Control and Cybernetics

vol. **32** (2003) No. 1

Discrete approximations of the Hamilton-Jacobi equation for an optimal control problem of a differential-algebraic system

by

J. Frédéric Bonnans[1], Philippe Chartier[2] and Housnaa Zidani[3]

[1] INRIA, B.P. 105, 78153 Rocquencourt, France
E-mail: Frederic.Bonnans@inria.fr.

[2] INRIA, Campus de Beaulieu 35042 Rennes, France
E-mail: Philippe.Chartier@inria.fr

[3] ENSTA, 32 boulevard Victor, 75739 Paris Cx 15,
and Project Sydoco, INRIA, France
E-mail: zidani@ensta.fr

Abstract: This paper discusses the numerical resolution of the Hamilton-Jacobi-Bellman equation associated with optimal control problem when the state equation is of algebraic differential type. We discuss two numerical schemes. The first reduces to the standard framework, while the second does not suppose any knowledge of the Jacobian of the data. We obtain some error estimates, and display numerical results obtained on a simple test problem.

Keywords: optimal control, differential-algebraic system, Hamilton-Jacobi-Bellman equation, dynamic programming, approximation schemes, finite differences, viscosity solutions.

1. Introduction

This paper is devoted to the discussion of numerical methods for solving an optimal control problem of a differential-algebraic dynamic system. As for the the case of optimal control of ordinary differential equations, the idea is based on dynamic programming, that leads to the Hamilton-Jacobi-Bellman (HJB) equation (Bellman, 1961), whose well-posedness can be proved in the viscosity sense, see Crandall and Lions (1983), and also Bardi and Capuzzo-Dolcetta (1997), and Barles (1994). The discretization of the HJB equation is a quite papers for such methods, in the case of optimal control problem of a differential dynamic system, let us mention the historical paper by Kružkov (1966), the analysis of time discretization in Capuzzo Dolcetta (1983) and Capuzzo Dolcetta and Ishii (1984), and the analysis of full discretization by Crandall and Lions (1984). More recent contributions on such algorithms, including the analysis of space discretization, may be found in Bardi and Capuzzo-Dolcetta (1997), Appendix by M. Falcone, Barles and Souganidis (1991), Camilli and Falcone (1999), Falcone (1987), Falcone and Ferretti (2002), and Rouy and Tourin (1992).

In principle, optimal control of differential-algebraic dynamic systems can be reduced to the standard framework of optimal control of an ordinary differential equation. The idea is to extract the algebraic variable from the time derivatives of the algebraic constraint, i.e., to express the algebraic variable as a function of state and control, and of some of their derivatives.

However, this cannot always be done. The main reason is that, often, the dynamics is available, but not its derivatives. This is indeed the main reason for the design of specific numerical schemes for integration of differential-algebraic dynamic system, see e.g. Hairer et al. (1980).

In that case, it may be effective to discretize the problem in a way that is coherent with the spirit of these specialized numerical schemes, except of course for the fact that one aims not to have high order accuracy, since the value function is in general not differentiable.

In this paper we introduce such a method, obtain error estimates, compare the new approach with the idea of reduction to the standard situation, and discuss numerical results for both methods on a simple example.

The paper is organized as follows. Section 2 presents the problem and the main hypotheses. The continuous problem is presented in Section 3, while Section 4 is devoted to the numerical analysis of the state equation. We obtain error estimates, in Section 5 for the discrete time optimal control problem, and in Section 6 for the fully discretized problem. Numerical results are presented in Section 7.

2. Setting of the problem. Preliminary results

Consider the following differential-algebraic dynamic system,

$$\begin{cases} \dot{y}_x(t) = f(y_x(t), z_x(t), u(t)) & t > 0, \\ \alpha_x = g(y_x(t)) & t > 0, \\ y_x(0) = x. \end{cases} \tag{1}$$

Here x stands for the initial condition for the state variable (or differential variable) $y_x(t) \in \mathrm{R}^n$, while $z_x(t) \in \mathrm{R}^q$ is called the algebraic variable; we have denoted $\alpha_x := g(x)$. Observe that we use the notation (y_x, z_x) for the solution initial condition x. The dependence of y_x and z_x with respect to the control variable $u(t) \in \mathrm{R}^m$ is understood and must be clear following the setting of the problem.

Consider the following infinite horizon optimal control problem:

$$V(x) = \inf_{u \in U_{ad}} \int_0^\infty \ell(y_x(t), z_x(t), u(x,t)) e^{-\lambda t}\, dt \quad \text{for } x \in \mathrm{R}^n. \tag{2}$$

Here U_{ad} denotes the set of all continuous and piecewise differentiable functions of $[0, \infty[$ with image in a nonempty compact subset U of R^m, ℓ is a function from $\mathrm{R}^n \times \mathrm{R}^q \times \mathrm{R}^m$ into R, and λ is a positive constant.

We assume that the functions ℓ, f, and g satisfy the following hypotheses:

H1 The function $\ell : \mathrm{R}^n \times \mathrm{R}^q \times \mathrm{R}^m \to \mathrm{R}$ is continuous, and satisfies the following estimate:

$$|\ell(y,z,u) - \ell(y',z',u)| \leq \Lambda_\ell(|y-y'| + |z-z'|), \quad |\ell(y,z,u)| \leq M_\ell,$$

for some $\Lambda_\ell, M_\ell > 0$, and for all $y, y' \in \mathrm{R}^n$, $\mathrm{z}, \mathrm{z}' \in \mathrm{R}^q$, $\mathrm{u} \in \mathrm{U}$.

H2 The function $f : \mathrm{R}^n \times \mathrm{R}^q \times \mathrm{R}^m \to \mathrm{R}^n$ is of class C^1, and satisfies the following estimate:

$$|f(y,z,u) - f(y',z',u)| \leq \Lambda_f(|y-y'| + |z-z'|), \quad |f(y,z,u)| \leq M_f,$$

for some $\Lambda_f, M_f > 0$, and for all $y, y' \in \mathrm{R}^n$, $\mathrm{z}, \mathrm{z}' \in \mathrm{R}^q$, $\mathrm{u} \in \mathrm{U}$.

H3 The function $g : \mathrm{R}^n \to \mathrm{R}^q$ is Lipschitz continuous, with constant $\Lambda_g > 0$.

H4 The $q \times q$ matrix $g'(y) f_z(y,z,u)$ is invertible for all $(y,z,u) \in \mathrm{R} \times \mathrm{R}^q \times \mathrm{U}$, and we have:

$$\left| [g'(y) f_z(y,z,u)]^{-1} \right| \leq M,$$

$$\forall\, (\Delta y_1, \Delta y_2) \in \mathrm{R}^n \times \mathrm{R}^n, \ |g''(y)(\Delta y_1, \Delta y_2)| \leq M^g_{yy} |\Delta y_1| |\Delta y_2|,$$

$$\forall\, \Delta z \in \mathrm{R}^q, \ |f_{zz}(y,z,u)(\Delta z, \Delta z)| \leq M^f_{zz} |\Delta z|^2.$$

for some positive constants M, M^g_{yy} and M^f_{zz} and for every $(y,z,u) \in \mathrm{R}^n \times \mathrm{R}^q \times \mathrm{U}$.

Hypotheses H1 to H3 are natural extensions of those classically used when studying the Hamilton-Jacobi-Bellman (HJB) equation for the value function of an optimal control problem. Computing the time derivative of the algebraic constraint along a trajectory, we obtain what is called the *hidden constraint*

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} g(y_x(t)) = g'(y_x(t)) f(y_x(t), z_x(t), u(t)).$$

Thanks to hypothesis H4, the implicit function theorem implies that (at least locally) we can extract the algebraic variable $z_x(t)$ from this equation or, equivalently, that we can obtain $\dot{z}_x(t)$ by differentiating a second time. This kind of algebraic-differential system is said to be of index 2. For the sake of simplicity, and Lipschitz continuous function Z from $\mathrm{R}^n \times \mathrm{R}^m$ into R^q such that for all $(y,u) \in \mathrm{R}^n \times \mathrm{R}^m$, $z = Z(y,u)$ is the *unique* solution of the hidden constraint:

$$0 = g'(y)f(y,z,u). \tag{3}$$

If the function $Z(y,u)$ can be computed numerically, then we can eliminate the algebraic variable and reduce the problem to the optimal control of an ordinary differential equation. However, it often occurs that f and g are available, but not their derivatives. It is still possible then to integrate numerically the state equation. The main contribution of this paper is to state a numerical scheme for solving the HJB equation associated with the optimal control problem.

3. Study of the continuous problem

Since our hypotheses allow to express the algebraic variable $z(t)$ as a Lipschitz function $Z(y(t),u(t))$, it is convenient to denote the "reduced" dynamics and running cost as

$$F(y,u) := f(y, Z(y,u), u); \quad L(y,u) := \ell(y, Z(y,u), u). \tag{4}$$

We can formulate the continuous optimal control problem as follows:

$$V(x) = \inf_{u \in U_{ad}} \int_0^\infty L(y_x(t), u(t)) e^{-\lambda t}\, dt \quad \text{for } x \in \mathrm{R}^n, \tag{5}$$

where $y_x(t)$ is solution of the ordinary differential equation

$$\begin{cases} \dot y_x(t) = F(y_x(t), u(t)) & t > 0, \\ y_x(0) = x. \end{cases} \tag{6}$$

Since the dynamics and running cost functions are Lipschitz and bounded, it is well known that the state equation has a unique solution in the space of absolutely continuous functions, and that the value function is finite and Hölder continuous as the next lemma tells. Let Λ_0 be defined by

$$\Lambda_0 := \sup \left\{ \frac{\|F(x',u) - F(x,u)\|}{\|x'-x\|}; \quad x' \neq x;\ u \in U \right\}.$$

LEMMA 3.1 *The value function is bounded. In addition, let γ be such that*

$$\begin{cases} \gamma = 1 & if\ \lambda > \Lambda_o; \\ \gamma = \frac{\lambda}{\Lambda_f} & if\ \lambda < \Lambda_o. \end{cases} \tag{7}$$

Then there exist $\Lambda_\gamma > 0$, and $\varepsilon_\gamma > 0$ such that, for all x and x' satisfying $|x'-x| \le \varepsilon_\gamma$, the following holds:

$$|V(x) - V(x')| \le \Lambda |x - x'|^\gamma. \tag{8}$$

Consider the Hamilton-Jacobi-Bellman (HJB) equation

$$\max_{u \in U} \{\lambda W(x) - F(x,u) \cdot D_x W(x) - L(x,u)\} = 0 \quad \forall x \in \mathrm{R}^n, \tag{9}$$

Here $D_x = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n} \right)$ and "$\cdot$" denotes the inner product in R^n.

For convenience, we recall here the definition of viscosity solution, see Crandall and Lions (1983).

DEFINITION 3.1 *A continuous function W on R^n is called a viscosity solution of (9) if, for every $\varphi \in C^1(\mathrm{R}^n)$, the following holds:*

(i) *If x_1 is a local maximum point of $W - \varphi$, then*

$$\max_{u \in U} \{\lambda W(x_1) - F(x_1,u) \cdot D_x \varphi(x_1) - L(x_1,u)\} \leq 0.$$

(ii) *If x_2 is a local minimum point of $W - \varphi$, then*

$$\max_{u \in U} \{\lambda W(x_2) - F(x_2,u) \cdot D_x \varphi(x_2) - L(x_2,u)\} \geq 0.$$

REMARK 3.1 *In fact, the boundedness and Lipschitz continuity of F and L imply (see, e.g., Barles, 1994) that the value function V is the unique bounded and uniformly continuous viscosity solution of the HJB equation.*

4. Numerical analysis

Let us briefly discuss two standard first-order schemes for solving the state equation. In each of these schemes, the control variable is supposed to be constant during the time step $h > 0$. Since we consider one step methods, it is sufficient to state the formula for computing the state after the first step. So, let us fix $u_0 \in U_{ad}$ and an initial condition $x \in \mathrm{R}^n$. The two schemes are

$$\begin{cases} y_x^h = x + hf(x, z_x^h, u_0) \\ 0 = g'(x) f(x, z_x^h, u_0) \end{cases} \tag{10}$$

on the one hand, and

$$\begin{cases} y_x^h = x + hf(x, z_x^h, u_0) \\ 0 = g(y_x^h) - g(x) \end{cases} \tag{11}$$

on the other hand. The first scheme is of explicit type while the second one is of implicit type.

In each of these schemes, one has to solve a system of nonlinear equations. In practice, this will mean using a variant of Newton's method, in which the Jacobian may be approximated using finite differences. The functions defining the equations to be solved, however, have to be computed with a good accuracy.

precise evaluation of f and g', whereas the second needs only to evaluate f and g. In other words, if only g is available and not its derivative, then the second scheme can be implemented, while the first cannot.

First scheme. In the first scheme we have that

$$y_x^h = x + hf(x, Z(x, u_0), u_0). \tag{12}$$

THEOREM 4.1 *Let $u_0 \in \mathrm{R}^m, \mathrm{x}_o \in \mathrm{R}^n$ be given and consider $w = Z(x, u_0)$ be the solution of*

$$g'(x)f(x, w, u_0) = 0.$$

For all $h \geq 0$, there exists a unique solution of (10). Moreover, the functions $x \mapsto (y_x^h, z_x^h)$, that map x to the unique solution of (10) *are Lipschitz continuous functions.*

Proof. The proof is classical and therefore omitted. ∎

Second scheme. We now turn to the study of the second scheme.

THEOREM 4.2 (Existence and uniqueness of the solution) *Let $u_0 \in \mathrm{R}^m$ and $x \in \mathrm{R}^n$ be given and consider $w = Z(x, u_0)$ the solution of*

$$g'(x)f(x, w, u_0) = 0.$$

There exist $h_0 > 0$, $\varepsilon > 0$ and $\eta > 0$, all independent of x such that for all $0 < h < h_0$ the system (11) has a unique solution (y_x^h, z_x^h) in the set $B(x, \frac{\varepsilon}{2}) \times B(w, \frac{\eta}{2})$.

Proof. The proof follows closely the ideas of Hairer et al. (1980). Consider the system

$$\left\{ \begin{array}{l} y(\tau) = x + hf(x, z(\tau), u_0) + h(\tau - 1)f(x, w, u_0) \\ 0 = g(y(\tau)) - g(x) \end{array} \right. . \tag{13}$$

For $\tau = 1$, (13) is equivalent to (11), while for $\tau = 0$, it has the obvious solution $y = x$ and $z = w$. Differentiating with respect to τ, we get

$$\left\{ \begin{array}{l} \dot{y} = hf_z(x, z, u_0)\dot{z} + hf(x, w, u_0) \\ 0 = g'(y)f_z(x, z, u_0)\dot{z} + g'(y)f(x, w, u_0) \end{array} \right. . \tag{14}$$

Now, due to H4, it is easily seen that $g'(y)f_z(x, z, u_0)$ is invertible for y in $B(x, \frac{\varepsilon}{2})$ with $\varepsilon = (M.M_{yy}^g)^{-1}$ and that

$$\forall y \in B\left(x, \frac{\varepsilon}{2}\right), \forall z \in \mathrm{R}^q, |(g'(y)f_z(x, z, u_0))^{-1}| \leq 2M$$

Within $B(x, \frac{\varepsilon}{2})$, (14) is equivalent to the differential system

$$\begin{cases} \dot{y} = hP(x,y,z)f(x,w,u_0) \\ \dot{z} = -\left(g'(y)f_z(x,z,u_0)\right)^{-1} g'(y)f(x,w,u_0) \end{cases},$$

where $P(x,y,z)$ is the projection $I - f_z(x,z,u_0)\left(g'(y)f_z(x,z,u_0)\right)^{-1} g'(y)$ and with initial conditions $y(0) = x$ and $z(0) = w$. This system has a unique solution which satisfies

$$|y(\tau) - x| = \left| \int_0^\tau \dot{y}(\zeta)d\zeta \right| \le \tau h(1 + 2\Lambda_f M \Lambda_g)M_f,$$

$$|z(\tau) - w| \le 2M \left| \underbrace{g'(x)f(x,w,u_0)}_{=0} \right.$$

$$\left. + \int_0^1 g''(x + t(y(\tau) - x))(f(x,w,u_0), y(\tau) - x)dt \right|$$

$$\le 2MM^g_{yy}M_f|y(\tau) - x|,$$

and thus remains in $B(x, \frac{\varepsilon}{2}) \times B(x, \frac{\eta}{2})$ for $\tau \le 1$, provided $h < h_0$, where

$$h_0 = \min\left(\frac{\varepsilon}{2(1 + 2\Lambda_f M \Lambda_g)M_f}, \frac{\eta}{4MM^g_{yy}M_f(1 + 2\Lambda_f M \Lambda_g)M_f} \right).$$

In order to prove uniqueness, we consider another solution $(\widetilde{y}, \widetilde{z})$ in $B(x, \frac{\varepsilon}{2}) \times B(w, \frac{\eta}{2})$ of (11). Writing $\Delta y = \widetilde{y} - y$, $\Delta z = \widetilde{z} - z$ and $\Delta f = f(x, \widetilde{z}, u_0) - f(x, z, u_0)$, we then have

$$\begin{aligned} 0 &= g(\widetilde{y}) - g(y) \\ &= g'(y)\Delta y + B^g(y, \widetilde{y})(\Delta y, \Delta y) \\ &= hg'(y)f_z(x,z,u_0)\Delta z + hg'(y)B^f(x,z,\widetilde{z})(\Delta z, \Delta z) + B^g(y,\widetilde{y})(\Delta y, \Delta y), \end{aligned}$$

where

$$B^g(y,\widetilde{y})(\Delta y, \Delta y) = \int_0^1 (1-t)g''(y + t\Delta y)(\Delta y, \Delta y)dt$$

and

$$B^f(x,z,\widetilde{z})(\Delta z, \Delta z) = \int_0^1 (1-t)f_{zz}(x, z + t\Delta z, u_0)(\Delta z, \Delta z)dt.$$

It follows that

$$\Delta z = (-g'(y)f_z(x,z,u_0))^{-1}(g'(y)B^f(x,z,\widetilde{z})(\Delta z, \Delta z)$$
$$+ hB^g(y,\widetilde{y})(\Delta f, \Delta f)).$$

We consequently get the estimate

Now, provided η is sufficiently small, the constant $M(\Lambda_g M^f_{zz} + H\Lambda_f^2 M^g_{yy})\eta$ is smaller than 1 and we necessarily have $\Delta z = 0$. The fact that $\Delta y = 0$ follows straightforwardly. ■

THEOREM 4.3 *For any $u_0 \in \mathrm{R}^m$ fixed and for $0 < h < h_0/2$ the functions*

$$x \mapsto y^h_x \text{ and } x \mapsto z^h_x \tag{15}$$

that map x to the unique solution of (11) *lying in $B(x, \frac{\varepsilon}{4}) \times B(Z(x, u_0), \frac{\eta}{4})$ are Lipschitz continuous function.*

Proof. Given x and $\widehat{x}$ in R^n, let $\Gamma(\tau) = \tau x + (1-\tau)\widehat{x}$ and consider the system

$$\left\{ \begin{array}{l} y = \Gamma(\tau) + hf(\Gamma(\tau), z, u_0) \\ 0 = g(y) - g(\Gamma(\tau)) \end{array} \right. , \tag{16}$$

whose solution is (y^h_x, z^h_x) for $\tau = 1$ and $(y^h_{\widehat{x}}, z^h_{\widehat{x}})$ for $\tau = 0$. As in previous theorem, differentiation with respect to τ leads to

$$\left\{ \begin{array}{l} \dot{y} = (x - \widehat{x}) + hf_y(\Gamma, z, u_0)(x - \widehat{x}) + hf_z(\Gamma, z, u_0)\dot{z} \\ 0 = \frac{1}{h}\left(g'(y) - g'(\Gamma) + hg'(y)f_y(\Gamma, z, u_0)\right)(x - \widehat{x}) + g'(y)f_z(\Gamma, z, u_0)\dot{z} \end{array} \right.$$

with initial conditions $y(0) = y_{\widehat{x}}$ and $z(0) = z^h_{\widehat{x}}$. Within the set

$$\mathcal{E} = \cup_{\tau \in [0,1]} B\left(\Gamma(\tau), \frac{\varepsilon}{2}\right),$$

the previous system is equivalent to the differential equation

$$\left\{ \begin{array}{l} \dot{y} = (x - \widehat{x}) + hf_y(\Gamma, z, u_0)(x - \widehat{x}) + hf_z(\Gamma, z, u_0)\dot{z} \\ \dot{z} = -(g'(y)f_z(\Gamma, z, u_0))^{-1}(\frac{1}{h}(g'(y) - g'(\Gamma) + hg'(y)f_y(\Gamma, z, u_0))(x - \widehat{x})) \end{array} \right.$$

with the same initial conditions as before. It has a unique solution for $\tau < \tau^*$ which satisfies

$$|y(\tau) - \Gamma(\tau)| \leq hM_f \leq \frac{\varepsilon}{2}$$

and thus can be extended up to $\tau = 1$. Now, we have

$$|z(1) - z^h_{\widehat{x}}| = \left| \int_0^1 \dot{z}(\zeta) d\zeta \right| \leq 2M \left(\frac{1}{h} M^g_{yy} \sup_{\tau \in [0,1]} |y - \Gamma| + \Lambda_g \Lambda_f \right) |x - \widehat{x}|,$$
$$\leq 2M(M^g_{yy} M_f + \Lambda_g \Lambda_f)|x - \widehat{x}|.$$

From this estimate, we then get

$$|z(1) - Z(x, u_0)| = |z(1) - z^h_{\widehat{x}} + z^h_{\widehat{x}} - Z(\widehat{x}, u_0) + Z(\widehat{x}, u_0) - Z(x, u_0)|$$
$$< |\underbrace{2M(M^g_{yy} M_f + \Lambda_g \Lambda_f)}_{K}|x - \widehat{x}| + \frac{\eta}{4} + L|x - \widehat{x}|$$

η

It follows that for $|x-\widehat{x}| \leq \frac{\eta}{4(K+L)}$, $|z(1)-Z(x,u_0)| < \frac{\eta}{2}$, so that $z(1)=z_x^h$ and

$$|z_x^h - z_{\widehat{x}}^h| \leq K|x-\widehat{x}|.$$

Now, if $|x-\widehat{x}| > \frac{\eta}{4(K+L)}$, we have

$$\begin{aligned}
&|z_x^h - z_{\widehat{x}}^h| \leq |z_x^h - Z(x,u_0) + Z(x,u_0) - Z(\widehat{x},u_0) + Z(\widehat{x},u_0) - z_{\widehat{x}}^h| \\
&\leq \eta + L|x-\widehat{x}| \\
&\leq 4(K+L)|x-\widehat{x}| + L|x-\widehat{x}| \\
&\leq (4K+5L)|x-\widehat{x}|.
\end{aligned}$$

The function $x \mapsto z_x^h$ is consequently Lipschitz continuous and so is $x \mapsto y_x^h$ in an obvious manner. ■

5. Analysis of the discrete-time optimal control problem

5.1. Case of reduction to the standard framework

Let h be a positive number, and consider the value function:

$$V_h^1(x) = \inf_{u \in U_{ad}^h} J_h(y_x^h, z_x^h, u_h); \quad x \in \mathrm{R}^n, \tag{17}$$

where U_{ad}^h denotes the subset of U_{ad} consisting of all controls u_h which take constant values u^k on each interval $[kh,(k+1)h[$, $k \in \mathrm{N}$, (y_x^h, z_x^h) denotes the sequence determined by the recursion

$$\begin{cases} y^{k+1} = y^k + hF(y^k,u^k) & k=0,1,2,\dots, \\ z_h^k = Z(y^k,u^k) & k=0,1,2,\dots, \\ y^0 = x, \end{cases} \tag{18}$$

and the cost function J_h is given by:

$$J_h(y,u) = h\sum_{i=0}^{\infty}(1+\lambda h)^{-(k+1)}L(y^k,u^k).$$

Then we have that the following dynamic principle holds:

$$\max_{u \in U}\{(1+\lambda h)V_h^1(x) - V_h^1(x+hF(x,u)) - h\,L(x,u)\} = 0. \tag{19}$$

As in Capuzzo Dolcetta and Ishii (1984), we prove that if $h < 1/\lambda$ then (19) has a unique bounded continuous solution V_h^1 and that $\{V_h^1\}$ converges locally uniformly in R^n, as h tends to 0, to the unique bounded uniformly continuous viscosity solution of (9). Equation (19) stands for an approximate problem of (9), and the following theorem follows from Capuzzo Dolcetta and Ishii (1984).

THEOREM 5.1 *Assume that H1-H3 hold. Let V and V_h^1 be the solutions of (9) and (19), respectively. Let $\gamma \in]0,1[$ be a Hölder exponent of V. Then*

$$\sup_{x\in \mathrm{R}^n} |V(x) - V_h^1(x)| \le Ch^{\gamma/2}, \tag{20}$$

for all h small enough, where $C > 0$ is independent of h.

5.2. Second scheme

In this section we discuss a discrete time optimal control problem associated with the second scheme (11).

Let h be a positive number, and consider the value function:

$$\widetilde{V}_h(x) = \inf_{u\in U_{ad}^h} J_h(y_x^h, z_x^h, u_h); \quad x \in \mathrm{R}^n, \tag{21}$$

where U_{ad}^h denotes the subset of U_{ad} consisting of all controls u_h which take constant values u^k on each interval $[kh,(k+1)h[$, for all $k \in \mathrm{N}$, and (y_x^h, z_x^h) denotes the sequence determined by the recursion

$$\begin{cases} y^{k+1} = y^k + hf(y^k, z^k, u^k) & k = 0,1,2,\dots, \\ y^0 = x \\ g(y^{k+1}) - g(y^k) = 0 & k = 0,1,2,\dots. \end{cases} \tag{22}$$

Here z^k can be expressed, as we have seen, as a function of y^k and u^k, that we denote $\widetilde{z}(y^k,u^k)$, and the cost function J_h is given by:

$$J_h(y_x^h, z_x^h, u_h) = h\sum_{k=0}^{\infty}(1+\lambda h)^{-(k+1)}\ell(y^k,z^k,u^k).$$

Observe that for every $(x,u) \in \mathrm{R}^n \times \mathrm{R}^m$, $z = \widetilde{z}(x,u)$ is the solution of the nonlinear system

$$g(x + hf(x,z,u)) - g(x) = 0,$$

then

$$\widetilde{z}(x,y) = Z(x,u) + O(h), \tag{23}$$

and consequently

$$f(x,\widetilde{z}(x,u),u) - f(x,Z(x,u),u) = O(h); \tag{24}$$

$$\ell(x,\widetilde{z}(x,u),u) - \ell(x,Z(x,u),u) = O(h). \tag{25}$$

So, it is convenient to consider the following abstract framework: consider two functions $F^h : \mathrm{R}^n \times \mathrm{R}^m \to \mathrm{R}^n$ and $L^h : \mathrm{R}^n \times \mathrm{R}^m \to \mathrm{R}$ and satisfying

$$F^h(x,u) - F(x,u) = O(h); \tag{26}$$

The state equation and cost function are, respectively,

$$\begin{cases} y^{k+1} = y^k + hF^h(y^k, u^k) \quad k = 0, 1, 2, \dots, \\ y^0 = x \end{cases} \tag{28}$$

$$\widetilde{V}_h(x) = \inf_{u \in U^h_{ad}} h \sum_{k=0}^{\infty} (1 + \lambda h)^{-(k+1)} L^h(y^k, u^k). \tag{29}$$

The corresponding dynamic principle is

$$\max_{u \in U} \left\{ (1 + \lambda h)\widetilde{V}_h(x) - \widetilde{V}_h(x + hF^h(x, u)) - hL^h(x, u) \right\} = 0. \tag{30}$$

LEMMA 5.1 *The solution of (30) satisfies*

$$|\widetilde{V}_h(x)| \le M_o, \quad |\widetilde{V}_h(x) - \widetilde{V}_h(x')| \le \Lambda_o |x - x'|^\gamma \tag{31}$$

for all $x, x' \in \mathrm{R}^n$, $h \in]0, 1/\lambda[$, *for some* $M_o > 0$ *independent of* h, *and where* γ *and* $\Lambda_o > 0$ *are as in Lemma 3.1.*

The following theorem holds.

THEOREM 5.2 *Assume that H1-H3 hold. Let* V *and* $\widetilde{V}_h$ *be the solutions of (9) and (30), respectively. Let* $\gamma \in]0, 1[$ *be a Hölder exponent of* V. *Then*

$$\sup_{x \in \mathrm{R}^n} |V(x) - \widetilde{V}_h(x)| \le C h^{\gamma/2}, \tag{32}$$

for all $h \in]0, 1/\lambda[$, *where* $C > 0$ *is independent of* h.

Proof. We adapt to our case the proof from Capuzzo Dolcetta and Ishii (1984). Given $0 < \varepsilon < 1$, set

$$\beta_\varepsilon(x) := -\varepsilon^{-2}|x|^2 \quad \text{for } x \in \mathrm{R}^n. \tag{33}$$

Define

$$\varphi(x, y) := \widetilde{V}_h(x) - V(y) + \beta_\varepsilon(x - y) \quad \text{for } (x, y) \in \mathrm{R}^n \times \mathrm{R}^n.$$

Let $\alpha \in (0, 1)$. Since V and $\widetilde{V}_h$ are bounded on R^n, there exists (x_1, y_1) in R^{2n} such that

$$\varphi(x_1, y_1) > \sup \varphi - \alpha.$$

Choose $\xi \in C_0^\infty(\mathrm{R}^{2n})$ so that and set

$$\psi(x,y) = \varphi(x,y) + \alpha\xi(x,y) \quad \text{for } (x,y) \in \mathrm{R}^{2n}.$$

Clearly, ψ attains its maximum value at some point (x_o, y_o) in the support of ξ. That is,

$$\psi(x_o,y_o) \geq \psi(x,y) \quad \text{for all } (x,y) \in \mathrm{R}^{2n}. \tag{34}$$

In particular, $y \longmapsto -\psi(x_o, y)$ attains its minimum at y_o. Therefore, by the definition of a viscosity solution of (9), there exists $u^* \in U$ such that

$$\lambda V(y_o) + F(y_o,u^*).\,(D\beta_\varepsilon(x_o - y_o) - \alpha D_y\xi(x_o,y_o)) - L(y_o,u^*) \geq 0. \tag{35}$$

Using (34) with $x = x_0 + hF^h(x_0,u^*)$ and $y = y_o$, we obtain

$$\begin{aligned}&\widetilde{V}_h(x_o + hF^h(x_0,u^*)) - \widetilde{V}_h(x_0) \leq \beta_\varepsilon(x_0 - y_0) + \alpha\xi(x_0,y_0)\\ &- \beta_\varepsilon(x_0 + hF^h(x_0,u^*) - y_0) - \alpha\xi(x_0 + hF^h(x_0,u^*)).\end{aligned}$$

Adding (30) to the previous inequality and using the definitions and properties of β_ε and ξ, we get

$$\begin{aligned}&\lambda h\widetilde{V}_h(x_0) \leq hL^h(x_0,u^*) + \beta_\varepsilon(x_0 - y_0) + \alpha\xi(x_0,y_0)\\ &- \beta_\varepsilon(x_0 + hF^h(x_0,u^*) - y_0) - \alpha\xi(x_0 + hF^h(x_0,u^*))\\ &\leq hL^h(x_0,u^*) - h\beta'_\varepsilon(x_0 - y_0)F^h(x_0,u^*) - \varepsilon^{-2}h^2|F^h(x_0,u^*)|^2\\ &+ \alpha h|F^h(x_0,u^*)|.\end{aligned}$$

Dividing by h and subtracting (35) leads to

$$\widetilde{V}_h(x_o) - V(y_o) \leq L^h(x_0,u^*) - L(y_0,u^*) \tag{36}$$

$$+ \beta'_\varepsilon(x_0 - y_0)(F(y_0,u^*) - F^h(x_o,u^*)) \tag{37}$$

$$+ C\left[\frac{h}{\varepsilon^2} + \alpha\right]. \tag{38}$$

Combining this inequality with the estimates

$$L^h(x_0,u^*) - L(y_0,u^*) = L^h(x_0,u^*) - L(x_0,u^*) \tag{39}$$

$$+ L(x_0,u^*) - L(y_0,u^*) \tag{40}$$

$$= O(h) + O(|x_0 - y_0|), \tag{41}$$

$$F^h(x_0,u^*) - F(y_o,u^*)) = O(h) + O(|x_0 - y_0|), \tag{42}$$

we obtain, whenever $\varepsilon \leq 1$,

$$\widetilde{V}_h(x_o) - V(y_o) \leq C\left[|x_o - y_o| + \frac{|x_o - y_o|^2}{\cdot} + h\frac{|x_o - y_o|}{\cdot} + \frac{h}{\cdot} + \alpha\right]. \tag{43}$$

Using $ab \leq \frac{1}{2}(a^2+b^2)$, we get

$$h|x_o - y_o| \leq \frac{1}{2}(h^2 + |x_o - y_o|^2)$$

as well as

$$|x_o - y_o| = \varepsilon \frac{|x_o - y_o|}{\varepsilon} \leq \frac{1}{2}(\varepsilon^2 + \frac{|x_o - y_o|^2}{\varepsilon^2}).$$

Then, by (43), and for $\varepsilon \leq 1$ and $h \leq 1$, and taking $\alpha = O(h)$, we obtain

$$\widetilde{V}_h(x_o) - V(y_o) \leq \frac{C}{\varepsilon^2}\left(|x_o - y_o|^2 + h\right). \tag{44}$$

Observe that if we choose $x = y = x_o$ in (34), we obtain first that $|x_0 - y_0| \rightarrow 0$; using the fact that V has Hölder constant γ, we get, using $\alpha < 1$,

$$\frac{1}{\varepsilon^2}|x_o - y_o|^2 \leq K|x_o - y_o|^\gamma + \alpha|x_o - y_o| \leq K|x_o - y_o|^\gamma.$$

Equivalently,

$$|x_o - y_o| \leq K\varepsilon^{\frac{2}{2-\gamma}}, \tag{45}$$

with K independent on ε and h. Thus, from (36) and 1(45)

$$\widetilde{V}_h(x_o) - V(y_o) \leq K\left[\varepsilon^{\frac{2\gamma}{2-\gamma}} + \frac{h}{\varepsilon^2}\right] = K\left[\varepsilon^{\frac{2\gamma}{2-\gamma}} + \frac{h}{\varepsilon^2}\right].$$

Choosing $\varepsilon = h^{(2-\gamma)/4}$, we are lead to:

$$\widetilde{V}_h(x_o) - V(y_o) \leq K(h^{\gamma/2}). \tag{46}$$

From (34), we finally obtain:

$$\widetilde{V}_h(x) - V(x) \leq Kh^{\gamma/2} \quad \text{for all } x \in \mathrm{R}^{\mathrm{n}}. \tag{47}$$

It remains to prove the opposite inequality. This can be done in a similar manner, by setting

$$\bar{\varphi}(x,y) := V(x) - \widetilde{V}_h(y) + \beta_\varepsilon(x-y).$$

Again, given $\alpha \in (0,1)$, there exists (x_2, y_2) such that $\bar{\varphi}(x_2,y_2) > \sup\varphi - \alpha$, and hence, given $\bar{\xi} \in C_0^\infty(\mathrm{R}^{2\mathrm{n}})$ such that $\bar{\xi}(x_2,y_2) = 1$, $0 \leq \bar{\xi} \leq 1$, and $|D\bar{\xi}| \leq 1$, and setting $\bar{\psi}(x,y) = \varphi(x,y) + \alpha\bar{\xi}(x,y)$ for $(x,y) \in \mathrm{R}^{2\mathrm{n}}$, we have that $\bar{\psi}$ attains its maximum value at some point $(\bar{x}_o, \bar{y}_o)$ in the support of ξ:

$$\bar{\psi}(\bar{x}_o, \bar{y}_o) \geq \bar{\psi}(x,y) \quad \text{for all } (x,y) \in \mathrm{R}^{2\mathrm{n}} \tag{48}$$

Since $x \to \bar{\psi}(x, \bar{y}_o)$ attains its maximum at $\bar{x}_o$, we have that, for each $u \in U$,

$$\lambda V(\bar{x}_o) + F(\bar{x}_o, u).\left(\beta'_\varepsilon(\bar{x}_o - \bar{y}_o) + \alpha D_y \bar{\xi}(\bar{x}_o, \bar{y}_o)\right) - L(\bar{x}_o, u) \le 0. \tag{49}$$

On the other hand, (19) implies that, for some $\bar{u} \in U$,

$$(1 + \lambda h)\widetilde{V}_h(\bar{y}_o) - \widetilde{V}_h(\bar{y}_o + hF^h(\bar{x}_0, \bar{u})) - hL^h(\bar{y}_o, \bar{u}) = 0. \tag{50}$$

Using (34) with $x = \bar{x}_o$ and $y = \bar{y}_o + hF^h(\bar{x}_o, \bar{u})$, we obtain

$$\begin{aligned}&\widetilde{V}_h(\bar{y}_o + hF^h(\bar{y}_o, \bar{u})) - \widetilde{V}_h(\bar{y}_o) \ge -\beta_\varepsilon(\bar{x}_o - \bar{y}_o) - \alpha\bar{\xi}(\bar{x}_o, \bar{y}_o)\\ &+ \beta_\varepsilon(\bar{x}_o + hF^h(\bar{x}_o, \bar{u}) - \bar{y}_o) + \alpha\bar{\xi}(\bar{x}_o + hF^h(\bar{x}_o, \bar{u}), \bar{y}_o).\end{aligned}$$

Adding equality (50), dividing by h and then subtracting (49), we obtain

$$\widetilde{V}_h(\bar{x}_o) - V(\bar{y}_o) \ge C\left[|\bar{x}_o - \bar{y}_o| + \frac{|\bar{x}_o - \bar{y}_o|^2}{\varepsilon^2} + \frac{h}{\varepsilon^2} + \alpha\right]. \tag{51}$$

The end of the proof parallels the one for proving (47). ■

6. Convergence of finite difference schemes

Consider the following finite difference scheme. Let $\delta_1, \dots, \delta_n$ be the (positive) space steps. With $j \in \mathrm{Z}^n$ the point $x_j \in \mathrm{R}^n$ is associated with coordinates $j_i\delta_i$. Denote by $e_1, \dots, e_n$ the natural basis of R^n. With $\varsigma \in \mathrm{R}^n$, whose each coordinate is either 0 or 1, we associate the spatial finite difference which, for the ith component, is on the right if $\varsigma_i = 1$, and on the left elsewhere:

$$(D^\varsigma v_j)_i = \frac{v_{j+\varsigma_i e_i} - v_{j+(\varsigma_i - 1)e_i}}{\delta_i}. \tag{52}$$

With this vector ς is also associated a subset of U:

$$U_\varsigma(x) := \{u \in U; F_i(x, u) \ge 0 \text{ if } \varsigma_i = 1,\ \ F_i(x, u) < 0 \text{ otherwise}\}.$$

A standard finite differences numerical scheme for computing the value function is

$$\lambda\widetilde{v}_j + \max_\varsigma \sup_{u \in U_\varsigma(x)} (-L(x_j, u) - D^\varsigma\widetilde{v}_j \cdot F(x_j, u)) = 0, \qquad j \in \mathrm{Z}^n. \tag{53}$$

This is the classical upwind scheme, where for each component, the spatial finite difference is on the right if the corresponding component of dynamics is nonnegative, and on the left otherwise. Consider now the case when the available data are the functions F^h and L^h satisfying (26)-(27). Introducing the "fictive" time step h, we can approximate (53) in the following way:

$$\lambda\widetilde{v}_j + \max \ \sup \ \left(-L^h(x_j, u) - D^\varsigma\widetilde{v}_j \cdot F^h(x_j, u)\right) = 0, \qquad j \in \mathrm{Z}^n. \tag{54}$$

Note that, in the case of the second scheme (11), h can be interpreted as a time step. In our abstract framework, however, the parameter h is just a measure of the quality of approximation of F and L. In particular, there is no stability condition linking h with the space steps.

THEOREM 6.1 *Assume that H1-H3 hold. Let V be the solution of (9), let (v_j) be the solution of the finite difference scheme* (54), *and let $V_{h\delta}$ be the piecewise linear function on R^n such that $V(x_j) = v_j$.*

Let $\gamma \in]0,1[$ be a Hölder exponent of V. Then

$$\sup_{x\in\mathrm{R}^n} |V(x) - V_{h\delta}(x)| \le C\left(h + \sum_i \delta_i\right)^{\gamma/2}, \tag{55}$$

for all $(h,\delta) \in \mathrm{R}^+ \times (\mathrm{R}^*_+)^n$, where $C > 0$ is independent of (h,δ).

Proof. Denote by R^n_δ the spatial grid, i.e. points of the form $(\delta_1 k_1, \dots, \delta_n k_n)$, with $k_1, \dots, k_n$ in Z^n. Given $0 < \varepsilon < 1$, set

$$\beta_\varepsilon(x) := -\varepsilon^{-2}|x|^2 \quad \text{for } x \in \mathrm{R}^n. \tag{56}$$

Define

$$\varphi(x,y) := V_{h\delta}(x) - V(y) + \beta_\varepsilon(x-y) \quad \text{for } (x,y) \in \mathrm{R}^n_\delta \times \mathrm{R}^n.$$

Let $\alpha \in (0,1)$. Since V and $V_{h\delta}$ are bounded on R^n and R^n_δ, respectively, there exists (x_1, y_1) in $\mathrm{R}^n_\delta \times \mathrm{R}^n$ such that

$$\varphi(x_1, y_1) > \sup \varphi - \alpha.$$

Choose $\xi \in C_0^\infty(\mathrm{R}^{2n})$ so that

$$\xi(x_1, y_1) = 1, \quad 0 \le \xi \le 1, \quad |D\xi| \le 1,$$

and set

$$\psi(x,y) = \varphi(x,y) + \alpha\xi(x,y) \quad \text{for } (x,y) \in \mathrm{R}^n_\delta \times \mathrm{R}^n.$$

Clearly, ψ attains its maximum value over $\mathrm{R}^n_\delta \times \mathrm{R}^n$ at some point (x_o, y_o) in the support of ξ. That is,

$$\psi(x_o, y_o) \ge \psi(x,y) \quad \text{for all } (x,y) \in \mathrm{R}^n_\delta \times \mathrm{R}^n. \tag{57}$$

In particular, $y \longmapsto -\psi(x_o, y)$ attains its minimum at y_o. Therefore, by the definition of a viscosity solution of (9), there exists $u^* \in U$ such that

Since x_o belongs to R^n_δ, there exists $j \in \mathrm{Z}^n$ such that $x_o = x_j$. Let $\varsigma \in \mathrm{R}^n$ be such that

$$\varsigma_i = 1 \quad \text{if } F^h(x_0, u^*)_i \geq 0, \quad 0 \text{ otherwise.}$$

Then

$$\lambda v_j \leq L^h(x_j, u) + \sum_{i;\varsigma_i=1} \frac{v_{j+e_i} - v_j}{\delta_i} F_i^h(x_j, u) + \sum_{i;\varsigma_i=0} \frac{v_j - v_{j-e_i}}{\delta_i} F_i^h(x_j, u). \tag{59}$$

Using (57) with $x = x_0 \pm \delta_i e_i$ and $y = y_o$, we obtain

$$\begin{aligned} v_{j\pm e_i} - v_j &\leq \beta_\varepsilon(x_0 - y_0) + \alpha\xi(x_0, y_0) - \beta_\varepsilon(x_0 \pm \delta_i e_i - y_0) - \alpha\xi(x_0 \pm \delta_i e_i) \\ &\leq -\beta_\varepsilon'(x_0 - y_0)(\pm\delta_i e_i) - \varepsilon^{-2}\delta_i^2 + \alpha\delta_i. \end{aligned}$$

If $F_i^h(x_j, u^*) \geq 0$, multiply this inequality (for $\pm = +$)) by $F_i^h(x_j, u^*)/\delta_i$; otherwise multiply this inequality (for $\pm = -$)) by $-F_i^h(x_j, u^*)/\delta_i$; adding these inequalities to (59), we obtain

$$\lambda v_j \leq L^h(x_j, u) - \beta_\varepsilon'(x_0 - y_0)F^h(x_j, u) - \varepsilon^{-2}\sum_{i=1}^{n} \delta_i + n\alpha. \tag{60}$$

Subtracting (58) from the previous inequality, we obtain

$$\begin{aligned} &\lambda(v_j - V(y_0)) \leq (L^h(x_0, u^*) - L(y_0, u^*)) \\ &+ \beta_\varepsilon'(x_0 - y_0)(F(y_0, u^*) - F^h(x_0, u^*)) + \frac{\sum_i \delta_i}{\varepsilon^2} + O(\alpha). \end{aligned}$$

Combining this inequality with the estimates

$$L^h(x_0, u^*) - L(y_0, u^*) = L^h(x_0, u^*) - L(x_0, u^*) \tag{61}$$
$$+ L(x_0, u^*) - L(y_0, u^*) \tag{62}$$
$$= O(h) + O(|x_0 - y_0|), \tag{63}$$
$$F^h(x_0, u^*) - F(y_o, u^*)) = O(h) + O(|x_0 - y_0|), \tag{64}$$

and taking $\alpha = O(h)$, we obtain,

$$V_{h\delta}(x_o) - V(y_o) \leq C\left[|x_o - y_o| + \frac{|x_o - y_o|^2}{\varepsilon^2} + h\frac{|x_o - y_o|}{\varepsilon^2} + \frac{\sum_i \delta_i}{\varepsilon^2} + h\right]. \tag{65}$$

From $ab \leq \frac{1}{2}(a^2 + b^2)$ we deduce that

$$h|x_o - y_o| \leq \frac{1}{2}(h^2 + |x_o - y_o|^2)$$

as well as

$$|x_o - y_o| \quad 1 \quad 2 \quad |x_o - y_o|^2$$

Then, by (43), and for $\varepsilon \leq 1$ and $h \leq 1$, we obtain

$$V_{h\delta}(x_o) - V(y_o) \leq C \left[\frac{|x_o - y_o|^2}{\varepsilon^2} + \frac{h + \sum_i \delta_i}{\varepsilon^2} + \alpha \right]. \tag{66}$$

Choosing $x = y = x_o$ in (57) we get $|x_0 - y_0| \to 0$; using the fact that V has Hölder constant γ, and using $\alpha < 1$, obtains

$$\frac{1}{\varepsilon^2}|x_o - y_o|^2 \leq K|x_o - y_o|^\gamma + \alpha|x_o - y_o| \leq K|x_o - y_o|^\gamma.$$

This is equivalent to

$$|x_o - y_o| \leq K\varepsilon^{\frac{2}{2-\gamma}}, \tag{67}$$

where K is independent of ε, h, and (δ_i). Thus, from (66) and (67)

$$V_{h\delta}(x_o) - V(y_o) \leq K \left[\varepsilon^{\frac{2\gamma}{2-\gamma}} + \frac{h + \sum_i \delta_i}{\varepsilon^2} \right] = K \left[\varepsilon^{\frac{2\gamma}{2-\gamma}} + \frac{h + \sum_i \delta_i}{\varepsilon^2} \right].$$

Choosing $\varepsilon = (h + \sum_i \delta_i)^{(2-\gamma)/4}$, we lead to:

$$V_{h\delta}(x_o) - V(y_o) \leq K \left(\left(h + \sum_i \delta_i \right)^{\gamma/2} \right). \tag{68}$$

From (34), we finally obtain:

$$V_{h\delta}(x) - V(x) \leq K \left(\left(h + \sum_i \delta_i \right)^{\gamma/2} \right) \quad \text{for all } x \in \mathrm{R}^n_\delta.$$

Since $\alpha \in]0, 1[$ is arbitrary, we thus have

$$V_{h\delta}(x) - V(x) \leq K \left(h + \sum_i \delta_i \right)^{\gamma/2} \quad \text{for all } x \in \mathrm{R}^n_\delta.$$

From the definition of $V_{h\delta}$ and the Hölder continuity of V, we conclude that

$$V_{h\delta}(x) - V(x) \leq K \left(h + \sum_i \delta_i \right)^{\gamma/2} \quad \text{for all } x \in \mathrm{R}^n. \tag{69}$$

It remains to prove the converse inequality. This can be done in a similar manner, by setting $\bar{\varphi} : \mathrm{R}^n \times \mathrm{R}^n_\delta \to \mathrm{R}$ defined by

$$\bar{\varphi}(x, y) := V(x) - V \ldots$$

Again, given $\alpha \in (0,1)$, there exists (x_2, y_2) such that $\bar{\varphi}(x_2, y_2) > \sup \bar{\varphi} - \alpha$, and hence, given $\bar{\xi} \in C_0^\infty(\mathrm{R}^{2n})$ such that $\bar{\xi}(\bar{x}_2, \bar{y}_2) = 1$, $0 \leq \bar{\xi} \leq 1$, and $|D\bar{\xi}| \leq 1$, and setting $\bar{\psi}(x,y) = \bar{\varphi}(x,y) + \alpha\bar{\xi}(x,y)$ for $(x,y) \in \mathrm{R}^n \times \mathrm{R}^n_\delta$, we have that $\bar{\psi}$ attains its maximum value at some point $(\bar{x}_o, \bar{y}_o)$ in the support of ξ:

$$\bar{\psi}(\bar{x}_o, \bar{y}_o) \geq \bar{\psi}(x,y) \quad \text{for all } (x,y) \in \mathrm{R}^n \times \mathrm{R}^n_\delta. \tag{70}$$

Since $x \to \bar{\psi}(x, \bar{y}_0)$ attains its maximum at $\bar{x}_0$, we have that, for each $u \in U$,

$$\lambda V(\bar{x}_o) + F(\bar{x}_o, u).\,(\beta'_\varepsilon(\bar{x}_o - \bar{y}_o) + \alpha D_y\xi(\bar{x}_o, \bar{y}_o)) - L(\bar{x}_o, u) \leq 0. \tag{71}$$

On the other hand, let $j \in \mathrm{Z}^n$ such that $y_o = x_j \in \mathrm{R}^n_h$. Then the equality (54) implies that, for some $\bar{u} \in U$,

$$\lambda V_{h\delta}(y_o) = L^h(x_j, \bar{u}) + \sum_{i;\varsigma_i=1} \frac{v_{j+e_i} - v_j}{\delta_i} F_i^h(x_j, \bar{u})$$
$$+ \sum_{i;\varsigma_i=0} \frac{v_j - v_{j-e_i}}{\delta_i} F_i^h(x_j, \bar{u}). \tag{72}$$

The end of the proof parallels the one for (69). ■

REMARK 6.1 *If we choose a "fictive" time step $h = O(\sum_i \delta_i)$, then the error estimate (55) is*

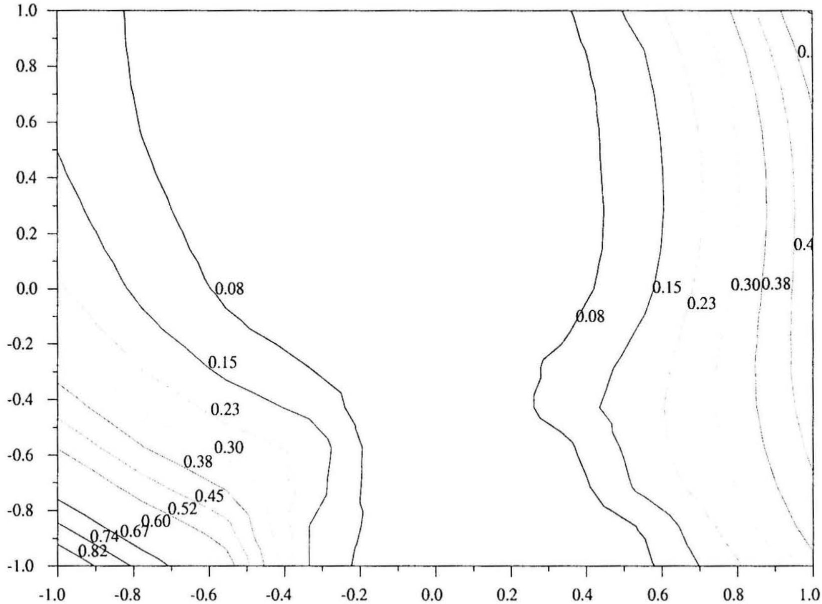
$$\sup_{x \in \mathrm{R}^n} |V(x) - V_{h\delta}(x)| \leq C(\sum_i \delta_i)^{\frac{1}{2}}. \tag{73}$$

7. Numerical test

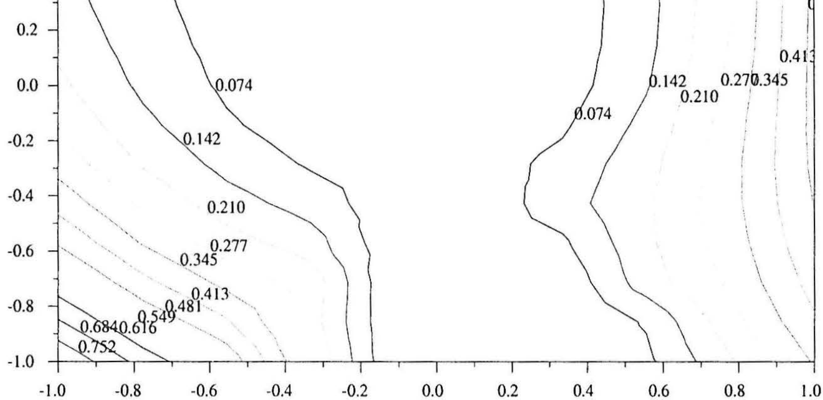
Consider a system described by the following differential-algebraic equation

$$\begin{cases} \dot{x_{s\xi}}(t) = y_{s\xi} + \sin(z_{s\xi}), & x_{s\xi}(s) = \xi_1, \\ \dot{y_{s\xi}}(t) = -x_{s\xi} + \cos(z_{s\xi}) + u, & y_{s\xi}(s) = \xi_2, \\ g(x_{s\xi}, y_{s\xi}) = \xi_1^2 + \xi_2^2, \end{cases} \tag{74}$$

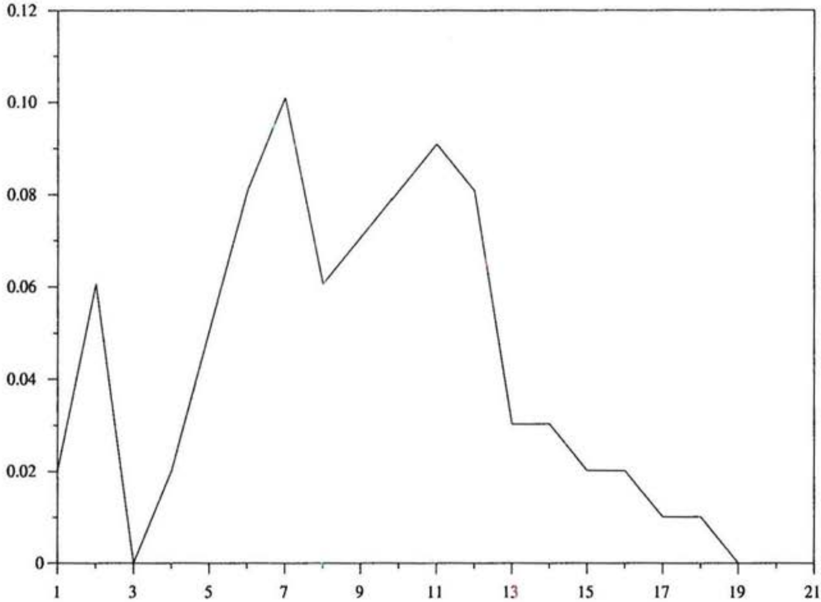
where $(x_{s\xi}, y_{s\xi}, z_{s\xi})$ denotes the state variable, and u is the control variable. The function g is defined by $g(x,y) = x^2 + y^2$, for every $(x,y) \in \mathrm{R}^2$. It is clear that the system (74) is of index 2 and that its associated hidden equation is the following:



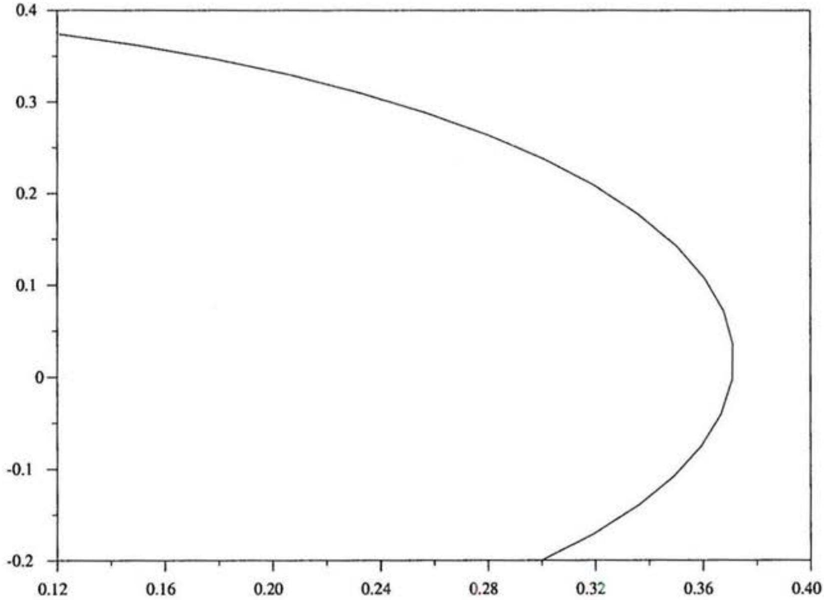
Scheme 1



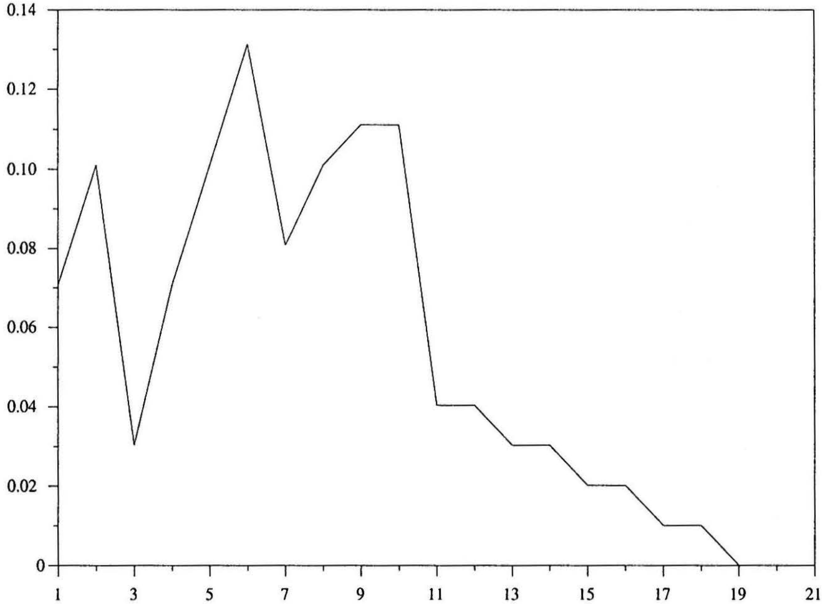
Scheme 2



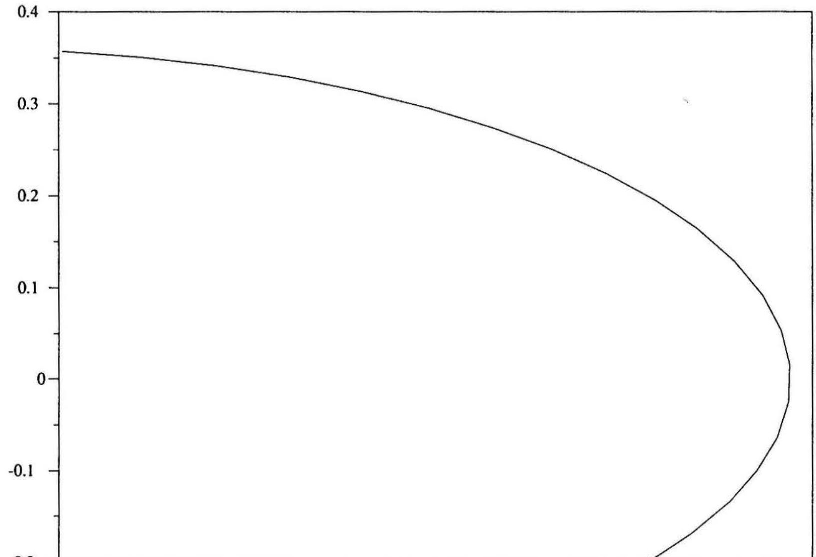
Computed optimal control for $s = 0, \xi = (0.3, -0.2)$



Computed optimal trajectory for $s = 0, \xi = (0.3, -0.2)$



Computed optimal control for $s = 0, \xi = (0.3, -0.2)$



Computed optimal trajectory for $s = 0, \xi = (0.3, -0.2)$

For every $s \in [0,1]$ and every $\xi = (\xi_1, \xi_2) \in \mathrm{R}^2$, we define a control problem $(P_{s\xi})$ by:

$$\begin{cases} \mathrm{Min} J(x_{s\xi}, y_{s\xi}, z_{s\xi}, u) = \int_s^1 \frac{1}{2}(x_{s\xi}^2 + u^2)\, dt \\ (x_{s\xi}, y_{s\xi}, z_{s\xi}, u) \text{ satisfies (74), and} \\ u(t) \in [0,1] \text{ a.e } t \in [0,1]. \end{cases}$$

We are interested in the computation of the value function for $s \in [0,1]$ and for ξ in the bounded domain $[-1,1]^2$. In order to avoid artificial boundary conditions, we use the classical method that consists in computing the value function on a big domain containing $[-1,1]^2$ and using the fact that the value function is known on R^2 when $s = 1$.

By this simple test, we remark that the value functions computed by the first scheme and the second one are very close to each other (see Fig. 1).

In Figs. 2–3, we represent the optimal trajectories starting at $s = 0$ with the initial condition $\xi = (0.3, 0.2)$. These trajectories seem to be very close. However, the trajectory computed by the scheme 1 satisfies the "hidden" constraint with precision of 10^{-16} but satisfies the algebraic constraint only with precision of 2.4×10^{-2}. On the contrary, the trajectory computed with the second scheme, satisfies the hidden constraint with precision of 2.9×10^{-2} and satisfies the algebraic constraint with precision of 10^{-16}. Note that, in view of the two algorithms, these results are not surprising. To conclude, it is important to note that the second scheme has the advantage of not needing the explicit knowledge of the derivative of the function g.

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