

On an operational model of single investment selection with information cost

by

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Abstract: An operational model of portfolio selection is presented in this paper. The target of a risk-neutral investor is to select the best portfolio composed of assets with the greatest rate of return. The term "best" means that probability of attaining the return required by an investor is the largest for all possible stopping times, in which he/she has the right to buy information concerning the random vector of returns from investments (assets).

Keywords: portfolio, information cost, Fisher information, optimal stopping time.

1. Introduction

All investments, and investments in securities in particular, are biased with risk connected with uncertainty of future returns. Models so far presented in the literature (Roy, 1952, Markowitz, 1952, Telser, 1956) can be generally characterised as tending to obtain possibly high returns at possibly low risk. They have another common feature – they do not take into account the costs to the investor of getting the information necessary to make the right decision. This is an assumption that simplifies the models considered in comparison with the real world. Elton and Gruber (1998) recommend using information contained in analysts' forecasts in constructing an investment portfolio. Tests, which they carried out, showed that information prepared by groups of experts was reflected in prices of shares. However, these tests do not answer the following question: are analysts' forecasts worth paying for them?

Some models in Banek (2000) and Banek, Kowalik (1999), Banek, Kowalik, Kozłowski (1999, 2000) considered a set of information available to the executive before making a decision and allowed for creating a specific plan of activity and for making a preliminary estimate of possible effects. On the other hand, this paper is an attempt to deal with the problem of purchase information by an investor during the decision process and to answer the question: how to

use the information coming to the investor. As a result of the decision process we obtain:

- how much will we spend on the purchase of information;
- how do we invest in securities available on the market.

By purchasing information, the investor obtains from the Information Seller a more concentrated distribution of the random vector of returns. It is obvious that the more precise is the analysis made by experts, the better are their prognoses, and more expensive as well. So, the investor during the decision process spends more and more money on purchase on information and obtains better estimates of parameters of distribution in return, but on the other hand less money remains for investing in the portfolio itself. A question connected with the previous problem appears: is it possible to find a "golden middle" between the amount of money spent during the process of purchase of information and the risk reduction obtained thanks to this purchase. The decision model presented in the paper works in the following way. The risk-neutral investor obtains from the Information Seller a more concentrated distribution of the random vector of returns and invests in assets with the highest rate of return (makes single investment selection). In the classical portfolio selection theory reduction of risk consists in selection of securities with different dynamics of returns (diversification). In the considered case the investor also tends to reduce the risk only by purchase of information about the distribution. Instead, in Banek and Kozłowski (2002) the case of a risk-aversion investor, who reduces the risk both by purchase of information and portfolio diversification, is considered.

In order to solve the operational model of portfolio selection with information cost, optimal principles of stopping random Markov sequences were used (see Shiryaev, 1976). The above principles are also used in models of pricing financial instruments (see Melnikov, 1997, Shiryaev, Kabanov, Kramkov, Melnikov 1994A, 1994B). The paper is organized as follows. In the next section we introduce our basic assumptions concerning the market of information. In Section 3 the principles of constructing the operational models of portfolio selection regarding the purchase of information are considered. In Section 4 a problem of purchase of information by a risk-neutral investor is considered. Next, a theorem which presents an optimal strategy of the decision maker is formulated. We close this section by proving the theorem. Some technical results are included in Appendix.

2. Basic assumptions

The assumption below concern the functioning of the information-selling institutions on the market. These assumptions can be found also in the book by Banek (2000). The work of analysts – information sellers consists in reflecting all data and news available on the market through the conditional distribution of a random vector ξ of rates of returns from particular assets. For gaussian

tation vector and the conditional covariance matrix. By t we will denote the duration of work of analysts who process available information. Let $c(t)$ denote the amount of money spent on purchase of information, where $c(t)$ is a continuous function, strictly increasing and $t \geq 0$, $c(0) = 0$. As a result of their investigations, the analysts create a vector $m(t)$ and a matrix $Q(t)$ whose values are corrected in comparison with $m(0)$ and $Q(0)$ by results of their analyses. In general, $m(\cdot)$, $Q(\cdot)$ are stochastic processes for which it seems to be rational to assume that $m(\cdot)$ is a martingale and $Q(\cdot) = (q_{ij}(\cdot))_{i,j=1,\dots,n}$ - a square, symmetrical matrix with differentiable elements such that

$$x^T \dot{Q}(t)x < 0 \text{ for all } x \neq 0 \quad (1)$$

The latter requirement results from the fact, that as the work of analysts lasts longer and longer, then mean-square error of $m(\cdot)$ estimation should decrease.

(A1) The Information Seller informs for free about the formula of the $Q(t)$ function for $t \geq 0$, confirming in this way the quality of the offered service because the covariance matrix is responsible for the estimation error. The Investor planning the purchase of information knows that for the price $c(t)$ he/she will obtain the distribution $N(m(t), Q(t))$ in which unknown $m(t)$ will be revealed only after paying the fee for the service.

(A2) Matrix $Q(\cdot)$ is deterministic and has the form

$$Q(t) = Q \left(I + Q \int_0^t H^T(s)H(s)ds \right)^{-1} \quad (2)$$

where $H(t)$ is some square, symmetrical matrix, with the trace $SpH^T(t)H(t) > 0$ and continuous elements $t \rightarrow h_{ij}(t)$. It is easy to see that inequality (1) is the consequence of the form of the matrix $Q(t)$. In the considered case we assume that the encoding matrix is constant $H(t) = H$, $t \geq 0$.

Let $G_n(z, m, Q)$ denote n -dimensional Gaussian density with the mean m and covariance matrix Q . The value of the Fisher information I_t included in the distribution $N(m(t), Q(t))$ is given by the following formula

$$\begin{aligned} I_t &= \int_{R^n} \frac{\|\nabla G_n(z, m, Q(t))\|^2}{G_n(z, m, Q(t))} dz \\ &= SpQ^{-1}(t) \left[\int_{R^n} [z - m][z - m]^T G_n(z, m, Q(t)) \right] Q^{-1}(t) \\ &= SpQ^{-1}(t) = Sp(Q^{-1} + tH^T H) \equiv a + bt \end{aligned}$$

what shows that for $Q(\cdot)$ as above I_t grows linearly with t . Hence, along with increasing t - the work time of analysts and experts - the amount of the Fisher information about the distribution of a random vector ξ also increases and.

(see Aivazian and Mkhitarian, 1998). We can also see, using (1) and (2), that dispersion of realisation of the random vector ξ decreases.

(A3) Activities performed by analysts should be rewarded. That is why the cost of purchase of information is proportional to I_t , what we adopt as our next assumption, i.e.

$$c(t) = c \cdot (I_t - I_0)$$

so that we obtain

$$c(t) = \alpha t, \quad \alpha = c \cdot SpH^T H \quad (3)$$

In the previously considered planning models (see Banek 2000, and Banek and Kowalik, 1999, Banek, Kowalik, Kozłowski, 1999, 2000) it was assumed that $m(t) = m(0) = m$, but here the following assumption is more justified.

(A4) In the processing of the set of information owned by the information seller in order to estimate (e.g. in the mean-square sense) the vector of returns ξ requires calculating the conditional expected value $E[\xi|Y_t]$, where $Y_t = \{y_s, 0 \leq s \leq t\}$ is a part of the set of information, used for the price $c(t)$. The model of such processing consists in defining the observation process $\{y_t, t \geq 0\}$ (see e.g. Lipcer and Shiryaev, 1981), where

$$y_t = \int_0^t H(s)\xi ds + b_t.$$

Matrix H is a matrix of encoding the new ξ , and $\{b_t, t \geq 0\}$ is a Wiener process, stochastically independent of ξ . From the theorem on Kalman–Bucy method linear filtration we know that:

1. the conditional distribution

$$P(\xi \leq a|Y_t)$$

is Gaussian with parameters $(m(t), Q(t))$.

2. optimal estimator ξ , which is the following

$$m(t) = E[\xi|Y_t]$$

and the error matrix

$$Q(t) = E[(\xi - m_t)(\xi - m_t)^T|Y_t]$$

are given by equations

$$m(t) = m + \int_0^t Q(s)H^T(s)dv_s \quad (4)$$

where v_t is an innovation process, defined as follows

$$v_t = \int_0^t [dy_s - H(s)m(s)ds],$$

which is a Wiener process (see Lipcer, Shiryaev, 1981).

3. Decision models

Let (Ω, A, P) be a probability space and

$$T = \{(t_0, t_1, \dots, t_N); 0 = t_0 \leq t_1 \leq \dots \leq t_N = c^{-1}(M)\}.$$

A stochastic process $(m(t), F_t)$, $t \in T$, is defined by formula (4), where $F = (F_t)_{t \in T}$ means a non-decreasing family of σ -fields $F_t = \sigma\{m(s) : 0 \leq s \leq t\}$. In each moment t_i , $i = 0, 1, \dots, N$ the investor can stop the observation of "behaviour" of securities and invest or can decide to continue the observation. Let the random variable τ

$$\Omega \ni \omega \rightarrow \tau(\omega) \in T$$

be a Markov moment with respect to $F = (F_t)_{t \in T}$, and Υ denote the class of all Markov moments defined on Ω with values in T with respect to the family $F = (F_t)_{t \in T}$.

Let us introduce some necessary notations. Let x be a vector of investment, J – an n -element vector of 1's, M – the total capital, $z > 0$ – a minimal aspiration level required by the investor. Let us assume that $U(x, m, Q)$ denotes a decision maker's utility function (see Elton and Gruber, 1998, Kruschwitz, 2000, Sharpe, Alexander, Bailey, 1999). The investor considers only portfolios with the largest utilities. So, the maximum of the utility function in each moment $t \in T$ is

$$f(t, m(t), Q(t)) \triangleq \max_{0 \leq (x, J) \leq M - c(t)} U(x, m(t), Q(t)). \quad (5)$$

Let us consider the following decision situation. The investor reviews information until the moment $\tau \in \Upsilon$, information contained in σ -fields F_τ , pays for it the amount of $c(\tau)$, where $c(\cdot)$ is a continuous increasing function and $c(0) = 0$. The remaining capital $M - c(\tau)$ is used to construct a portfolio by maximising the utility function. The decision maker wants to invest the capital so that average portfolio utility over all the stopping times is the largest. We can present the operational model as follows

$$\sup_{\tau \in \Upsilon} E f(\tau, m(\tau), Q(\tau)). \quad (6)$$

Instead, if the investor also uses the Roy criterion (see Roy, 1952) when constructing a portfolio, then the operational model can be presented as

$$\sup_{\tau \in \Upsilon} P(f(\tau, m(\tau), Q(\tau)) \geq z). \quad (7)$$

In this case, the decision maker chooses such a moment of stopping the process of purchase of information, for which the probability of exceeding the aspiration level z by the maximum of a utility function is the largest.

4. Purchase of information by a risk-neutral investor

We assume that the investor's utility function is the following:

$$U(x, m(t), Q(t)) = \langle x, m(t) \rangle - \beta x^T Q(t)x \quad (8)$$

where β denotes a risk aversion coefficient (see Sharpe, Alexander, Bailey, 1999). The solution of the operational model (6) for an investor with risk aversion ($\beta > 0$) can be found in Banek, Kozłowski (2002). The problem of purchase of information by a risk-neutral investor ($\beta = 0$) is also interesting. We will deal with it in the further part of this section.

We assume that a risk-neutral investor uses the operational model (7), the utility function is of the form (8) and $\beta = 0$. Additionally, we exclude the short-selling from the portfolio selection process. So, the mathematical model of this problem is the following:

$$\sup_{\tau \in \mathcal{T}} P\left(\max_{0 \leq \langle x, J \rangle \leq M - c(\tau)} \langle x, m(\tau) \rangle \geq z\right) \quad (9)$$

where $x \in R_+^k$ is a vector of investment of the Executive. The investor selects the stopping time τ so that probability of attaining the profit from the investment on the level z would be the greatest for all stopping times. Simultaneously, he takes under consideration only such a portfolio which gives the greatest rate of return (the portfolio composed of assets with the greatest rate of return, he makes single investment selection).

It is easy to see that the set of admissible solutions

$$D^t = \{x \in R_+^k : 0 \leq \langle x, J \rangle \leq M - c(t)\}$$

is a convex polyhedron with vertices $x^0 = (0, 0, \dots, 0)$, $x^1 = (M - c(t), 0, \dots, 0)$, $x^2 = (0, M - c(t), \dots, 0)$, \dots , $x^k = (0, 0, \dots, M - c(t))$.

So, the solution of the linear programming problem

$$\max_{0 \leq \langle x, J \rangle \leq M - c(t)} \langle x, m(t) \rangle \quad (10)$$

is attained in one of vertices of the polyhedron D^t (the solution can be in exactly one vertex or on an edge). If the solution of the problem (10) is attained in vertices x^i and x^j , we choose the vertex $x^{\min(i,j)}$ (such choice is connected with the construction of sets D_i , $1 \leq i \leq k$ see Lemma 1). Hence, for a fixed $t \in [0, c^{-1}(M)]$

$$\max \langle x, m(t) \rangle = \max \langle x^i, m(t) \rangle.$$

Let us introduce a transformation $\Xi : R^k \rightarrow \{x^0, x^1, \dots, x^k\}$ for which

$$\max_{i \in \{0, 1, \dots, k\}} \langle x^i, m(t) \rangle = \langle \Xi m(t), m(t) \rangle.$$

The transformation Ξ assigns a vertex of the polyhedron D^t to the vector $m(t)$ (see Banek, 2000, Chapter 8.1). So the problem (9) can be reduced to the form

$$\sup_{\tau \in \Upsilon} P(\langle \Xi m(\tau), m(\tau) \rangle \geq z) \quad (11)$$

or

$$\sup_{\tau \in \Upsilon} P[(M - c(\tau)) \langle \Theta \Xi m(\tau), m(\tau) \rangle \geq z] \quad (12)$$

where the transformation $\Theta x = \frac{x}{\|x\|}$ means normalisation of any vector $x \in R^k$.

Before we present the main result of the considered model, we will introduce the necessary notations. Let

$$g(t, m) = I_{\{(M - c(t)) \langle \Theta \Xi m, m \rangle \geq z\}} \quad (13)$$

where

$$I_{\{y \geq z\}} = \begin{cases} 1, & \text{when } y \geq z \\ 0, & \text{when } y < z \end{cases}.$$

So, the problem (12) can be reduced to the following form

$$\sup_{\tau \in \Upsilon} E[g(\tau, m(\tau))]. \quad (14)$$

The problem (14) consists in finding a possibly maximal mean prize (see Melnikov, 1997, Shiryaev, 1976) and such a time τ^* for which the following formula holds

$$Eg(\tau^*, m(\tau^*)) = \sup_{\tau \in \Upsilon} E[g(\tau, m(\tau))].$$

Let us introduce additional notations. Let for arbitrary $s, t \in T, s < t$

$$\psi(m(s), s, t) = \sum_{n=1}^k U_n(m(s), s, t) W_n(m(s), s, t) \quad (15)$$

where

$$U_n(m(s), s, t) = G\left(\frac{(M - c(t))m_n(s) - z}{(M - c(t))\sqrt{a_{n-1}(s) - a_{n-1}(t)}}\right), \quad (16)$$

$m(t) = \text{col}(m_1(t), \dots, m_k(t))$, $G(\cdot)$ is a distribution function of $N(0, 1)$, and $q_{ij}(s)$, $1 \leq i, j \leq k$, are elements of the matrix $Q(s)$. Next,

$$W_n(m(s), s, t) = \int_0^\infty \cdots \int_0^\infty \frac{1}{(2\pi)^{\frac{k-1}{2}} \sqrt{\det V_n(s, t)}} \times \exp \left\{ -\frac{1}{2} (u - M^n(s))^T V_n^{-1}(s, t) (u - M^n(s)) \right\} du_1 \dots du_{k-1} \quad (17)$$

where

$$V_n(s, t) = [v_{ij}^n(s, t)] \text{ for } i, j \in \{1, \dots, n-1, n+1, \dots, k\}$$

$$v_{ij}^n(s, t) = \int_s^t \sum_{l=1}^k (b_{nl}(r) - b_{il}(r))(b_{nl}(r) - b_{jl}(r)) dr$$

$b_{ij}(s)$, $1 \leq i, j \leq k$ are elements of the matrix $Q(s)H^T(s)$, and $u = \text{col}(u_1, \dots, u_{k-1})$

$$M^n(s) = \text{col}(m_n(s) - m_1(s), \dots, m_n(s) - m_{n-1}(s), \\ m_n(s) - m_{n+1}(s), \dots, m_n(s) - m_k(s)).$$

In Theorem 1 below an explicit solution of the problem (9) is given – an optimal moment τ^* of stopping the observation process (the purchase of information), an optimal portfolio x_{opt} and the highest probability of attaining the return z .

THEOREM 1 *If assumptions (A1), (A2), (A3), (A4) are satisfied, the solution of the problem*

$$\sup_{\tau \in \mathcal{T}} P \left(\max_{0 \leq \langle x, j \rangle \leq M - c(\tau)} \langle x, m(\tau) \rangle \geq z \right)$$

is:

a)

$$\tau^* = \min \{ t \in T : t \in A \cup B \} \quad (18)$$

where

$$A = \{ t_0 \leq t_i \leq t_N : (M - c(t_i)) \langle \Theta \Xi m(t_i), m(t_i) \rangle \geq z \} \\ B = \{ t_0 \leq t_i \leq t_N : (M - c(t_i)) \langle \Theta \Xi m(t_i), m(t_i) \rangle < z, \\ \psi(m(t_i), t_i, t_j) = 0 \text{ for } j = i + 1, \dots, N \}$$

where $\psi(\cdot)$ is defined by formulas (15)–(17).

b)

$$\tau^* = \min \{ t \in T : (M - c(\tau^*)) \langle \Theta \Xi m(\tau^*), m(\tau^*) \rangle \geq z \} \quad (19)$$

c)

$$\sup_{\tau \in \mathcal{T}} P\left(\max_{0 \leq \langle x, J \rangle \leq M - c(\tau)} \langle x, m(\tau) \rangle \geq z\right) = \psi(m(0), 0, \tau^*).$$

Proof. The stochastic process $(m(t_i), F_{t_i}, P_i)$ $t_i \in T$, $0 \leq i \leq N$ denotes a Markov chain with values in R^k and $F = (F_{t_i})$, $t_i \in T$ denotes a non-decreasing family of σ -fields $F_{t_i} = \sigma\{m(s) : 0 \leq s \leq t_i\}$ generated by the process

$$m(t) = \text{col}(m_1(t), \dots, m_k(t))$$

defined with the formula (4). In order to simplify notations we put $F_i = F_{t_i}$ for $0 \leq i \leq N$.

In R^k for $1 \leq i \leq k$ we define the sets

$$D_i = \{(x_1, \dots, x_k) : x_i > x_1, \dots, x_i > x_{i-1}, x_i \geq x_{i+1}, \dots, x_i \geq x_k\}.$$

From the Chapman–Kolmogorov equation for an arbitrary moment t_i , $0 \leq i \leq N$ we obtain:

$$\begin{aligned} & P((M - c(t_i)) \langle \Theta \Xi m(t_i), m(t_i) \rangle \geq z | F_0) \\ &= \int_R I_{\{M - c(t_i)y \geq z\}} dP(t_0, m; t_i, y) \end{aligned} \quad (20)$$

where $\frac{\partial P(t_0, m; t_i, y)}{\partial y}$ is a density function of the random variable $\langle \Theta \Xi m(t_i), m(t_i) \rangle$ for each fixed t_i . By Lemma 2 (see appendix), we have

$$\begin{aligned} & \frac{\partial P(\langle \Theta \Xi m(t_i), m(t_i) \rangle \leq y | F_0)}{\partial y} \\ &= \sum_{n=1}^k P(m(t_i) \in D_n | F_0) \frac{1}{\sqrt{2\pi(q_{nn}(0) - q_{nn}(t_i))}} \\ & \times \exp\left\{-\frac{(y - m_n)^2}{2(q_{nn}(0) - q_{nn}(t_i))}\right\} \end{aligned} \quad (21)$$

where $P(m(t_i) \in D_n | F_0)$, $n = 1, \dots, k$, $i = 1, \dots, N$ is given by the formula (40). By substituting the formula (21) to (20) we obtain

$$P((M - c(t_i)) \langle \Theta \Xi m(t_i), m(t_i) \rangle \geq z | F_0) = E[g(t_i, m(t_i)) | F_0] \quad (22)$$

where the distribution $\langle \Theta \Xi m(t_i), m(t_i) \rangle$ is given by the formula (21).

According to the optimal stopping principle we define an operator T_i , $i = 1, \dots, N$, acting on $g(0, m)$ in the following way

$$T_i g(0, m) = E[g(t_i, m(t_i)) | F_0]$$

Thus, the investor being in the moment of observation $t_j, j < N$ of the stochastic process $(m(t), F_t)$ has already paid $c(t_j)$ and obtained from the Information Seller a distribution. Next, he/she decides to buy some more information until the moment $t_{j+i}, i = 0, \dots, N - j$ and will pay $c(t_{j+i}) - c(t_j)$ more. Then, the density function $\frac{\partial P(t_j, m(t_j); t_{j+i}, y)}{\partial y}$ of the random variable $\langle \Theta \Xi m(t_{j+i}), m(t_{j+i}) \rangle$ for each fixed t_j is of the form

$$\begin{aligned} & \frac{\partial P(\langle \Theta \Xi m(t_{j+i}), m(t_{j+i}) \rangle \leq y | F_j)}{\partial y} \\ &= \sum_{n=1}^k P(m(t_{j+i}) \in D_n | F_j) \frac{1}{\sqrt{2\pi(q_{nn}(t_j) - q_{nn}(t_{j+i}))}} \\ & \times \exp\left\{-\frac{(y - m_n(t_j))^2}{2(q_{nn}(t_j) - q_{nn}(t_{j+i}))}\right\} \end{aligned} \tag{24}$$

where $P(m(t_{j+i}) \in D_n | F_j), n = 1, \dots, k$ is given by (40). So,

$$\begin{aligned} T_i g(t_j, m(t_j)) &= E[g(t_{j+i}, m(t_{j+i})) | F_j] \\ &= P((M - c(t_{j+i})) \langle \Theta \Xi m(t_{j+i}), m(t_{j+i}) \rangle \geq z | F_j) \end{aligned} \tag{25}$$

Using Lemma 2 we have

$$T_i g(t_j, m(t_j)) = \psi(m(t_j), t_j, t_{j+i})$$

where $\psi(\cdot)$ is defined by formulas (15)–(17).

From the general theory of optimal stopping principles (see Shiryaev, 1976) it is known that the Markov moment

$$\begin{aligned} \tau^* &= \min\{t_i : g(t_i, m(t_i)) \\ &= \max(g(t_i, m(t_i)), E[g(t_{i+1}, m(t_{i+1})) | F_i], \dots, E[g(t_N, m(t_N)) | F_i])\} \end{aligned}$$

is optimal where $g(t_i, m(t_i)), i = 1, \dots, N$ is defined by (13). ■

Let us look once again at Theorem 1. We can see that the optimal moment of stopping the process of purchasing the information is defined as the smallest of moments in which the value of the utility function $g(t, m(t))$ is:

* equal to 1

* equal to 0, when expected probabilities of attaining return z in later moments are equal to zero.

Purchased information (see assumption (A2)) results in obtaining a new, more concentrated distribution of a random vector of returns. So, according to the construction of the operational model (9) we can see that the Investor should invest the entire remaining capital $M - c(\tau^*)$ in the security with the highest return, where the vector of returns $m(\tau^*)$ is revealed by the Information

5. Summary

In this paper a specific operational model of portfolio selection, which takes into account paying for information essential for the investor was discussed. Money spent on purchase of information gives better estimates of parameters of a random vector of returns. On the other hand, more money spent during the purchase of information means less money spent on "direct" investments in securities. That is why the purpose of the investor is to find an optimal moment of stopping the observation process (the process of purchase of information). In this paper a precise recipe of finding an optimal stopping moment and constructing a portfolio is given.

It seems that the presented operational model requires some additional restrictions, namely, when the Investor is very requiring, even greedy (sets a high level of z), it may happen that the optimal stopping moment is $\tau = t_N$, then there is no money left to invest in the securities. From the practical point of view, such a situation does not make sense. The problem can be solved thanks to either of the two additional assumptions:

- a. maximal possible cost of purchase of information in amount $c(T)$ until the moment T is a fixed part of the capital M owned by the Investor (fixed earlier),
- b. first, the investor considers a planning model of (9) and then fixes a moment T until which he/she can perform observations.

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Appendix

LEMMA 1 For arbitrary natural $n \geq 2$ the space $R^n = \bigcup_{i=1}^n D_i$ where $D_i \cap D_j = \emptyset$ for $i \neq j$ and

$$D_i = \{(x_1, \dots, x_n) : x_i > x_1, \dots, x_i > x_{i-1}, x_i \geq x_{i+1}, \dots, x_i \geq x_n\}.$$

Proof. Let us introduce the necessary notations. Let

$$x = (x_1, x_2, x_3, \dots, x_n)$$

$$w_0 = (1, 1, 1, \dots, 1)$$

$$w_1 = (1, 0, 0, \dots, 0)$$

.....

$$w_n = (0, 0, 0, \dots, 1).$$

For an arbitrary $0 \leq i < j \leq n$

$$p(i, j) = |\text{col}(x, w_0, w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_{j-1}, w_{j+1}, \dots, w_n)|$$

where $|\cdot|$ denotes the determinant of a matrix, $\text{col}(x, w_0, w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_{j-1}, w_{j+1}, \dots, w_n)$ is a matrix of the $n \times n$ size. Thus, the plane

is spanned on $n - 1$ vectors $w_0, w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_{j-1}, w_{j+1}, \dots, w_n$ and is of the form

$$\pi_{i,j} : x_i - x_j = 0.$$

Let us consider the set D_1 bounded with $n - 1$ planes

$$\pi_{1,2}, \pi_{1,3}, \dots, \pi_{1,n}$$

which is of the form $D_1 = \{x \in R^n : x_1 \geq x_2, x_1 \geq x_3, \dots, x_1 \geq x_n\}$.

By analogous reasoning, we obtain that for an arbitrary $1 < k < n$ the set D_k is bounded by $n - 1$ planes

$$\pi_{1,k}, \pi_{2,k}, \dots, \pi_{k-1,k}, \pi_{k,k+1}, \dots, \pi_{k,n}$$

and is of the form $D_k = \{x \in R^n : x_k > x_1, \dots, x_k > x_{k-1}, x_k \geq x_{k+1}, \dots, x_k \geq x_n\}$. On the other hand the set $D_n = \{x \in R^n : x_n > x_1, x_n > x_2, \dots, x_n > x_{n-1}\}$.

Thus, the sets constructed in this way are disjoint and

$$R^n = \bigcup_{i=1}^n D_i. \quad \blacksquare$$

LEMMA 2 *If $(m(t), F_t)_{0 \leq t < \infty}$ is a k -dimensional stochastic process described with (4) and (A2) is satisfied, then for arbitrary $0 \leq s < t < \infty$*

$$\begin{aligned} & P((M - c(t))(\Theta \Xi m(t), m(t)) \geq z | F_s) \\ &= \sum_{n=1}^k P(m(t) \in D_n | F_s) P((M - c(t))m_n(t) \geq z | F_s) \end{aligned}$$

where

$$P((M - c(t))m_i(t) \geq z | F_s) = G\left(\frac{(M - c(t))m_i(s) - z}{(M - c(t))\sqrt{q_{ii}(s) - q_{ii}(t)}}\right)$$

$G(\cdot)$ is a distribution function of $N(0, 1)$, and for $1 \leq i, j \leq k$, $q_{ij}(s)$ are elements of the matrix $Q(s)$.

$$\begin{aligned} P(m(t) \in D_n | F_s) &= \int_0^\infty \dots \int_0^\infty \frac{1}{(2\pi)^{\frac{k-1}{2}} \sqrt{\det V_n(s, t)}} \\ &\times \exp\left\{-\frac{1}{2}(x - M^n(s))^T V_n^{-1}(s, t)(x - M^n(s))\right\} dx_1 \dots dx_{k-1} \end{aligned}$$

where

$$V_n(s, t) = [v_{ij}^n(s, t)] \quad \text{for } i, j \in \{1, \dots, n-1, n+1, \dots, k\}$$

$$v_{ij}^n(s, t) = \int_s^t \sum_{l=1}^k (b_{nl}(u) - b_{il}(u))(b_{nl}(u) - b_{jl}(u)) du$$

for $1 \leq i, j \leq k$ $b_{ij}(u)$ are elements of the matrix $Q(u)H^T(u)$, and $x \in R^{k-1}$

$$M^n(s) = \begin{pmatrix} m_n(s) - m_1(s) \\ \dots \\ m_n(s) - m_{n-1}(s) \\ m_n(s) - m_{n+1}(s) \\ \dots \\ m_n(s) - m_k(s) \end{pmatrix}$$

Proof. A k - dimensional stochastic process $(m(t), F_t)$, $0 \leq t < \infty$

$$m(t) = m(0) + \int_0^t Q(u)H^T(u)dv_u \quad (26)$$

where $m(t) = \text{col}(m_1(t), \dots, m_k(t))$, $Q(u)$ is a square symmetrical matrix and $H(u)$ is an encoding matrix, symmetrical with a positive trace, and v_s is the k -dimensional innovation (k -dimensional Wiener process). By definition, the process $(m(t), F_t)$, $0 \leq t < \infty$ is a martingale.

According to the definition of operators Θ and Ξ , by definition of Lemma 1 and formula of entire probability we obtain for arbitrary $0 \leq s < t < \infty$

$$\begin{aligned} &P((M - c(t))\langle \Theta \Xi m(t), m(t) \rangle \geq z | F_s) \\ &= \sum_{i=1}^k P(m(t) \in D_i | F_s) P((M - c(t))m_i(t) \geq z | F_s) \end{aligned} \quad (27)$$

Let us introduce the following notations. Let $B(u) = Q(u)H^T(u)$ and

$$b(u) = \begin{pmatrix} b_{11}(u) & b_{12}(u) & \dots & b_{1k}(u) \\ b_{21}(u) & b_{22}(u) & \dots & b_{2k}(u) \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1}(u) & b_{k2}(u) & \dots & b_{kk}(u) \end{pmatrix} = \begin{pmatrix} b^1(u) \\ b^2(u) \\ \vdots \\ b^k(u) \end{pmatrix} \quad (28)$$

and $b^i(u) = (b_{i1}(u), b_{i2}(u), \dots, b_{ik}(u))$, $1 \leq i \leq k$.

According to (26) for each $1 \leq i \leq k$

$$m_i(t) = m_i(0) + \sum_{j=1}^k \int_0^t b_{ij}(u)dv_j(u) = m_i(0) + \int_0^t \langle b^i(u), dv_u \rangle. \quad (29)$$

Thanks to the properties of stochastic integrals, the conditional expected value of the process $(m_i(t), F_t)$ for each $1 \leq i \leq k$, $0 \leq s < t < \infty$

$$E(m_i(t) | F_s) = m_i(s) \quad (30)$$

and the variance

$$Var(m_i(t) | F_s) = \int_s^t \langle b^i(u), b^i(u) \rangle du.$$

According to (A2)

$$Q'(u) = -Q^T(u)H^T(u)H(u)Q(u) \text{ and } Q(0) = Q$$

and formula (28) we obtain $Q'(u) = -B(u)B^T(u)$ for $0 \leq u < \infty$. According to the above

$$Q'(u) = [\dot{q}_{ij}(u)]_{0 \leq i, j \leq k} = [-\langle b^i(u), b^j(u) \rangle]_{0 \leq i, j \leq k}. \tag{31}$$

Finally, we have

$$Var(m_i(t)|F_s) = \int_s^t -\dot{q}_{ii}(u)du = \int_t^s \dot{q}_{ii}(u)du = q_{ii}(s) - q_{ii}(t). \tag{32}$$

The conditional distribution $m_i(t)$ with respect to F_s is a normal distribution $N(m_i(s), q_{ii}(s) - q_{ii}(t))$. So,

$$P((M - c(t))m_i(t) \geq z|F_s) = G\left(\frac{(M - c(t))m_i(s) - z}{(M - c(t))\sqrt{q_{ii}(s) - q_{ii}(t)}}\right) \tag{33}$$

where $G(\cdot)$ is a distribution function of $N(0, 1)$.

Let $y = col(y_1, y_2, \dots, y_k)$. For each $1 \leq i \leq k$ let us consider an operator $\phi_i : R^k \rightarrow R^{k-1}$ defined in the following way

$$\begin{aligned} \phi_i(y) &= \begin{pmatrix} -1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 & 1 & 0 & \dots & 0 \\ \cdot & \cdot \\ 0 & 0 & \dots & -1 & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & -1 & \dots & 0 \\ \cdot & \cdot \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_{i-1} \\ y_i \\ y_{i+1} \\ \dots \\ y_k \end{pmatrix} \\ &= \begin{pmatrix} y_i - y_1 \\ y_i - y_2 \\ \dots \\ y_i - y_{i-1} \\ y_i - y_{i+1} \\ \dots \\ y_i - y_k \end{pmatrix}. \end{aligned} \tag{34}$$

The operator defined in such way, $\phi_i, 1 \leq i \leq k$, is a linear operator. Let us denote for any $1 \leq j \leq k, i \neq j$

$$\phi_i : z(u) = u_i - u_j. \tag{35}$$

It is necessary to find the values $P(m(t) \in D_i | F_s)$ for $1 \leq i \leq k$. According to the definition of sets D_i (see Lemma 1)

$$\begin{aligned} & P(m(t) \in D_i | F_s) \\ &= P(\phi_{i,1}(m(t)) > 0, \dots, \phi_{i,i-1}(m(t)) > 0, \\ & \quad \phi_{i,i+1}(m(t)) \geq 0, \dots, \phi_{i,k}(m(t)) \geq 0 | F_s) \end{aligned}$$

and $1 < i, j \leq k, i \neq j$ (see (28), (29), (35))

$$\phi_{i,j}(m(t)) = \phi_{i,j}(m(0)) + \int_0^t (b^i(u) - b^j(u), dv_u). \quad (36)$$

The expected value is

$$E(\phi_{i,j}(m(t)) | F_s) = \phi_{i,j}(m(s)), \quad 0 \leq s < t < \infty, \quad (37)$$

covariance matrix is of the form

$$V_i(s, t) = [v_{jh}^i(s, t)] \quad (38)$$

where $1 \leq i, j, h \leq k, i \neq j, i \neq h$

$$\begin{aligned} v_{jh}^i(s, t) &= E[(\phi_{i,j}(m(t)) - E\phi_{i,j}(m(t)))(\phi_{i,h}(m(t)) - E\phi_{i,h}(m(t))) | F_s] \\ &= \int_s^t \langle b^i(u) - b^j(u), b^i(u) - b^h(u) \rangle du. \end{aligned} \quad (39)$$

The operator $\phi_i : R^k \rightarrow R^{k-1}$ is a linear operator, so $\phi_i(m(t))$ is a $k - 1$ -dimensional stochastic process.

By defining $x = \text{col}(x_1, \dots, x_{k-1})$ and using the formula for the joint distribution we obtain

$$\begin{aligned} P(m(t) \in D_i | F_s) &= \int_0^\infty \dots \int_0^\infty \frac{1}{(2\pi)^{\frac{k-1}{2}} \sqrt{\det V_i(s, t)}} \\ & \times \exp\left\{-\frac{1}{2}[x - \phi_i(m(s))]^T V_i^{-1}(s, t)[x - \phi_i(m(s))]\right\} dx_1 \dots dx_{k-1}. \end{aligned} \quad (40)$$

By substituting formulae (33) and (40) to (27) we finish the proof. ■