepsilon = \varphi^{v_\varepsilon^*}$ (v_ε^* optimal in (P^ε)). We have on a subsequence $\{\varepsilon_n\}$

$$\begin{array}{lll} v_{\varepsilon_n}^* \to v^0 & \text{weakly in} & L^\infty(0,T), \\ \tilde{w}_{\varepsilon_n}^{v_{\varepsilon_n}^*} \to w^0 & \text{strongly in} & C[0,T], \\ \tilde{u}_{\varepsilon_n} \to u & \text{strongly in} & L^2(0,T,H^1(\Omega)), \\ \tilde{\varphi}_{\varepsilon_n} \to \varphi & \text{strongly in} & L^2(0,T,H^1(\Omega)), \end{array}$$

where (u,φ,w^0,v^0) satisfy (1.1)–(1.2) and (1.4), i.e., $(u,\varphi,w^0) = (u^{v^0},\varphi^{v^0},w^{v^0})$. We have therefore

$$\int_Q \chi_0(\varphi^{v^0}(t,x))^2 dxdt + \int_0^T (w^{v^0}(t))^2 dt \leq \inf P$$

and since $\{\varepsilon_n\}$ was arbitrarily chosen (3.3) follows. Now, since v_ε^* is an optimal controller for problem (P^ε) it follows that

$$\begin{aligned} &\int_Q \chi_0(\varphi_\varepsilon^{v_\varepsilon^*}(t,x))^2 dxdt + \int_0^T (w_\varepsilon^{v_\varepsilon^*}(t))^2 dt \\ &\leq \int_Q \chi_0(\varphi_\varepsilon^{v}(t,x))^2 dxdt + \int_0^T (w_\varepsilon^{v}(t))^2 dt \quad \forall v \in \mathcal{U}. \end{aligned}$$

But, as we have seen above,

$$j(v^*) \leq \liminf_{\varepsilon\to 0} \int_Q \chi_0(\varphi_\varepsilon^{v_\varepsilon^*}(t,x))^2 dxdt + \int_0^T (w_\varepsilon^{v_\varepsilon^*}(t))^2 dt$$

and thus, along with above inequality, we have

$$j(v^*) \leq \lim_{\varepsilon\to 0} \int_Q \chi_0(\varphi_\varepsilon^{v}(t,x))^2 dxdt + \int_0^T (w_\varepsilon^{v}(t))^2 dt \quad \forall v \in \mathcal{U}.$$

Hence

$$j(v^*) \leq j(v) \quad \forall v \in \mathcal{U}$$

i.e., the weak limit point v^* is a suboptimal controller for problem (P). This completes the proof of Theorem 3.1.

4. Approximating problems. Optimality conditions

In order to establish the optimality conditions for problem (P^ε) we consider $(u^\varepsilon,\varphi^\varepsilon,w,v)$ the solution of (1.5)–(1.8) and the corresponding variations

$$u^\lambda = u^\varepsilon + \lambda\tilde{u}^\varepsilon, \quad \varphi^\lambda = \varphi^\varepsilon + \lambda\tilde{\varphi}^\varepsilon, \quad w^\lambda = w + \lambda\tilde{w}, \tag{4.1}$$

($T_{\mathcal{U}}(v)$ is the tangent cone at $\mathcal{U}$ in v). For $(u^\lambda, \varphi^\lambda, w^\lambda, v^\lambda)$ we have

$$\begin{aligned} & u_t^\lambda + \frac{\ell}{2}\varphi_t^\lambda = k\Delta u^\lambda && \text{in } Q_i^\varepsilon, \\ & \frac{\partial u^\lambda}{\partial \nu} + hu^\lambda = w^\lambda(t)g(x) && \text{on } \Sigma_i^\varepsilon, \\ & u^\lambda(0,x) = u_0(x) && x \in \Omega, \end{aligned} \tag{4.2}$$

$$\begin{aligned} & (w^\lambda(t))' = \beta w^\lambda(t) + v^\lambda(t) \quad t \in [0,T], \\ & w^\lambda(0) = 0, \end{aligned} \tag{4.3}$$

$$\begin{aligned} & \tau\varphi_t^\lambda = \xi^2\Delta\varphi^\lambda + \frac{1}{2a}\varphi^\lambda + 2u^\lambda + \frac{1}{2a} && \text{in } Q_i^\varepsilon, \\ & \varphi^\lambda = 0 && \text{on } \Sigma_i^\varepsilon, \\ & \varphi_+^\lambda(i\varepsilon, x) = z^\lambda(\varepsilon, \varphi_-^\lambda(i\varepsilon, x)), \end{aligned} \tag{4.4}$$

where z^λ is the solution of

$$\begin{aligned} & (z^\lambda(s))' + \frac{1}{2a}(z^\lambda(s)+1)^3 = 0 \quad s \in [0,T], \\ & z^\lambda(0) = \varphi_-^\lambda(i\varepsilon, x), \end{aligned} \tag{4.5}$$

computed at $s = \varepsilon$, for $i = \overline{0, M_\varepsilon - 1}$.

By (4.1) we get

$$\begin{aligned} & \tilde{u}^\varepsilon = \lim_{\lambda\to 0} \frac{u^\lambda - u^\varepsilon}{\lambda}, \quad \tilde{\varphi}^\varepsilon = \lim_{\lambda\to 0} \frac{\varphi^\lambda - \varphi^\varepsilon}{\lambda}, \\ & \tilde{w} = \lim_{\lambda\to 0} \frac{w^\lambda - w}{\lambda} \quad \tilde{z} = \lim_{\lambda\to 0} \frac{z^\lambda - z}{\lambda} \quad \tilde{v} = \lim_{\lambda\to 0} \frac{v^\lambda - v}{\lambda}, \end{aligned} \tag{4.6}$$

$$\begin{aligned} \eta &= \lim_{\lambda\to 0} \frac{z^\lambda(\varepsilon, \varphi_-^\lambda(i\varepsilon, x)) - z(\varepsilon, \varphi_-^\varepsilon(i\varepsilon, x))}{\lambda} \\ &= z(\varepsilon, \varphi_-^\varepsilon(i\varepsilon, x))\tilde{\varphi}_-^\varepsilon(i\varepsilon, x) + \tilde{z}(\varepsilon, \varphi_-^\varepsilon(i\varepsilon, x)). \end{aligned} \tag{4.7}$$

Subtracting now (1.5)–(1.8) from (4.2)–(4.5), letting λ tend to zero and taking into account (4.6), (4.7), we obtain the system in variations

$$\begin{aligned} & \tilde{u}_t^\varepsilon + \frac{\ell}{2}\tilde{\varphi}_t^\varepsilon = k\Delta\tilde{u}^\varepsilon && \text{in } Q_i^\varepsilon, \\ & \frac{\partial \tilde{u}^\varepsilon}{\partial \nu} + h\tilde{u}^\varepsilon = \tilde{w}(t)g(x) && \text{on } \Sigma_i^\varepsilon, \\ & \tilde{u}^\varepsilon(0,x) = 0 && x \in \Omega, \end{aligned} \tag{4.8}$$

$$\tilde{w}'(t) = \beta\tilde{w}(t) + \tilde{v}(t) \quad t \in [0,T], \tag{4.9}$$

$$\begin{aligned} &\tau\tilde{\varphi}^\varepsilon_t = \xi^2\Delta\tilde{\varphi}^\varepsilon + \frac{1}{2a}\tilde{\varphi}^\varepsilon + 2\tilde{u}^\varepsilon && \text{in } Q_i^\varepsilon, \\ &\tilde{\varphi}^\varepsilon = 0 && \text{on } \Sigma_i^\varepsilon, \\ &\tilde{\varphi}^\varepsilon_+(i\varepsilon, x) = \eta(\varepsilon) && x \in \Omega, \end{aligned} \tag{4.10}$$

for $i = \overline{0, M_\varepsilon - 1}$, where $\eta(\varepsilon)$ is the solution

$$\begin{aligned} &\eta'(s) + \frac{3}{2a}(z+1)^2\eta(s) = 0 && s \in [0,T], \\ &\eta(0) = \tilde{\varphi}^\varepsilon_-(i\varepsilon, x), && \tilde{\varphi}^\varepsilon_-(0,x) = 0. \end{aligned} \tag{4.11}$$

By (4.11) we have

$$\eta(\varepsilon) = \exp\left(-\int_0^\varepsilon \frac{3}{2a}(z(t,\cdot)+1)^2 dt\right)\tilde{\varphi}^\varepsilon_-(i\varepsilon, x) \tag{4.12}$$

and then (4.10_3) is equivalent to

$$\tilde{\varphi}^\varepsilon_-(i\varepsilon, x) = \exp\left(\int_0^\varepsilon \frac{3}{2a}(z(t,\cdot)+1)^2 dt\right)\tilde{\varphi}^\varepsilon_+(i\varepsilon, x). \tag{4.13}$$

We now introduce the adjoint state system.

The equations (4.8_1), (4.10_1) can be written in the form

$$\frac{\partial}{\partial t}\begin{pmatrix}\tilde{u}^\varepsilon \\ \tilde{\varphi}^\varepsilon\end{pmatrix} = A\begin{pmatrix}\tilde{u}^\varepsilon \\ \tilde{\varphi}^\varepsilon\end{pmatrix} \text{ in } Q_i^\varepsilon$$

where

$$A = \begin{pmatrix} k\Delta - \frac{\ell}{\tau} & -\frac{\ell\xi^2}{2\tau}\Delta - \frac{\ell}{4a\tau} \\ \frac{2}{\tau} & \frac{\xi^2}{\tau}\Delta + \frac{1}{2a\tau} \end{pmatrix}.$$

Then

$$A^* = \begin{pmatrix} k\Delta - \frac{\ell}{\tau} & \frac{2}{\tau} - \frac{\ell\xi^2}{2\tau}\Delta - \frac{\ell}{4a\tau} & \frac{\xi^2}{\tau}\Delta + \frac{1}{2a\tau} \end{pmatrix},$$

and the adjoint state system is

$$\frac{\partial}{\partial t}\begin{pmatrix}p \\ q\end{pmatrix} = -A^*\begin{pmatrix}p \\ q\end{pmatrix} + \chi_0\begin{pmatrix}0 \\ \varphi^\varepsilon\end{pmatrix}, \tag{4.14}$$

i.e. the optimality conditions for problem (P^ε) can be written as

$$\begin{aligned} &p_t + k\Delta p - \frac{\ell}{\tau}p + \frac{2}{\tau}q = 0 && \text{in } Q_i^\varepsilon, \\ &\frac{\partial p}{\partial \nu} + hp = 0 && \text{on } \Sigma_i^\varepsilon, \end{aligned} \tag{4.15}$$

$$
\begin{aligned}
& q_t - \frac{\ell\xi^2}{2\tau}\Delta p - \frac{\ell}{4a\tau}p + \frac{\xi^2}{\tau}\Delta q + \frac{1}{2a\tau}q = \chi_0\varphi^\varepsilon && \text{in } Q_i^\varepsilon, \\
& q = \frac{\ell}{2}p && \text{on } \Sigma_i^\varepsilon, \\
& q^-((i+1)\varepsilon,\cdot) \\
& = \exp\Big(-\int_0^\varepsilon \frac{3}{2a}(z(t,\cdot)+1)^2 dt\Big) q^+((i+1)\varepsilon,\cdot) && x \in \Omega, \\
& q^-(T,\cdot) = 0 && x \in \Omega,
\end{aligned}
\tag{4.16}
$$

for $i = M_\varepsilon - 2, M_\varepsilon - 3, \ldots, 1, 0$, where $z(t,\cdot)$ is the solution of (1.8).

Let us introduce the cost functional

$$
J^\varepsilon(v) = \frac{1}{2}\int_Q \chi_0(\varphi_\varepsilon^v(t))^2 dxdt + \frac{1}{2}\int_0^T w^2(t)dt + I_{\mathcal{U}}(v), \tag{4.17}
$$

where $I_{\mathcal{U}}(v)$ is the indicator function of the set $\mathcal{U}$. If v_ε^* is an optimal controller for problem (P^ε) then

$$
\frac{J^\varepsilon(v_\varepsilon^* + \lambda\tilde{v}) - J^\varepsilon(v_\varepsilon^*)}{\lambda} \geq 0, \quad \forall\, \lambda > 0.
$$

Thus, letting λ tend to zero we get

$$
\int_{Q_0} \varphi^\varepsilon\tilde{\varphi}^\varepsilon dxdt + \int_0^T w\tilde{w}dt + I'_{\mathcal{U}}(v_\varepsilon^*, \tilde{v}) \geq 0, \ \forall\, \tilde{v} \in T_{\mathcal{U}}(v_\varepsilon^*), \tag{4.18}
$$

where $Q_0 = [0,T] \times \Omega_0$.

Multiplying (4.15_1) by $\tilde{u}^\varepsilon$ and (4.16_1) by $\tilde{\varphi}^\varepsilon$, using integration by parts and Green's formula, yields

$$
\begin{aligned}
& \int_{Q_i^\varepsilon} p_t\tilde{u}^\varepsilon dxdt + k\int_{Q_i^\varepsilon} p\Delta\tilde{u}^\varepsilon dxdt - \frac{\ell}{\tau}\int_{Q_i^\varepsilon} p\tilde{u}^\varepsilon dxdt \\
& + \frac{2}{\tau}\int_{Q_i^\varepsilon} q\tilde{u}^\varepsilon dxdt + k\int_{\Sigma_i^\varepsilon}\Big(\frac{\partial p}{\partial\nu}\tilde{u}^\varepsilon - p\frac{\partial\tilde{u}^\varepsilon}{\partial\nu}\Big)dxdt = 0;
\end{aligned}
\tag{4.19}
$$

$$
\begin{aligned}
& \int_{Q_i^\varepsilon} q_t\tilde{\varphi}^\varepsilon dxdt + \frac{\xi^2}{\tau}\int_{Q_i^\varepsilon} q\Delta\tilde{\varphi}^\varepsilon dxdt + \frac{\xi^2}{\tau}\int_{\Sigma_i^\varepsilon}\Big(\frac{\partial q}{\partial\nu}\tilde{\varphi}^\varepsilon - q\frac{\partial\tilde{\varphi}^\varepsilon}{\partial\nu}\Big)dxdt \\
& + \frac{1}{2a}\int_{Q_i^\varepsilon} q\tilde{\varphi}^\varepsilon dxdt - \frac{\ell\xi^2}{2\tau}\int_{Q_i^\varepsilon} p\Delta\tilde{\varphi}^\varepsilon dxdt - \frac{\ell\xi^2}{2\tau}\int_{\Sigma_i^\varepsilon}\Big(\frac{\partial p}{\partial\nu}\tilde{\varphi}^\varepsilon - p\frac{\partial\tilde{\varphi}^\varepsilon}{\partial\nu}\Big)dxdt \\
& - \frac{\ell}{4a\tau}\int_{Q_i^\varepsilon} p\tilde{\varphi}^\varepsilon dxdt = \int_{Q_i^\varepsilon} \varphi^\varepsilon\tilde{\varphi}^\varepsilon dxdt.
\end{aligned}
\tag{4.20}
$$

Now we multiply (4.8_2) by p, (4.15_2) by $\tilde{u}^\varepsilon$ and, by subtraction, we get

Adding (4.19)-(4.20) and taking into account (4.10_2), (4.16_2), (4.21), we obtain, after some calculations

$$\int_{Q_i^\varepsilon} p_t\tilde{u}^\varepsilon dxdt + \int_{Q_i^\varepsilon} q_t\tilde{\varphi}^\varepsilon dxdt$$
$$+\int_{Q_i^\varepsilon} p\left[k\Delta\tilde{u}^\varepsilon - \frac{\ell}{\tau}\tilde{u}^\varepsilon - \frac{\ell\xi^2}{2\tau}\Delta\tilde{\varphi}^\varepsilon - \frac{\ell}{4a\tau}\tilde{\varphi}^\varepsilon\right]dxdt$$
$$+\int_{Q_i^\varepsilon} q\left[\frac{\xi^2}{\tau}\Delta\tilde{\varphi}^\varepsilon + \frac{1}{2a\tau}\tilde{\varphi}^\varepsilon + \frac{2}{\tau}\tilde{u}^\varepsilon\right]dxdt - k\int_{\Sigma_i^\varepsilon} p\tilde{w}(t)g(x)dxdt$$
$$= \int_{Q_i^\varepsilon} \varphi^\varepsilon\tilde{\varphi}^\varepsilon dxdt$$

i.e., taking into account (4.8_1), (4.10_1),

$$\int_{Q_i^\varepsilon} p_t\tilde{u}^\varepsilon dxdt + \int_{Q_i^\varepsilon} q_t\tilde{\varphi}^\varepsilon dxdt + \int_{Q_i^\varepsilon} p\tilde{u}_t^\varepsilon dxdt + \int_{Q_i^\varepsilon} q\tilde{\varphi}_t^\varepsilon dxdt -$$
$$-k\int_{\Sigma_i^\varepsilon} p\tilde{w}(t)g(x)dxdt = \int_{Q_i^\varepsilon} \varphi^\varepsilon\tilde{\varphi}^\varepsilon dxdt.$$

By Fubini's theorem and definition of distributional derivative, the latter implies

$$-k\int_{\Sigma_i^\varepsilon} p\tilde{w}(t)g(x)dxdt = \int_{Q_i^\varepsilon} \varphi^\varepsilon\tilde{\varphi}^\varepsilon dxdt$$

and then (4.18) becomes

$$\int_\Sigma [-kpg(x) + w(t)]\tilde{w}(t)dxdt + I'_{\mathcal{U}}(v_\varepsilon^*, \tilde{v}) \geq 0 \quad \forall\, \tilde{v} \in T_{\mathcal{U}}(v_\varepsilon^*), \tag{4.22}$$

where $\tilde{w}(t) = \int_0^t e^{\beta(t-s)}\tilde{v}(s)ds$. From (4.22) we get

$$\int_0^T r(t)\tilde{v}(t)dt + I'_{\mathcal{U}}(v_\varepsilon^*, \tilde{v}) \geq 0 \quad \forall\, \tilde{v} \in T_{\mathcal{U}}(v_\varepsilon^*), \tag{4.23}$$

where

$$r(t) = \int_t^T \left(\int_{\partial\Omega} (w(s) - kp(s,x)g(x))dx\right)e^{\beta(s-t)}ds.$$

From (4.23) we obtain $-r(t) \in \partial I_{\mathcal{U}}(v_\varepsilon^*)$, a.e. $t \in [0,T]$, where ∂ denotes the subdifferential, and

$$v_\varepsilon^*(t) = \begin{cases} R, & \text{if } r(t) < 0 \\ 0, & \text{if } r(t) > 0. \end{cases} \tag{4.24}$$

Summarizing, we have proved the following maximum principle for problem (P^ε)

THEOREM 4.1 *Let* $(u_\varepsilon^*, \varphi_\varepsilon^*, w_\varepsilon^*, v_\varepsilon^*)$ *be optimal in problem* (P^ε). *Then the optimal control is given by* (4.24) *where* (p, q) *satisfy along with* $u_\varepsilon^*, \varphi_\varepsilon^*$ *the dual system (4.15)-(4.16).*

5. A numerical algorithm and a numerical example

The aim of this section is to give a conceptual algorithm in order to compute the approximating suboptimal control of problem (P) stated in the first section, i.e. (see Theorem 3.1), to compute the approximating optimal control of problem (P^ε) given by (4.24). For simplicity, we have proposed a gradient type method in this sense (see also Barbu, 1988, and Moroşanu, 1993). For a much better and faster algorithm in this area we refer to Sachs (1994).

Algorithm CPHT-2D (**C**ontrol **PH**ase **T**ransition-**2D**)

P0. Choose $v^{\varepsilon,(0)} \in \mathcal{U}$; set $iter := 0$.
P1. Compute $w^{\varepsilon,(iter)}$ from (1.6), i.e.
$$w_t^{\varepsilon,(iter)} = \beta w^{\varepsilon,(iter)} + v^{\varepsilon,(iter)} \text{ on } [0,T], \quad w^{\varepsilon,(iter)}(0) = 0;$$
P2. Compute $(u^{\varepsilon,(iter)}, \varphi^{\varepsilon,(iter)})$ from (1.5), (1.7), (1.8), i.e.

P2.1. Compute z from (1.8);

P2.2. Compute $(u^{\varepsilon,(iter)}, \varphi^{\varepsilon,(iter)})$ solving the linear system (1.5), (1.7);

P3. Compute $(p^{\varepsilon,(iter)}, q^{\varepsilon,(iter)})$ from (4.15)-(4.16);
P4. For $t \in [0,T]$, compute $r^{\varepsilon,(iter)}(t)$ and $\tilde{v}^{\varepsilon,(iter)}(t)$ given by
$$r^{\varepsilon,(iter)}(t) = \int_t^T \left(\int_{\partial\Omega} (w^{\varepsilon,(iter)}(s) - kp^{\varepsilon,(iter)}(s,x)g(x))dx \right) \exp(\beta(s-t))ds,$$
$$\tilde{v}^{\varepsilon,(iter)}(t) = \begin{cases} R, & \text{if } r^{\varepsilon,(iter)} < 0, \\ 0, & \text{if } r^{\varepsilon,(iter)} > 0. \end{cases}$$
P5. Compute $\lambda_{iter} \in [0,1]$ - the steplenght of the gradient method, solution of the minimization process
$$\min\{j^\varepsilon(\lambda v^{\varepsilon,(iter)} + (1-\lambda)\tilde{v}^{\varepsilon,(iter)}), \quad \lambda \in [0,1]\}.$$
Set $v^{\varepsilon,(iter+1)} := \lambda_{iter} v^{\varepsilon,(iter)} + (1-\lambda_{iter})\tilde{v}^{\varepsilon,(iter)}$.
P6. (the "*Stopping Criterion*")
if $\|v^{\varepsilon,(iter+1)} - v^{\varepsilon,(iter)}\| \le \eta$
then STOP (the algorithm is convergent)
else $iter := iter + 1$; Go to **P1**.

REMARK 5.1 *The "Stopping Criterion" in* **P6** *may be also*

$$\|j^\varepsilon(v^{\varepsilon,(iter+1)}) - j^\varepsilon(v^{\varepsilon,(iter)})\| \le \eta$$

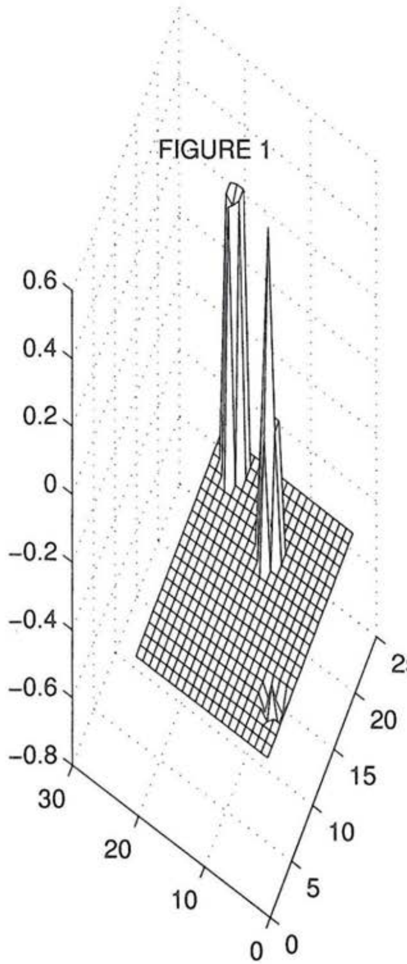


Figure 1. The initial value $u_0(x, y)$

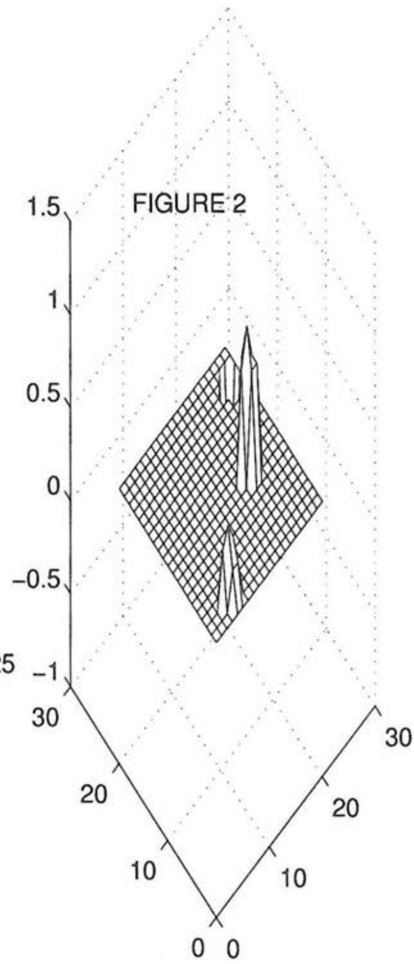


Figure 2. The initial value $\varphi_0(x, y)$

Let us briefly discuss the main steps in algorithm **CPHT-2D**. For approximating the solution of the nonlinear parabolic system (1.1)–(1.3), we have used a numerical method of fractional steps type (see Moroşanu, 1997). This method, expressed in step **P2**, avoids the iterative process required by the classical approaches (e.g., Newton's type method) in passing from a time level to another (see also Moroşanu, 1997 for additional details). Moreover, we point out that the equation (1.8) in **P2.1** can be solved directly, by separation of variables. The values of λ_{iter} from **P5** are chosen from the sequence

$$1 - d, \quad 1 - 2 * d, \quad 1 - 3 * d, \ldots, 0$$

have a finite number of options for λ, we choose $\lambda_{iter} = \lambda$, the value from the above sequence which minimizes

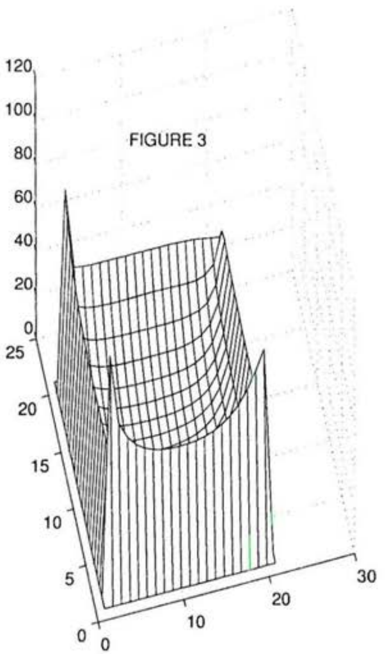
$$j^{\varepsilon}(\lambda v^{\varepsilon,(iter)} + (1-\lambda)\tilde{v}^{\varepsilon,(iter)}), \quad \lambda \in [0,1].$$

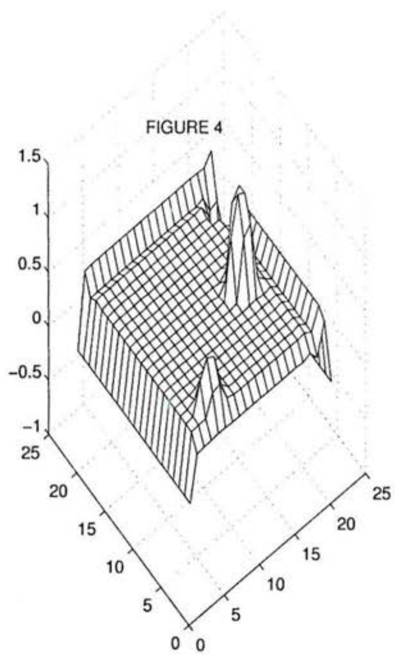
Next, for completenes, we ilustrate our algorithm by a concrete numerical computation. For this, let the domain $\Omega = \Omega_0$ be $[0,1] \times [0,1] \subset \mathrm{R}^2$ and the time interval be $[0,2]$. For the time as well as spatial discretization we use uniform discretizations. The time step chosen was $\varepsilon = 0.4$ (i.e. $M_\varepsilon = 5$). The triangulation of the spatial domain is obtained by dividing x- and y- axis into equidistant subintervals of length $dx = dy = 0.048$ and then dividing each of the resulting subsequares into two triangles. The values of parameters are: $\xi = 0.5$, $a = \sqrt{\xi}$, $\tau = 1.0e + 5 * \xi^2$, $l = 0.6$, $k = 0.06$, $h = 1.2$, $\beta = 1$.

We shall present now the numerical example. Figs. 1 and 2 show the initial conditions. We choose $R = 200$ and $v_i^{\varepsilon,(0)} = R/2$, $i = 1, 2, \ldots, M_\varepsilon$. In Table 1 we present the results obtained:

iter	$j^{\varepsilon}(v_\varepsilon)$
1	4.162858e+03
3	2.091130e+03
5	8.291100e+02

Table 1.





The optimal value $j^{\varepsilon}(v_{\varepsilon}^{*}) = j(v^{*}) = 5.127619e+01$ was obtained in 8 iterations. The corresponding approximating optimal control is

$$v_{\varepsilon}^{*} = (9.739e-01, 9.739e-01, 9.739e-01, 9.739e-01, 9.739e-01).$$

So, the reader easily can appreciate that the algorithm **CPHT-2D** is functional and, in consequence, may be implemented on a computer with higher performance, permiting the use of finer discretization.

Figures 3 and 4 show the approximate solution u_{ε}^{*} and $\varphi_{\varepsilon}^{*}$, respectively, corresponding to the approximating optimal control v_{ε}^{*} (see Theorem 4.1).

Finally, we underline that the numerical results do not have in view the physical aspects. They are given only to implement the conceptual algorithm. The comparison between the algorithm **CPHT-2D** and other numerical algorithms for approximation of the optimal control of problem (P), as well as numerical results regarding the physical nature of this problem, are a matter for further investigation.

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