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Controllability of a time-discrete dynamical system with the aid of the solution of an approximation problem

by

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Abstract: This paper is concerned with time-discrete dynamical systems whose dynamics is described by a system of vector difference equations involving state and control vector functions. It is assumed that the uncontrolled system (in which the control vectors are put to zero) admits steady states. The aim is to reach such a steady state by a suitable choice of control functions within finitely many time-steps starting with an initial state at the time zero. We first give sufficient conditions for the solvability of this problem of controllability. Then, we develop a stepwise game-theoretical method for its solution. In the cooperative case this method can be combined with the solution of a suitable approximation problem and thereby leads to a solution of the problem of controllability within the smallest number of time steps, if the problem is solvable. Finally, we present a stepwise non-cooperative game theoretical solution.

Keywords: time-discrete control problems, approximation problem, controllability.

1. Introduction

We consider a controlled dynamical system whose dynamics is described by the difference equations of the form

$$x_i(t+1) = x_i(t) + f_i(x(t), u(t)) \tag{1.1}$$

for $i = 1, \ldots, n$ and $t \in \mathrm{N}_0 = \mathrm{N} \cup \{0\}$, where $x_i : \mathrm{N}_0 \to \mathrm{R}^{l_i}$ and $u_i : \mathrm{N}_0 \to \mathrm{R}^{m_i}$ for $i = 1, \ldots, n$ are state and control vector functions, respectively, which are composed of the vector functions

$$x(t) = (x_1(t), \ldots, x_n(t)) \quad \text{and} \quad u(t) = (u_1(t), \ldots, u_n(t)) \quad \text{for } t \in \mathrm{N}_0.$$

Further,

$$f_i : \prod_{j=1}^{n} \mathrm{R}^{l_j} \times \prod_{j=1}^{n} \mathrm{R}^{m_j} \to \mathrm{R}^{l_j}$$

are given vector functions for $i = 1, \dots, n$.

In addition, we assume, for every $i = 1, \dots, n$, non-empty sets $X_i \subseteq \mathrm{R}^{l_i}$ and $U_i \subseteq \mathrm{R}^{m_i}$ to be given and require control conditions of the form

$$u_i(t) \in U_i \quad \text{for all } i = 1, \dots, n \quad \text{and} \quad t \in \mathrm{N}_0 \tag{1.2}$$

as well as state constraints of the form

$$x_i(t) \in X_i \quad \text{for all } i = 1, \dots, n \quad \text{and} \quad t \in \mathrm{N}_0. \tag{1.3}$$

Finally, we require initial conditions of the form

$$x_i(0) = x_{0i} \quad \text{for } i = 1, \dots, n \tag{1.4}$$

where $x_{0i} \in X_i$ for $i = 1, \dots, n$ are given.

If one choses n control functions $u_i : \mathrm{N}_0 \to U_i$ for $i = 1, \dots, n$, then, by (1.1) and (1.4), n state vector functions $x_i : \mathrm{N}_0 \to \mathrm{R}^{l_i}$ are uniquely determined. We now make the following

Assumptions

1. For every $i = 1, \dots, n$ the zero vector Θ_{m_i} of R^{m_i} belongs to U_i.
2. The nonlinear system
$$f_i(\hat{x}_1, \dots, \hat{x}_n, \Theta_{m_1}, \dots, \Theta_{m_n}) = \Theta_{l_i} \quad \text{for } i = 1, \dots, n \tag{1.5}$$
has at least one solution
$$\hat{x} = (\hat{x}_1, \dots, \hat{x}_n) \in \prod_{j=1}^{n} X_j.$$

Under these assumptions we formulate the

Problem of Controllability

Let

$$\hat{x} = (\hat{x}_1, \dots, \hat{x}_n) \in \prod_{j=1}^{n} X_j$$

be a solution of (1.5). Then we are looking for control functions $u_i : \mathrm{N}_0 \to \mathrm{R}^{m_i}$ for $i = 1, \dots, n$ and some $N \in \mathrm{N}_0$ such that under the conditions (1.1)–(1.4) it is true that

$$u_i(t) = \Theta_{m_i} \quad \text{and} \quad x_i(t) = \hat{x}_i \tag{1.6}$$
$$\text{for all} \quad i = 1, \dots, n \quad \text{and} \quad t \geq N.$$

In words:

Given an initial state $x_0 \in \prod_{j=1}^{n} X_j$ of the dynamical system under consideration, find control functions which satisfy (1.2) and steer the system, under the conditions (1.3), into a steady state of the uncontrolled system whose dynamics is described by

$$x_i(t+1) = x_i(t) + f_i(x(t), \Theta) \tag{1.7}$$
$$\text{for} \quad i = 1, \ldots, n \quad \text{and} \quad t \in \mathrm{N}_0$$

where $\Theta = (\Theta_{m_1}, \ldots, \Theta_{m_n})$. The main purpose of this paper is to develop methods for solving this problem via the solution of an approximation problem. We begin, however, with

2. Sufficient conditions for the solvability of the problem of controllability

In addition to the Assumptions 1 and 2 of Section 1 we assume that

$$\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n) \in \prod_{j=1}^{n} X_j$$

(as a solution of (1.5)) is a globally attractive steady state of the uncontrolled system (1.7) which means that, for every $i = 1, \ldots, n$ and for every $x_{0i} \in \mathrm{R}^{l_i}$ and every solution $x = x(t)$ of (1.7) with

$$x_i(0) = x_{0i} \quad \text{for} \quad i = 1, \ldots, n$$

it is true that

$$\lim_{t \to \infty} x_i(t) = \hat{x}_i \quad \text{for} \quad i = 1, \ldots, n.$$

For every $N \in \mathrm{N}_0$ we denote by $S(N)$ the set of all initial states

$$x_0 = (x_{01}, \ldots, x_{0n}) \in \prod_{j=1}^{n} \mathrm{R}^{l_i}$$

such that there exist control functions $u_i : \mathrm{N}_0 \to \mathrm{R}^{m_i}$ satisfying (1.2) for which the corresponding solutions $x_i : \mathrm{N}_0 \to \mathrm{R}^{l_i}$ of (1.1) and (1.4) satisfy

$$x_i(N) = \hat{x}_i \quad \text{for} \quad i = 1, \ldots, n.$$

Let

$$S = \bigcup_{N \in \mathrm{N}_0} S(N). \tag{2.1}$$

Obviously it follows that $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n) \in S$. The following theorem now formulates sufficient conditions for controllability

THEOREM 2.1 *We assume that*

$$X_i = \mathrm{R}^{l_i} \quad \textit{for} \quad i = 1, \ldots, n \tag{2.2}$$

and that

$$\hat{x} \in \prod_{j=1}^{n} \mathrm{R}^{l_j}$$

is a globally attractive steady state of the uncontrolled system (1.7). Further we assume that $\hat{x}$ is an interior point of the set S of controllability defined by (2.1).

Then the problem of controllability is solvable for every $x_0 \in \prod_{j=1}^{n} \mathrm{R}^{l_j}$.

Proof. Since $\hat{x}$ is an interior point of S, there is a neighborhood $V_\vartheta(\hat{x})$ with $V_\vartheta(\hat{x}) \subseteq S$. Since $\hat{x}$ is a globally attractive steady state of (1.7), for every $x_0 \in \prod_{j=1}^{n} \mathrm{R}^{l_j}$ there is some $N_1 \in \mathrm{N}_0$ such that $x(N_1) \in V_\vartheta(\hat{x})$. This implies the existence of a control function

$$u^* : \mathrm{N}_0 \to \prod_{j=1}^{n} \mathrm{R}^{m_j} \quad \text{with} \quad u^*(t) \in \prod_{j=1}^{n} U_j \quad \text{for all } t \in \mathrm{N}_0$$

and some time $N_2 \in \mathrm{N}_0$ such that the corresponding trajectory $x^* = x^*(t)$ of (1.1) for $u = u^*$ and $x^*(0) = x(N_1)$ satisfies $x^*(N_2) = \hat{x}$. If one defines a control function

$$\bar{u} : \mathrm{N}_0 \to \prod_{j=1}^{n} \mathrm{R}^{m_j}$$

by

$$\bar{u}(t) = \begin{cases} \Theta = (\Theta_{m_1}, \ldots, \Theta_{m_n}) & \text{for } t = 0, \ldots, N_1 \\ u^*(t - N_1 - 1) & \text{for } t = N_1 + 1, \ldots, N_1 + N_2 + 1 \\ \Theta = (\Theta_{m_1}, \ldots, \Theta_{m_n}) & \text{for } t \geq N_1 + N_2 + 2 \end{cases}$$

then the corresponding trajectory $\hat{x} = \hat{x}(t)$ of (1.1) which satisfies the initial conditions (1.4), we have $x(N) = \hat{x}$ for $N = N_1 + N_2 + 2$. This completes the proof. ∎

Let us conclude this section with *sufficient* conditions for $\hat{x}$ to be an interior point of the set S of controllability defined by (2.1). For this purpose we write the system (1.1) in the form

$$x(t+1) = x(t) + f(x(t), u(t)) \tag{2.3}$$

where $f : \mathrm{R}^l \times \mathrm{R}^m \to \mathrm{R}^l$ with

$$l = \sum^{n} l_i \quad \text{and} \quad m = \sum^{n} m_i. \tag{2.4}$$

We assume that $f \in C^1(\mathrm{R}^l \times \mathrm{R}^m)$. We further assume that all sets $U_i \subseteq \mathrm{R}^{m_i}$, $i = 1, \ldots, n$, are open and hence the set $U = \prod_{i=1}^{n} U_i$ is open in R^m.

Then we can prove the following

THEOREM 2.2 *Let the matrix $I_l + f_x(\hat{x}, \theta_m)$ with I_l being the $l \times l$-unit matrix and $f_x(\hat{x}, \theta_m)$ the Jacobi matrix of f with respect to x at $(\hat{x}^T, \theta_m^T)^T \in \mathrm{R}^l \times \mathrm{R}^m$ be non-singular.*

Further let, for some $N \in \mathrm{N}$,

$$rank(f_u(\hat{x}, \theta_m)|(I_l + f_x(\hat{x}, \theta_m))^{-1} f_u(\hat{x}, \theta_m)| \\ \ldots |(I_l + f_x(\hat{x}, \theta_m))^{-N+1} f_u(\hat{x}, \theta_m)| = l \tag{2.5}$$

where $f_u(\hat{x}, \theta_m)$ is the Jacobi matrix of f with respect to u at $(\hat{x}^T, \theta_m^T)^T \in \mathrm{R}^l \times \mathrm{R}^m$.

Then $\hat{x}$ is an interior point of S (2.1).

The proof of this theorem will appear in Krabs (2003).

3. A stepwise, cooperative game-theoretical solution

We assume, for some $t \in \mathrm{N}_0$, the vectors

$$x_i(t) \in X_i \quad \text{for} \quad i = 1, \ldots, n$$

to be given. For $t = 0$ we choose

$$x_i(0) = x_{0i} \quad \text{for} \quad i = 1, \ldots, n$$

with $x_{0i} \in X_i$ being the initial values in (1.4).

For every vector $u \in \prod_{j=1}^{n} \mathrm{R}^{m_j}$ we define

$$x_i(u)(t+1) = x_i(t) + f_i(x(t), u) \quad \text{for} \quad i = 1, \ldots, n \tag{3.1}$$

where $x(t) = (x_1(t), \ldots, x_n(t))$ and

$$a_i^t(u) = \|x_i(u)(t+1) - \hat{x}_i\|_2^2 + \|u_i\|_2^2 \quad \text{for} \quad i = 1, \ldots, n \tag{3.2}$$

where $\|\cdot\|_2$ denotes the Euclidean norm. For every $i = 1, \ldots, n$ we consider the function

$$a_i^t : \prod_{j=1}^{n} \mathrm{R}^{m_j} \to \mathrm{R}_+$$

as payoff function for the i-th player of a game in which the i-th player has the set U_i at his disposal as the set of strategies by which he tries to control the game. The players are, however, linked through the set

$$Z_t = \left\{ u \in \prod_{j=1}^{n} U_j | x_i(u)(t+1) \in X_i \quad \text{for all} \quad i = 1, \ldots, n \right\} \tag{3.3}$$

of feasible controls. Every player endeavours to minimize the value $a_i^t(u)$ of his own payoff function. This value, however, depends on all controls $u_1, \dots, u_n$ and therefore cannot be determined by the i-th player alone. Let us assume that all players cooperate and try to minimize the common payoff function

$$\varphi_t(u) = \sum_{i=1}^{n} a_i^t(u), \quad u \in Z_t, \tag{3.4}$$

with $a_i^t(u)$ for $i = 1, \dots, n$ being defined by (3.2). Then they have to solve the following

PROBLEM: Find $u^t \in Z_t$ such that

$$\varphi_t(u^t) \leq \varphi_t(u) \quad \text{for all} \quad u \in Z_t. \tag{3.5}$$

Let $u^t \in Z_t$ be a solution of this problem. Then we have to distinguish the cases:

(a) $\varphi_t(u^t) = 0$

Then it follows necessarily that

$$u_i^t = \Theta_{m_i} \quad \text{and} \quad x_i(u^t)(t+1) = \hat{x}_i \quad \text{for } i = 1, \dots, n.$$

If we put $N = t + 1$ and define control functions $u_i : \mathrm{N}_0 \to \mathrm{R}^{m_i}$ for $i = 1, \dots, n$ by

$$u_i(s) = \begin{cases} u_i^s & \text{for } s = 0, \dots, N-1\ , \\ \Theta_{m_i} & \text{for } s \geq N, \end{cases}$$

and state functions

$$x_i : \mathrm{N}_0 \to \mathrm{R}^{l_i} \quad \text{for} \quad i = 1, \dots, n \quad \text{by} \tag{3.6}$$

$$x_i(0) = x_{0i},$$

$$x_i(s) = \begin{cases} x_i(u^{s-1})(s) & \text{for } s = 1, \dots, N-1, \\ \hat{x}_i & \text{for } s \geq N, \end{cases}$$

then we have gained a solution of the problem of controllability.

(b) $\varphi_t(u^t) > 0$

Then we put

$$x_i(t+1) = x_i(u^t)(t+1) \quad \text{for} \quad i = 1, \dots, n$$

and solve problem (3.5) with $t+1$ instead of t.

The cooperation of the players is expressed by the following

THEOREM 3.1 *Every solution $u^t \in Z_t$ of problem (3.5) is a so called* **Pareto Optimum**, *i.e., for every $u \in Z_t$ with*

$$a_i^t(u) \leq a_i^t(u^t) \quad \textit{for all} \quad i = 1, \dots, n \tag{3.7}$$

it follows that

$$a_i^t(u) = a_i^t(u^t) \quad \textit{for all} \quad i = 1, \dots, n. \tag{3.8}$$

Proof. From (3.7) we infer $\varphi_t(u) \leq \varphi_t(u^t)$ where φ_t is the common payoff function defined by (3.4). This in turn implies $\varphi_t(u) = \varphi_t(u^t)$ because of the [illegible] if (3.8) holds true. ∎

The assertion of Theorem 3.1 has the consequence that there is no $u \in Z_t$ with (3.7) and

$$a^t_{i_0}(u) < a^t_{i_0}(u^t) \quad \text{for at least one} \quad i_0 \in \{1, \dots, n\}.$$

In many applications there appears the following Special Case:

We assume that all the functions on the right hand side of (1.1) are of the form

$$f_i(x,u) = f_{0i}(x) + \sum_{j=1}^{n} f_{ij}(x) u_j \tag{3.9}$$

where f_{ij} is a $l_i \times m_j$-matrix function and (3.10)

$$x \in \prod_{j=1}^{n} \mathrm{R}^{l_j}, \quad u \in \prod_{j=1}^{n} \mathrm{R}^{m_j} \quad \text{for} \quad i = 1, \dots, n.$$

Further we assume the sets $U_i \subseteq \mathrm{R}^{m_i}$ for $i = 1, \dots, n$ to be convex and compact and the sets $X_i \subseteq \mathrm{R}^{l_i}$ for $i = 1, \dots, n$ to be convex and closed. Then, for every $t \in \mathrm{N}_0$, the set Z_t of feasible controls defined by (3.3) is convex and compact, if it is non-empty.

Further the common payoff function $\varphi_t : \prod_{j=1}^{n} \mathrm{R}^{m_i} \to \mathrm{R}_+$ given by (3.4) has the form

$$\varphi_t(u) = \sum_{i=1}^{n} \left(\left\| \sum_{j=1}^{n} f_{ij}(x(t)) u_j + f_{0i}(x(t)) + x_i(t) - \hat{x}_i \right\|_2^2 + \|u_i\|_2^2 \right)$$

for $u \in \prod_{j=1}^{n} \mathrm{R}^{m_j}$ and is convex and hence also continuous. Consequently, for every $t \in \mathrm{N}_0$, there is one $u^t \in Z_t$ satisfying (3.5), if Z_t is non-empty. This is the case, for instance, if (2.2) holds true, for then

$$Z_t = \prod_{j=1}^{n} U_j \tag{3.11}$$

is non-empty for every $t \in \mathrm{N}_0$. For every $t \in \mathrm{N}_0$ and $u \in Z_t$ we define

$$\nabla \varphi_t(u) = (\nabla_1 \varphi_t(u), \dots, \nabla_n \varphi_t(u))$$

where

$$\nabla_s \varphi_t(u) = \left(\frac{\partial \varphi_t}{\partial u_{s1}}(u), \dots, \frac{\partial \varphi_t}{\partial u_{sm_s}}(u) \right) \quad \text{for} \quad s = 1, \dots, n$$

and

$$\frac{\partial \varphi_t}{\partial u_{sr}}(u) = 2 \Big\{ \sum_{i=1}^{n} \sum_{l=1}^{l_i} \Big(\sum_{j=1}^{n} \sum_{k=1}^{m_j} (f_{ij}(x(t)))_{lk} u_{jk}$$

for $r = 1, \ldots, m_s$ and $s = 1, \ldots, n$.

Then it is well known that (3.5) is equivalent to

$$\langle \nabla \varphi_t(u^t), u - u^t \rangle \geq 0 \quad \text{for all} \quad u \in Z_t \tag{3.12}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\prod_{j=1}^{n} \mathrm{R}^{m_j}$.

In particular for

$$U_j = \{u_j \in \mathrm{R}^{m_j} | \|u_j\|_2 \leq M_j\} \quad \text{for} \quad j = 1, \ldots, n$$

and given numbers $M_j > 0$ and Z_t being of the form (3.11) condition (3.12) turns out to be equivalent to

$$-\sum_{s=1}^{n} \langle \nabla_s \varphi_t(u^t), u_s^t \rangle \geq -\sum_{s=1}^{n} M_S \|\nabla_s \varphi_t(u^t)\|_2$$

which implies

$$u_s^t = \begin{cases} \frac{M_S}{\|\nabla_s \varphi_t(u^t)\|_2} \nabla_s \varphi_t(u^t), & \text{if } \nabla_s \varphi_t(u^t) \neq \Theta_{m_s}, \\ \Theta_{m_s}, & \text{if } \nabla_s \varphi_t(u^t) = \Theta_{m_s}, \end{cases} \tag{3.13}$$

for $s = 1, \ldots, n$.

This representation of u^t can be used in order to determine u^t by an iteration procedure. For numerical results we refer to Pickl (2000).

4. An approximation problem

We assume condition (2.2) to hold and choose some $N \in \mathrm{N}$. Then we consider the following

Approximation Problem A:
Find control functions $u_i : \mathrm{N}_0 \to \mathrm{R}^{m_i}$ with $u_i(t) \in U_i$ for $t = 0, \ldots, N-1$ and $i = 1, \ldots, n$ such that under the conditions

$$x_i(t+1) = x_i(t) + f_i(x(t), u(t)), t = 0, \ldots, N-1 \quad \text{and} \quad x_i(0) = x_{0i}$$

for $i = 1, \ldots, n$ the function value

$$\varphi_N(u) = \sum_{i=1}^{n} (\|x_i(N) - \hat{x}_i\|_2^2 + \|u_i(N-1)\|_2^2)$$

is as small as possible.

If the problem of controllability has a solution, then there is some $N \in \mathrm{N}$ such that for every solution of the approximation problem A it necessarily follows that so that by solving the problem A one also obtains a solution of the *problem of controllability.* The solution of the approximation problem can be achieved by iteration as follows:

We choose control functions $u_i^0 : \mathrm{N}_0 \to \mathrm{R}^{m_i}$ with

$$u_i^0(t) \in U_i \quad \text{for} \quad t = 0, \dots, N-1 \quad \text{and} \quad i = 1, \dots, n$$

$$(\text{for instance } u_i^0(t) = \Theta_{m_i} \quad \text{for} \quad t = 0, \dots, N-1 \quad \text{and} \quad i = 1, \dots, n)$$

and calculate

$$x_i^0(t+1) = x_i^0(t) + f_i(x^0(t), u^0(t))$$

for $t = 0, \dots, N-1$ with $x_i^0(0) = x_{0i}$ for $i = 1, \dots, n$.

Then we construct a sequence

$$(u^k)_{k \in \mathrm{N}_0} \quad \text{in} \quad \left\{ u : \{0, \dots, N-1\} \to \prod_{j=1}^n U_j \right\}$$

and a sequence

$$(x^k)_{k \in \mathrm{N}_0} \quad \text{in} \quad \left\{ x : \{0, \dots, N\} \to \prod_{j=1}^n \mathrm{R}^{l_j} \right\}$$

as follows: If u^k and x^k are given for some $k \in \mathrm{N}_0$, then we determine

$$u_i^{k+1}(t) \in U_i \quad \text{for} \quad t = 0, \dots, N-1 \quad \text{and} \quad i = 1, \dots, n$$

such that under the conditions

$$\tilde{x}^{k+1}(t+1) = \tilde{x}^{k+1}(t) + f(x^k(t), u^{k+1}(t)) \quad \text{for} \quad t = 0, \dots, N-1$$

and

$$\tilde{x}^{k+1}(0) = x_0$$

the function value

$$\varphi_N^k(u^{k+1}) = \sum_{i=1}^n (\|\tilde{x}_i^{k+1}(N) - \hat{x}_i\|_2^2 + \|u_i^{k+1}(N-1)\|_2^2) \tag{4.1}$$

becomes minimal.

With this representation we obtain the following transformed objective function taking advantage of the special structure of the discrete dynamics:

$$\varphi_N^k(u^{k+1}) = \sum^n \Big(\Big\| \sum^{N-1} f_i(x^k(t), u^{k+1}(t)) + x_{0i} - \hat{x}_i \Big\|^2 + \|u_i^{k+1}(N-1)\|^2 \Big)$$

If $u^{k+1}(t)$ has been determined for $t = 0, \ldots, N-1$, then we calculate

$$x^{k+1}(t+1) = x^{k+1}(t) + f(x^{k+1}(t), u^{k+1}(t))$$

$$\text{for} \quad t = 0, \ldots, N-1 \quad \text{where} \quad x^{k+1}(0) = x_0.$$

If $x^{k+1}(t) = \tilde{x}^{k+1}(t)$ for all $t = 0, \ldots, N$, then we have found a solution of the problem A. Otherwise we proceed with u^{k+1} and x^{k+1} instead of u^k and x^k, respectively. Let us make the assumption that all functions

$$f_i : \prod_{j=1}^{n} \mathrm{R}^{l_j} \times \prod_{j=1}^{n} \mathrm{R}^{m_j} \to \mathrm{R}^{l_i}$$

for $i = 1, \ldots, n$ are continuous.

Then we have the following

THEOREM 4.1 *If for every $t \in \{0, \ldots, N-1\}$, there is some*

$$u(t) \in \prod_{j=1}^{n} U_j \quad \textit{with} \quad u(t) = \lim_{k \to \infty} u^k(t)$$

then $u_i(t)$ for $t = 0, \ldots, N-1$ and $i = 1, \ldots, n$ solve the problem A.

Proof. For $t = 0$ it follows from

$$x^{k+1}(1) = x_0 + f(x_0, u^{k+1}(0))$$

that the limit

$$x(1) = \lim_{k \to \infty} x^{k+1}(1) = x_0 + f(x_0, u(0))$$

exists.

We assume that, for some $t \in \{1, \ldots, N-1\}$, the limit

$$x(t) = \lim_{k \to \infty} x^{k+1}(t) = x(t-1) + f(x(t-1), u(t-1))$$

exists.

Then it follows from

$$x^{k+1}(t+1) = x^{k+1}(t) + f(x^{k+1}(t), u^{k+1}(t))$$

that the limit

$$x(t+1) = \lim_{k \to \infty} x^{k+1}(t+1) = x(t) + f(x(t), u(t))$$

exists. By the principle of induction, for every $t \in \{0, \ldots, N-1\}$, with respect to t it therefore follows that the limit exists and is given by

$$x(t+1) = x(t) + f(x(t), u(t))$$

for every $t \in \{0, \dots, N-1\}$.

This implies, because of

$$\tilde{x}^{k+1}(N) = x_0 + \sum_{t=0}^{N-1} f(x^k(t), u^{k+1}(t))$$

that

$$\lim_{k\to\infty} \tilde{x}^{k+1}(N) = x_0 + \sum_{t=0}^{N-1} f(x(t), u(t)) = x(N)$$

and hence

$$\lim_{k\to\infty} \varphi_N^k(u^{k+1}) = \varphi_N(u).$$

Now, let $\tilde{u} : \{0, \dots, N-1\} \to \prod_{j=1}^{n} U_j$ be chosen arbitrarily.

Then it also follows that $\lim_{k\to\infty} \varphi_N^k(\tilde{u}) = \varphi_N(\tilde{u})$.

Further, we have, for every $k \in \mathrm{N}_0$,

$$\varphi_N^k(u^{k+1}) \leq \varphi_N^k(\tilde{u})$$

and hence

$$\varphi_N(u) = \lim_{k\to\infty} \varphi_N^k(u^{k+1}) \leq \lim_{k\to\infty} \varphi_N^k(\tilde{u}) = \varphi_N(\tilde{u}).$$

This shows that $u_i(t)$ for $t = 0, \dots, N-1$ and $i = 1, \dots, n$ solve the approximation problem A. ■

We again consider the special case in which the functions

$$f_i : \prod_{j=1}^{n} \mathrm{R}^{l_j} \times \prod_{j=1}^{n} \mathrm{R}^{m_j} \to \mathrm{R}^{l_i} \quad \text{for} \quad i = 1, \dots, n$$

are of the form (3.9) and the sets $U_j \subseteq \mathrm{R}^{m_j}$ for $j = 1, \dots, n$ are given by

$$U_j = \{u_j \in \mathrm{R}^{m_j} \,|\, \|u_j\|_2^2 \leq M_j\}$$

with given numbers $M_j > 0$. With (3.3) let us define

$$Z_N = \prod_{t=0}^{N-1} Z_t = \Big(\prod_{j=1}^{n} U_j\Big)^N. \tag{4.2}$$

Then, for $u^{k+1} \in Z_N$, the statement

$$\varphi_N^k(u^{k+1}) \leq \varphi_N^k(u) \quad \text{for all} \quad u \in Z_N \tag{4.3}$$

is equivalent to

$$\langle \nabla \varphi_N^k(u^{k+1}), u - u^{k+1} \rangle \geq 0 \quad \text{for all} \quad u \in Z_N \tag{4.4}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\left(\prod_{j=1}^{n} \mathrm{R}^{m_j} \right)^N$ and

$$\begin{aligned} &\nabla \varphi_N^k(u) \\ &= (\nabla_{10}\varphi_N^k(u), \ldots, \nabla_{n0}\varphi_N^k(u), \ldots, \nabla_{1N-1}\varphi_N^k(u), \ldots, \nabla_{nN-1}\varphi_N^k(u)) \end{aligned}$$

with

$$\nabla_{st}\varphi_N^k(u) = \left(\frac{\partial \varphi_N^k}{\partial u_{s1}(t)}(u), \ldots, \frac{\partial \varphi_N^k}{\partial u_{sm_s}(t)}(u) \right), \quad u \in \left(\prod_{j=1}^{n} \mathrm{R}^{m_j} \right)^N,$$

for $s = 1, \ldots, n$ and $t = 0, \ldots, N-1$. From the definition of the scalar product $\langle \cdot, \cdot \rangle$ in $\left(\prod_{j=1}^{n} \mathrm{R}^{m_j} \right)^N$ it follows that (4.4) is equivalent to

$$\sum_{s=1}^{n} \sum_{t=0}^{N-1} \langle \nabla_{st}\varphi_N^k(u^{k+1}), u_s(t) - u_s^{k+1}(t) \rangle_{\mathrm{R}^{m_s}} \geq 0$$

for all $u \in Z_N$, where $\langle \cdot, \cdot \rangle_{\mathrm{R}^{m_s}}$ denotes the scalar product in R^{m_s} for $s = 1, \ldots, n$. This, in turn, is equivalent to

$$-\sum_{s=1}^{n} \sum_{t=0}^{N-1} \langle \nabla_{st}\varphi_N^k(u^{k+1}), u_s^{k+1}(t) \rangle_{\mathrm{R}^{m_s}} \geq -\sum_{s=1}^{n} \sum_{t=0}^{N-1} M_s \|\nabla_{st}\varphi_N^k(u^{k+1})\|_2$$

which is satisfied if

$$u_s^{k+1}(t) = \begin{cases} \frac{M_s}{\|\nabla_{st}\varphi_N^k(u^{k+1})\|_2} \nabla_{st}\varphi_N^k(u^{k+1}) & , \text{if } \nabla_{st}\varphi_N^k(u^{k+1}) \neq \Theta_{m_s} \\ \Theta_{m_s} & , \text{if } \nabla_{st}\varphi_N^k(u^{k+1}) = \Theta_{m_s} \end{cases}$$

for $s = 1, \ldots, n$ and $t = 0, \ldots, N-1$.

This representation can be used in order to determine u^{k+1} by an iteration procedure. The approximation problem A can be coupled with the stepwise, cooperative game-theoretical solution of the problem of controllability described in Section 3.

Let $u^N \in Z_N$ (4.2) be a solution of the approximation problem A for some $N \in \mathrm{N}$. Then we put

$$x^N(t+1) = x^N(t) + f(x^N(t), u^N(t))$$

For every $u \in \prod_{j=1}^{n} \mathrm{R}^{m_j}$ we define

$$x(u)(N+1) = x^N(N) + f(x^N(N), u)$$

and determine $u_N \in \prod_{j=1}^{n} U_j$ so that

$$\varphi_{N+1}(u_N) \leq \varphi_{N+1}(u) \quad \text{for all} \quad u \in \prod_{j=1}^{n} U_j$$

where

$$\varphi_{N+1}(u) = \sum_{i=1}^{n} (\|x_i(u)(N+1) - \hat{x}_i\|_2^2 + \|u_i\|_2^2) \quad \text{for} \quad u \in \prod_{j=1}^{n} U_j.$$

After having determined u_N we define

$$u^0(t) = u^N(t) \quad \text{for} \quad t = 0, \ldots, N-1, \quad u^0(N) = u_N$$

and

$$x^0(t) = x^N(t) \quad \text{for} \quad t = 0, \ldots, N-1, \quad x^0(N+1) = x(u_N)(N+1)$$

and perform the above iteration procedure for solving the approximation problem with $N+1$ instead of N.

For $N = 1$ the approximation problem coincides with the problem to be solved in the algorithm of stepwise solution for $t = 0$.

If the problem of controllability is solvable for some $N \in \mathrm{N}$ and if the iteration procedure converges, then the combination of the two methods guarantees finding of a solution for the smallest possible $N \in \mathrm{N}$.

5. A stepwise, non-cooperative game theoretical solution

We assume that the players do not cooperate and everybody tries to minimize his own payoff function (3.2). This, however, is in general not possible so that the players have to compromise. A compromise solution is a so called **Nash equilibrium** $u^t \in Z_t$ (3.3) for which the following condition is satisfied:

$$a_i^t(u^t) \leq a_i^t(u_1^t, \ldots, u_{i-1}^t, u_i, u_{i+1}^t, \ldots, u_n^t) \tag{5.1}$$
$$\text{for all} \quad (u_1^t, \ldots, u_{i-1}^t, u_i, u_{i+1}^t, \ldots, u_n^t) \in Z_t \quad \text{and} \quad i = 1, \ldots, n.$$

Let $u^t \in Z_t$ be a Nash equilibrium. Then we have to distinguish two cases:

(a) $a_i^t(u^t) = 0$ for all $i = 1, \ldots, n$

Then it follows that

$$u_i^t = \Theta_{m_i} \quad \text{and} \quad x_i(u^t)(t+1) = \hat{x}_i \quad \text{for} \quad i = 1, \ldots, n.$$

If we put $N = t+1$ and define control functions $u_i : \mathrm{N}_0 \to \mathrm{R}^{m_i}$ and state functions $x_i : \mathrm{N}_0 \to \mathrm{R}^{l_i}$ for $i = 1, \ldots, n$ by (3.6), then we obtain a solution

(b) There is some $i_t \in \{1,\dots,n\}$ with $a^t_{i_t}(u^t) > 0$ Then we put

$$x_i(t+1) = x_i(u^t)(t+1) \quad \text{for} \quad i = 1,\dots,n$$

and determine a Nash equilibrium for the step $t+1$ instead of t.

In order to guarantee the existence of Nash equilibria we make the following

ASSUMPTIONS

1. All the sets $U_i \subseteq \mathrm{R}^{m_i}$ are compact and convex and all sets $X_i \subseteq \mathrm{R}^{l_i}$ are closed and convex for $i = 1,\dots,n$.
2. The set Z_t (3.3) of feasible controls is non empty.
3. All functions $u \to f_i(x(t),u)$, $u \in \prod_{j=1}^{n} \mathrm{R}^{m_i}$, are affine-linear for $i = 1,\dots,n$, i.e., of the form (3.9) for $x = x(t)$ and the matrices $f_{ij}(x)$ have full rank for all $x \in \prod_{j=1}^{n} \mathrm{R}^{l_j}$.

CONCLUSION 1 *For every $u^* \in Z_t$ and every $i = 1,\dots,n$ there is exactly one vector*

$$T_i u^* = (u_1^*,\dots,u_{i-1}^*,(T_i u^*)_i,u_{i+1}^*,\dots,u_n^*) \in Z_t \tag{5.2}$$

with

$$a_i^t(T_i u^*) \le a_i^t(u_1^*,\dots,u_{i-1}^*,u_i,u_{i+1}^*,\dots,u_n^*) \tag{5.3}$$

for all $(u_1^*,\dots,u_{i-1}^*,u_i,u_{i+1}^*,\dots,u_n^*) \in Z_t$.

Proof. By the assumptions 1 and 3 the set

$$Z_t^{i_*} = \{u_i \in U_i | (u_1^*,\dots,u_{i-1}^*,u_i,u_{i+1}^*,\dots,u_n^*) \in Z_t\} \tag{5.4}$$

is compact and convex for every $i = 1,\dots,n$ and the function

$$u_i \to a_i^t(u_1^*,\dots,u_{i-1}^*,u_i,u_{i+1}^*,\dots,u_n^*), \quad u_i \in \mathrm{R}^{m_i}$$

is strictly convex, hence continuous, as a consequence of Assumption 3. This implies the assertion of Conclusion 1. ■

CONCLUSION 2 *The mapping $T = T_n \circ T_{n-1} \circ \cdots \circ T_1 : Z_t \to Z_t$ with $T_i : Z_t \to Z_t$ defined by virtue of (5.2) and (5.3) for $i = 1,\dots,n$, is continuous.*

Proof. Let $(u^k)_{k\in \mathrm{N}}$ be a sequence in Z_t with $u^k \to u^*$ for some $u^* \in Z_t$. Then, for every $k \in \mathrm{N}$ and every $i = 1,\dots,n$ it follows that

$$\begin{aligned} &a_i^t(u_1^k,\dots,u_{i-1}^k,(T_i u^k)_i,u_{i+1}^k,\dots,u_n^k) \\ &\le a_i^t(u_1^k,\dots,u_{i-1}^k,u_i,u_{i+1}^k,\dots,u_n^k) \end{aligned} \tag{5.5}$$

Further we have, for every $i = 1, \ldots, n$,

$$a_i^t(u_1^*, \ldots, u_{i-1}^*, (T_i u^*)_i, u_{i+1}^*, \ldots, u_n^*) \leq a_i^t(u_1^*, \ldots, u_{i-1}^*, u_i, u_{i+1}^*, \ldots, u_n^*)$$

for all $u_i \in Z_t^{i*}$. Choose $i \in \{1, \ldots, n\}$ arbitrarily. Then there is a subsequence $(u^{k_l})_{l\in\mathbb{N}}$ and some $\hat{u}_i \in Z_t^{i*}$ such that

$$\lim_{l\to\infty} (T_i u^{k_l})_i = \hat{u}_i.$$

From (5.5) it therefore follows that

$$a_i^t(u_1^*, \ldots, u_{i-1}^*, \hat{u}_i, u_{i+1}^*, \ldots, u_n^*) \leq a_i^t(u_1^*, \ldots, u_{i-1}^*, u_i, u_{i+1}^*, \ldots, u_n^*)$$

for all $u_i \in Z_t^{i*}$, which implies

$$\hat{u}_i = (T_i u^*)_i.$$

In the same way one shows that for every subsequence $(u^{k_l})_{l\in\mathbb{N}}$ there exists a subsequence $(u^{k_{l_m}})_{m\in\mathbb{N}}$ with

$$\lim_{m\to\infty} (T_i u^{k_{l_m}})_i = (T_i u^*)_i$$

from which we infer that

$$\lim_{k\to\infty} (T_i u^k)_i = (T_i u^*)_i$$

holds true. Hence, every $T_i : Z_t \to Z_t$ for $i = 1, \ldots, n$ is continuous and therefore also

$$T = T_n \circ T_{n-1} \circ \ldots \circ T_1 : Z_t \to Z_t.$$

Since $Z_t \subseteq \prod_{j=1}^{n} \mathbb{R}^{m_j}$ is convex and compact, Brouwer's fixed point theorem implies the existence of a fixed point $u^t \in Z_t$ of

$$T : Z_t \to Z_t.$$

Each such fixed point $u^t \in Z_t$ is a Nash equilibrium for $Tu^t = u^t$, which is equivalent to (5.1). ∎

As a result of Conclusion 1 and 2 we obtain the following

THEOREM 5.1 *Under the Assumptions (1),(2),(3) there exists a Nash equilibrium.*

For the calculation of a Nash equilibrium the following iteration procedure is conceivable: Starting with $u^0 \in Z_t$ one defines recursively a sequence $(u^k)_{k\in\mathbb{N}_0}$ in Z_t by virtue of

$$u^{k+1} = Tu^k \tag{5.6}$$

where $T = T_n \circ T_{n-1} \circ \ldots T_1$ defined by (5.2) and (5.3) for every $i = 1, \ldots, n$.

THEOREM 5.2 *If the sequence* $(u^k)_{k\in \mathrm{N}_0}$ *defined by (5.6) converges to some*

$$u^t \in \prod_{j=1}^{n} \mathrm{R}^{m_j},$$

then $u^t \in Z_t$ *and* u^t *is a Nash equilibrium.*

Proof. $u^t \in Z_t$ follows from the closedness of Z_t and $u^t = Tu^t$ is a consequence of the continuity of T. If the Assumptions (1), (2), (3) are not necessarily satisfied, then the iteration (5.6) has to be performed as follows: Starting with $k = 0, i = 1$ and some $u^k \in Z_t$ we determine $\hat{u}_i \in Z_t^{ik}$ such that

$$a_i^t(u_1^k, \dots, u_{i-1}^k, \hat{u}_i, u_{i+1}^k, \dots, u_n^k) \le a_i^t(u_1^k, \dots, u_{i-1}^k, u_i, u_{i+1}^k, \dots, u_n^k) \quad (5.7)$$

for all $u_i \in Z_t^{ik}$, put

$$u^{k+1} = (u_1^k, \dots, u_{i-1}^k, \hat{u}_i, u_{i+1}^k, \dots, u_n^k) \quad (5.8)$$

and replace i by $i+1$ modulo n. ■

Instead of (1), (2), (3) we now make the assumptions:

(a) All sets $U_i \subseteq \mathrm{R}^{m_i}$ are compact and all sets $X_i \subseteq \mathrm{R}^{l_i}$ are closed for $i = 1, \dots, n$.

(b) For every $u = (u_1, \dots, u_n) \in Z_t$ and $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_n) \in Z_t$ and every $i \in \{1, \dots, n\}$ it follows that
$$(\tilde{u}_1, \dots, \tilde{u}_{i-1}, u_i, \tilde{u}_{i+1}, \dots, \tilde{u}_n) \in Z_t.$$

(c) All functions $u \to f_i(x(t), u)$, $u \in \prod_{j=1}^{n} \mathrm{R}^{m_j}$, are continuous for $i = 1, \dots, n$.

If Z_t is non-empty, then the Sssumptions (a) and (b) guarantee, for every $i \in \{1, \dots, n\}$ and $k \in \mathrm{N}_0$, the existence of $\hat{u}_i \in Z_t$ with (5.7).

Under the Assumptions (a), (b), (c) the following statement holds

THEOREM 5.3 *If the sequence* $(u^k)_{k\in N_0}$ *in* Z_t *defined by (5.7) and (5.8) converges to some*

$$u^t \in \prod_{j=1}^{n} \mathrm{R}^{m_j},$$

then $u^t \in Z_t$ *and* u^t *is a Nash-equilibrium.*

Proof. The assumptions (a) and (c) imply that Z_t is closed (even compact). Therefore $u^t \in Z_t$. Let us assume that u^t is not a Nash-equilibrium, i.e., (5.1) is violated for some $i \in \{1, \dots, n\}$. Then there is some

$$\tilde{u}^i = (u_1^t, \dots, u_{i-1}^t, \tilde{u}, u_{i+1}^t, \dots, u_n^t) \in Z_t$$

with $a_i^t(\tilde{u}^i) < a_i^t(u^t)$. Since $u \to a_i^t(u), u \in \prod_{j=1}^{n} \mathrm{R}^{m_j}$ is continuous (as a consequence of Assumption (c)), it follows that

If we put $\delta = \frac{1}{2}(a_i^t(u^t) - a_i^t(\tilde{u}^i))$, then we can infer that

$$a_i^t(u^{k+1}) > a_i^t(\tilde{u}^i) + \delta \quad \text{for all} \quad k \geq k_1(\delta).$$

If one puts, for every $k \in \mathrm{N}$,

$$(\tilde{u}^k)^i = (u_1^k, \ldots, u_{i-1}^k, \tilde{u}_i, u_{i+1}^k, \ldots, u_n^k)$$

then it follows that

$$\tilde{u}^i = \lim_{k\to\infty} (\tilde{u}^k)^i$$

and hence

$$a_i^t((\tilde{u}^k)^i) < a_i^t(\tilde{u}^i) + \delta < a_i^t(\tilde{u}^{k+1}) \quad \text{for all} \quad k \geq k_2(\delta)$$

which contradicts (5.7) and (5.8), since because of assumption (b), for every $k \in \mathrm{N}_0$, it follows from

$$\tilde{u}^i = (u_1^t, \ldots, u_{i-1}^t, \tilde{u}_i, u_{i+1}^t, \ldots, u_n^t) \in Z_t$$

and

$$u^{k+1} = (u_1^k, \ldots, u_{i-1}^k, \hat{u}_i, u_{i+1}^k, \ldots, u_n^k) \in Z_t$$

that also

$$(\tilde{u}^k)^i = (u_1^k, \ldots, u_{i-1}^k, \tilde{u}_i, u_{i+1}^k, \ldots, u_n^k) \in Z_t.$$

Hence, the assumption that u^t is not a Nash equilibrium is false. ■

In Pickl (1998) algorithms are implemented in order to treat an actual economic model, namely a Joint-Implementation Program which can be used to simulate and diminish the CO_2-emissions.

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