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Holdability and stabilizability of 2D Roesser model

by

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Abstract: The holdability and stabilizability problem of 2D Roesser model is formulated and solved. Conditions for the existence of solution to the problem are established. Two procedures for computation of a gain matrix of the state-feedback are proposed and illustrated by a numerical example.

Keywords: holdability, 2D Roesser model, state-feedback, stabilizability.

1. Introduction

Roughly speaking, the feedback holdability problem can be formulated as follows: given a discrete-time linear system, find a linear state-feedback such that the closed-loop system trajectory is nonnegative (positive) whenever the initial conditions are nonnegative (positive). The feedback holdability problem for standard continuous-time linear system has been considered in Berman and Stern (1987), Berman, Neumann and Stern (1989), for standard discrete-time linear systems it has been studied in Rumchev (2001) and for singular discrete-time linear systems in Kaczorek (2000).

The feedback holdability problem arises in the dynamic modeling of capacity planning in manufacturing systems, see Kaczorek (2001), Rumchev (2001). The feedback and positive feedback holdability problems have applications in chemical and production engineering, population dynamics, economics, biology and medicine, see for instance Caccetta, Foulds and Rumchev (2001), Rumchev (2001).

The most popular models of two-dimensional (2D) systems were introduced by Roesser (1975), Fornasini and Marchesini (1976, 1978) and Kurek (1985). These models have been generalized for singular 2D models in Kaczorek (1992, In this paper the holdability and stabilizability problem will be formulated and solved for the 2D Roesser model. To the best knowledge of the author the holdability and stabilizability problem for the 2D Roesser model has not been considered yet.

2. The preliminaries and the problem formulation

Let $\mathrm{R}^{n\times m}$ be the set of $n\times m$ real matrices and $\mathrm{R}^n = \mathrm{R}^{n\times 1}$. The set of $n\times m$ real matrices with nonnegative entries will be denoted by $\mathrm{R}_+^{n\times m}$ and $\mathrm{R}_+^n = \mathrm{R}_+^{n\times 1}$. Z_+ will denote the set of nonnegative integers and I_k will stand for the $k\times k$ identity matrix.

Consider the 2D Roesser model, Kaczorek (1985), Roesser (1975):

$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = A \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} + Bu_{ij}, \quad y_{ij} = C \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} + Du_{ij}, \qquad (1)$$
$$i,j \in \mathrm{Z}_+$$

where $x_{ij}^h \in \mathrm{R}^{n_1}$ and $x_{ij}^v \in \mathrm{R}^{n_2}$ are horizontal and vertical state vectors at the point $(i,j) \in \mathrm{Z}_+ \times \mathrm{Z}_+$, $u_{ij} \in \mathrm{R}^m$ is the input vector, $y_{ij} \in \mathrm{R}^p$ is the output vector and

$$\begin{aligned} &A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad A_{kl} \in \mathrm{R}^{n_k\times n_l}, \\ &B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad B_k \in \mathrm{R}^{n_k\times m}, \\ &C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \qquad C_k \in \mathrm{R}^{p\times n_k}, \qquad D \in \mathrm{R}^{p\times m}, \qquad k,l = 1,2. \end{aligned} \qquad (2)$$

DEFINITION 1 (Kaczorek 2001, 1996). The Roesser model (1) is called internally positive (shortly: positive) if for any nonnegative boundary conditions

$$x_{0j}^h \in \mathrm{R}_+^{n_1} \text{ for } j \in \mathrm{Z}_+ \quad \text{and} \quad x_{i0}^v \in \mathrm{R}_+^{n_2} \text{ for } i \in \mathrm{Z}_+ \qquad (3)$$

and all inputs $u_{ij} \in \mathrm{R}_+^m$, $i,j \in \mathrm{Z}_+$ we have $x_{ij} = \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} \in \mathrm{R}_+^n$, $n = n_1 + n_2$ and $y_{ij} \in \mathrm{R}_+^p$ for all $i,j \in \mathrm{Z}_+$.

THEOREM 1 (Kaczorek 2001, 1996). *The Roesser model (1) is internally positive if and only if*

$$A \in \mathrm{R}_+^{n\times n}, \qquad B \in \mathrm{R}_+^{n\times m}, \qquad C \in \mathrm{R}_+^{p\times n}, \qquad D \in \mathrm{R}_+^{p\times m}. \qquad (4)$$

DEFINITION 2. The Roesser model (1) is called asympotically stable if for the zero input $u_{ij} = 0$, $i,j \in \mathrm{Z}_+$ and finite

$$x^h = \sup_j \|x_{0j}^h\| \qquad \text{and} \qquad x^v = \sup_i \|x_{i0}^v\| \qquad (\|x\| \text{ is a norm of } x) \qquad (5)$$

THEOREM 2 (Kaczorek 1985, Kurek 1984). *The positive Roesser model (1) is asymptotically stable if and only if*

$$A_{11} \;\; \textit{and} \;\; A_{22} + A_{21}\left[I_{n_1} - A_{11}\right]^{-1} A_{12} \quad \textit{are asymptotically stable} \tag{6}$$

or equivalently

$$A_{22} \;\; \textit{and} \;\; A_{11} + A_{12}\left[I_{n_2} - A_{22}\right]^{-1} A_{21} \quad \textit{are asymptotically stable.} \tag{7}$$

Consider the Roesser model (1) with the state-feedback

$$\begin{aligned} &u_{ij} = K \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix}, \qquad K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}, \\ &K_l \in \mathrm{R}^{m \times n_l}, \qquad l = 1, 2. \end{aligned} \tag{8}$$

Substitution of (8) into (1) yields

$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = A_c \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} \tag{9}$$

where

$$A_c = \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix} = \begin{bmatrix} A_{11} + B_1 K_1, & A_{12} + B_1 K_2 \\ A_{21} + B_2 K_1, & A_{22} + B_2 K_2 \end{bmatrix}.$$

The problem can be stated as follows: given the matrices A and B of (1), find a gain matrix $K \in \mathrm{R}^{m \times n}$ of the state feedback (8) such that the state vector x_{ij} of the closed-loop system (9) satisfies the conditions

$$x_{ij} \in R_+^n \;\text{ for all boundary conditions (3) and } i, j \in \mathrm{Z}_+ \tag{10}$$

and

$$\lim_{i,j \to \infty} x_{ij} = 0 \;\text{ for all boundary conditions (3).} \tag{11}$$

In this paper conditions for the solvability of the problem will be established and two procedures for computation of the gain matrix K will be proposed.

3. Problem solution

Let Ω_K be the set of gain matrices K such that $A_c \in \mathrm{R}_+^{n \times n}$. If the set Ω_K is empty then the problem has no solution.

To check if the set Ω_K is not empty the elimination procedure presented in Kuhn (1956) can be applied. The necessary and sufficient conditions for the solvability of the inequalities were also established in Kuhn (1956).

1. *the i-th row A_i $(i = 1, \dots, n)$ of the matrix $A = [a_{ij}]$ contains at least one entry $a_{ij} < 0$ and the i-th row $\overline{B}_i$ of the matrix B is zero $(\overline{B}_i = 0)$;*
2. *the j-th column $(j = 1, \dots, n)$ of the matrix $A = [a_{ij}]$ contains at least two entries $a_{ij} < 0, a_{kj} < 0$ and corresponding entries of the matrix $B = b = [b_1 b_2 \dots b_n]^T$ $(m = 1$, T denotes the transpose) satisfy the conditions $b_i b_k < 0$.*

Proof. If at least one entry $a_{ij} < 0$ and $\overline{B}_i = 0$ then $A_i + \overline{B}_i K = A_i$ and there does not exist K such that $A_c \in \mathrm{R}_+^{n \times n}$.

For $A_c = A + bK = [\overline{a}_{ij}]$ and $K = [k_1 k_2 \dots k_n]$ we have

$$\overline{a}_{ij} = a_{ij} + b_i k_j \quad \text{and} \quad \overline{a}_{kj} = a_{kj} + b_k k_j. \tag{12}$$

From (12) it follows that if $a_{ij} < 0$, $a_{kj} < 0$ $(i \neq k)$ and $b_i b_k < 0$, then there does not exist k_j such that $\overline{a}_{ij} \geq 0$ and $\overline{a}_{kj} \geq 0$. ■

Let us assume that the set Ω_K is not empty. The transition matrix T_{ij} of the closed-loop system (9) is defined by (see Kaczorek, 1992, 1985, Klamka, 1991, Roesser, 1975):

$$T_{ij} = \begin{cases} I_n & \text{for } i = j = 0 \\ \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ 0 & 0 \end{bmatrix} & \text{for } i = 1,\ j = 0 \\ \begin{bmatrix} 0 & 0 \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix} & \text{for } i = 0,\ j = 1 \\ T_{10} T_{i-1,j} + T_{01} T_{i,j-1} & \text{for } i, j \in \mathrm{Z}_+,\ i + j > 0 \\ 0 \text{ (the zero matrix)} & \text{for } i < 0 \text{ or/and } j < 0. \end{cases} \tag{13}$$

Using (13) it is easy to show that if $A_c \in \mathrm{R}_+^{n \times n}$, then $T_{ij} \in \mathrm{R}_+^{n \times n}$ for $i, j \in \mathrm{Z}_+$, Kaczorek (2001, 1996). The solution to the equation (9) with (3) is given by

$$x_{ij} = \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} = \sum_{k=0}^{i} T_{i-k,j} \begin{bmatrix} 0 \\ x_{k0}^v \end{bmatrix} + \sum_{l=0}^{j} T_{i,j-l} \begin{bmatrix} x_{0l}^h \\ 0 \end{bmatrix}. \tag{14}$$

From (14) it follows that if $A_c \in \mathrm{R}_+^{n \times n}$, then the condition (10) is satisfied.

Therefore, the following lemma has been proved:

LEMMA 2. *Let Ω_K be not empty. The condition (10) is satisfied if and and only if a gain matrix K is chosen so that $A_C \in \mathrm{R}_+^{n \times n}$.*

Proof. The necessity will by proved by contradiction. Assume that the condition (10) is satisfied but $\bar{a}_{rs} < 0$ for some $r, s \in [1, \dots, n_1]$ and $\bar{a}_{ij} = 0$ for $i \neq r$ and $j \neq s$. Take $x_{ij} = e_s$ (the basic unit vector with 1 in its s-th position and all other entries equal to zero). Then, $x_{i+1,j}^h = \bar{a}_{rs} < 0$, and we got a contradiction.

To find $K \in \Omega_K$ satisfying (11) we shall use the following theorem, Kaczorek (2001),

THEOREM 3. *The system*

$$x_{i+1} = \widehat{A}x_i, \qquad i \in \mathrm{Z}_+ \tag{15}$$

is asymptotically stable if and only if

1. *all the coefficients of the polynomial*

$$\det\left[I_n\left(z+1\right)-\widehat{A}\right] = z^n + \widehat{a}_{n-1}z^{n-1} + \cdots + \widehat{a}_1 z + \widehat{a}_0$$

are positive, $\widehat{a}_i > 0$, $i = 0, 1, \ldots, n-1$, *or, equivalently,*

2. *the principal minors of the matrix* $\tilde{A} := I_n - \widehat{A} = [\tilde{a}_{ij}]$ *are positive, i.e.*

$$|\tilde{a}_{11}| > 0, \quad \begin{vmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{vmatrix} > 0, \quad \ldots, \quad \det \tilde{A} > 0. \tag{16}$$

Let the set Ω_K be non empty. Then we are looking for a gain matrix $K \in \Omega_K$ such that for the closed-loop matrix $A_c \geq 0$ the condition (6) or (7) is satisfied, i.e.

$$\begin{array}{l} \overline{A}_{11} \quad \text{and} \quad \overline{A'}_{22} := \overline{A}_{22} + \overline{A}_{21}\left[I_{n_1} - \overline{A}_{11}\right]^{-1}\overline{A}_{12} \\ \text{are asymptotically stable} \end{array} \tag{17}$$

or

$$\begin{array}{l} \overline{A}_{22} \quad \text{and} \quad \overline{A'}_{11} := \overline{A}_{11} + \overline{A}_{12}\left[I_{n_2} - \overline{A}_{22}\right]^{-1}\overline{A}_{21} \\ \text{are asymptotically stable.} \end{array} \tag{18}$$

Using Theorem 3 we may replace the condition (17) (or (18)) by the condition 1.:
the polynomials

$$\det\left[I_{n_1}\left(z+1\right)-\overline{A}_{11}\right] = z^{n_1} + \overline{a}_{n_1-1}z^{n_1-1} + \ldots + \overline{a}_1 z + \overline{a}_0 \tag{19}$$

$$\det\left[I_{n_2}\left(z+1\right)-\overline{A'}_{22}\right] = z^{n_1} + \overline{a'}_{n_2-1}z^{n_2-1} + \ldots + \overline{a'}_1 z + \overline{a'}_0 \tag{20}$$

have positive coefficients $\overline{a}_i > 0$, $i = 0, 1, \ldots, n_1 - 1$; $\overline{a'}_j > 0$, $i = 0, 1, \ldots, n_2 - 1$.
or by the condition 2.:

$$|\tilde{a}_{11}| > 0, \quad \begin{vmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{vmatrix} > 0, \quad \ldots, \quad \det \tilde{A}_{11} > 0 \tag{21}$$

and

$$|\tilde{a'}_{11}| > 0, \quad \begin{vmatrix} \tilde{a'}_{11} & \tilde{a'}_{12} \\ \tilde{a'}_{21} & \tilde{a'}_{22} \end{vmatrix} > 0, \quad \ldots, \quad \det \tilde{A'}_{22} > 0 \tag{22}$$

where

$$\tilde{A}_{11} = I_{n_1} - \overline{A}_{11} = [\tilde{a}_{ij}], \qquad \tilde{A'}_{22} = I_{n_2} - \overline{A'}_{22} = \left[\tilde{a'}_{ij}\right]$$

and $\overline{a}_i = \overline{a}_i(k)$, $i = 0, 1, \dots, n_1 - 1$; $\overline{a'}_j = \overline{a'}_j(k)$; $j = 0, 1, \dots, n_2$; $\tilde{a}_{kl} = \tilde{a}_{kl}(k)$, $k, l = 1, \dots, n_1$, $\tilde{a'}_{pq} = \tilde{a'}_{pq}(k)$, $p, q = 1, \dots, n_2$.

Similar (dual) relations can be written for (18).

For $m > 1$ the coefficients $\overline{a}_i, \overline{a}_j, \tilde{a}_{kl}$ and $\tilde{a}_{pq}$ depend nonlinearly on the entries of the gain matrix K.

Therefore, the following theorem has been proved:

THEOREM 4. *Let the set Ω_K be non empty. Then, the problem has a solution if there exists a $K \in \Omega_K$ such that the polynomials (19) and (20) have positive coefficients or, equivalently, the conditions (21) and (22) are satisfied.*

The dual theorem can be obtained by replacing the condition (18) by the conditions 1 and 2 of Theorem 3.

From the above considerations we have the following procedure for computation of the gain matrix K.

PROCEDURE 1. STEP 1. Find the set Ω_K such that $A_c \in \mathrm{R}_+^{n \times n}$. If the set Ω_K is empty the problem has no solution.
STEP 2. Find a $K \in \Omega_K$ such that the conditions (21) and (22) are satisfied or, equivalently, such that the coefficients of polynomials (19) and (20) are positive.

In some cases the problem can be solved by its decomposition into the following two subproblems.

SUBPROBLEM 1. Given A_{11}, B_1, find a gain matrix K_1 such that $K = [K_1\ 0] \in \Omega_K$ and the matrix $\overline{A}_{11} = A_{11} + B_1 K_1$ is asymptotically stable. Let the pair (A_{11}, B_1) satisfy the stabilizability condition

$$\operatorname{rank}\left[\begin{array}{cc} I_{n_1} z - A_{11} & B_1 \end{array}\right] = n_1 \qquad \text{for all } |z| \geq 1 \tag{23}$$

and $\overline{K}_1 \in \mathrm{R}^{m \times n_1}$ be a solution of the Subproblem 1.

Then we may proceed to the second subproblem.

SUBPROBLEM 2. Given $\overline{A}_{11}$, A_{12}, A_{21}, A_{22}, find a gain matrix K_2 such that $K = \left[\begin{array}{cc} \overline{K}_1 & K_2 \end{array}\right] \in \Omega_K$ and the matrix

$$\begin{aligned}\tilde{A}_{22} &= A_{22} + B_2 K_2 + \left(A_{21} + B_2 \overline{K}_1\right)\left[I_{n_1} - \overline{A}_{11}\right]^{-1}(A_{12} + B_1 K_2) \\ &= \widehat{A}_{22} + \widehat{B}_2 K_2\end{aligned}$$

is asymptotically stable, where

$$\begin{aligned}\widehat{A}_{22} &= A_{22} + \left(A_{21} + B_2 \overline{K}_1\right)\left[I_{n_1} - \overline{A}_{11}\right]^{-1} A_{12}, \\ \widehat{B}_2 &= B_2 + \left(A_{21} + B_2 \overline{K}_1\right)\left[I_{n_1} - \overline{A}_{11}\right]^{-1} B_1\end{aligned} \tag{24}$$

It is assumed that the pair $(\widehat{A}_{22}, \widehat{B}_2)$ is stabilizable, i.e.

$$\operatorname{rank}\left[\begin{array}{cc} I_{n_2}z - \widehat{A}_{22} & \widehat{B}_2 \end{array}\right] = n_2 \quad \text{for all } \ |z| \geq 1. \tag{25}$$

LEMMA 3. *Let $\overline{K}_1$ be chosen so that $\overline{A}_{11} = A_{11} + B_1\overline{K}_1$ is asymptotically stable. Then the matrix $I_{n_1} - \overline{A}_{11}$ is nonsingular and the condition (25) is equivalent to the condition*

$$\operatorname{rank}\left[\begin{array}{cc|c} I_{n_1} - \overline{A}_{11} & A_{12} & -B_1 \\ A_{21} + B_2\overline{K}_1 & I_{n_2}z - \overline{A}_{22} & B_2 \end{array}\right] = n_1 + n_2 \tag{26}$$

for all $|z| \geq 1$.

Proof. It is easy to verify that

$$\begin{aligned} &\left[\begin{array}{cc} I_{n_1} & 0 \\ -\left(A_{21} + B_2\overline{K}_1\right)\left[I_{n_1} - \overline{A}_{11}\right]^{-1} & I_{n_2} \end{array}\right] \\ &\times \left[\begin{array}{cc|c} I_{n_1} - \overline{A}_{11} & A_{12} & -B_1 \\ A_{21} + B_2\overline{K}_1 & I_{n_2}z - \overline{A}_{22} & B_2 \end{array}\right] \\ &= \left[\begin{array}{cc|c} I_{n_1} - \overline{A}_{11} & A_{12} & -B_1 \\ 0 & I_{n_2}z - \widehat{A}_{22} & \widehat{B}_2 \end{array}\right] \end{aligned} \tag{27}$$

where $\widehat{A}_{22}$ and $\widehat{B}_2$ are defined by (24).

From (27) it follows that (25) holds if and only if the condition (26) is satisfied since rank $\left[I_{n_1} - \overline{A}_{11}\right] = n_1$ and the first (left) matrix in (27) is nonsingular. ■

REMARK. It is well known, see Kaczorek (1992), that if there are no restrictions on K then there exist K_1 and K_2 such that the matrices $\overline{A}_{11}$ and $\tilde{A}_{22}$ are asymptotically stable if and only if the conditions (23) and (25) are satisfied. From the considerations presented we have the following procedure for computation of the gain matrix $K \in \Omega_K$.

PROCEDURE 2. STEP 1. The same as in Procedure 1.
STEP 2. Compute K_1 such that $K = \left[\begin{array}{cc} K_1 & 0 \end{array}\right] \in \Omega_K$ and the matrix $\overline{A}_{11} = A_{11} + B_1K$ is asymptotically stable.
If the Subproblem 1 has a solution $\overline{K}_1$, go to the step 3. Otherwise Procedure 2 does not allow for finding a solution to the problem.
STEP 3. Compute K_2 such that $\left[\begin{array}{cc} \overline{K}_1, & K_2 \end{array}\right] \in \Omega_K$ and the matrix $\tilde{A}_{22} = \widehat{A}_{22} + \widehat{B}_2K_2$ is asymptotically stable.

4. Examples

EXAMPLE 1. For the Roesser model (1) with

$$A = \left[\begin{array}{cc|c} -1 & 1 & 2 \\ 0 & -2 & 1 \\ \hline \end{array}\right], \qquad B = \left[\begin{array}{c} 1 \\ 0 \\ \hline \end{array}\right]$$

find a gain matrix $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$ such that the closed-loop system satisfies the conditions (10) and (11).

In this case the second row of A has the negative entry -2 and the second row of B is zero. Hence by the condition 1 of Lemma 1, the set Ω_K is empty and the problem has no solution.

EXAMPLE 2. For the Roesser model (1) with

$$A = \left[\begin{array}{cc|c} -1 & 1 & 2 \\ 0 & -2 & 2 \\ \hline -2 & 1 & 1 \end{array}\right], \qquad B = \left[\begin{array}{c} 1 \\ 1 \\ \hline -1 \end{array}\right]$$

find a gain matrix $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$ such that the closed-loop system satisfies the conditions (10) and (11).

In this case the first column of A has two negative entries $(-1, -2)$ and the corresponding entries of B are 1 and -1. Hence by the condition 2 of Lemma 1, the set Ω_K is empty and the problem has no solution.

EXAMPLE 3. For the Roesser model (1) with

$$A = \left[\begin{array}{cc|c} -1.5 & 0 & -1 \\ -2 & 0.2 & -0.2 \\ \hline -1.8 & 0.1 & -0.8 \end{array}\right], \qquad B = \left[\begin{array}{c} 1 \\ 1 \\ \hline 1 \end{array}\right] \tag{28}$$

find a gain matrix $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$ such that the closed-loop system satisfies the conditions (10) and (11).

In this case $n_1 = 2$, $n_2 = 1$ and $m = 1$. Using the Procedure 1 we obtain

STEP 1

In this case the closed-loop matrix (9)

$$A_c = A + BK = \left[\begin{array}{cc|c} k_1 - 1.5 & k_2 & k_3 - 1 \\ k_1 - 2 & k_2 + 0.2 & k_3 - 0.2 \\ \hline k_1 - 1.8 & k_2 + 0.1 & k_3 - 0.8 \end{array}\right] \in \mathrm{R}_+^{n \times n} \tag{29}$$

for $k_1/ge2$, $k_2 \geq 0$, $k_3 \geq 1$. Hence the set Ω_K is not empty and it has the form

$$\Omega_K\colon\ \left\{K = \begin{bmatrix} K_1 & k_2 & k_3 \end{bmatrix} : k_1 \geq 2,\, k_2 \geq 0,\, k_3 \geq 1\right\}. \tag{30}$$

STEP 2

Using the conditions (21), (22) and (29) we obtain

$$\tilde{A}_{11} = I_{n_1} - \overline{A}_{11} = \begin{bmatrix} 2.5 - k_1 & -k_2 \\ 2 - k_1 & 0.8 - k_2 \end{bmatrix},$$

and

$$
\begin{aligned}
&|\tilde{a}_{11}| = |2.5 - k_1| > 0 \\
&\det \tilde{A}_{11} = 2 - 0.5k_2 - 0.8k_1 > 0 \\
&\det \tilde{A}'_{22} = \frac{5.6k_1 + 9.6k_2 + 6.1k_3 - 20.1}{8k_1 + 5k_2 - 20} > 0.
\end{aligned} \tag{31}
$$

It is easy to check that

$$
K = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \in \Omega_K \text{ defined by (30)} \tag{32}
$$

and it satisfies the conditions (31).

Using the Procedure 2 we obtain

STEP 1

The set Ω_K is given by (30).

STEP 2

In this case $K_1 = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$

$$
\overline{A}_{11} = \begin{bmatrix} -1.5 & 0 \\ -2 & 0.2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} k_1 - 1.5 & k_2 \\ k_1 - 2 & k_2 + 0.2 \end{bmatrix}
$$

and

$$
\begin{aligned}
&\det[I_{n_1} z - \overline{A}_{11}] = \begin{vmatrix} z - k_1 + 1.5 & -k_2 \\ 2 - k_1 & z - k_2 - 0.2 \end{vmatrix} \\
&= z^2 + (1.3 - k_1 - k_2)z + 0.2k_1 + 0.5k_2 - 0.3.
\end{aligned}
$$

The matrix $\overline{A}_{11}$ is asymptotically stable if $k_1 = 2$ and $k_2 = 0$, since $\det[I_{n_1} z - \overline{A}_{11}] = z^2 - 0.7z + 0.1$. Hence $\overline{K} = \begin{bmatrix} 2 & 0 \end{bmatrix}$.

STEP 3

Using (24) we obtain

$$
\begin{aligned}
&\widehat{A}_{22} = A_{22} + (A_{21} + B_2 \overline{K}_1)[I_{n_1} - \overline{A}_{11}]^{-1} A_{12} \\
&= -0.8 + \begin{bmatrix} 0.2 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.8 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -0.2 \end{bmatrix} = -1.225, \\
&\widehat{B}_2 = B_2 + (A_{21} + B_2 \overline{K}_1)[I_{n_1} - \overline{A}_{11}]^{-1} B_1 \\
&= 1 + \begin{bmatrix} 0.2 & 0.1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.8 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1.525.
\end{aligned}
$$

Hence for $K_2 = [k_3]$ we have

$$
\tilde{A}_{22} = \widehat{A}_{22} + \widehat{B}_2 K_2 = -1.225 + 1.525k_3
$$

For $k_3 = 1$ the matrix $\tilde{A}_{22} = 0.3$ is asymptotically stable.

The desired gain matrix is equal to $K = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix}$ and it coincides with (32) obtained by the use of Procedure 1.

For (32) the matrix (29) has the form

$$A_c = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.2 & 0.8 \\ 0.2 & 0.1 & 0.2 \end{bmatrix}$$

and by Theorem 2 the closed-loop system is asymptotically stable since the matrices

$$\overline{A}_{11} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.2 \end{bmatrix} \quad \text{and} \quad \overline{A}_{22} + \overline{A}_{21}[I_{n_1} - \overline{A}_{11}]^{-1}\overline{A}_{12} = 0.3$$

are asymptotically stable.

5. Concluding remarks

The holdability and stabilizability problem for 2D Roesser model has been formulated. Conditions for solvability of the problem have been established. Two procedures for computation of the gain matrix of the state-feedback have been proposed and illustrated by a numerical example. The considerations also can be applied to the first 2D Fornasini–Marchesini model which is a particular case of the 2D Roesser model, Kaczorek (1992, 1985). With slight modifications the considerations can be extended for the second 2D Fornasini–Marchesini model and the general 2D model, Kaczorek (1992, 1985). An extension of the considerations for singular 2D models is an open problem and it will be considered in a forthcoming paper.

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