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Static and dynamic equilibria in stochastic games with continuum of players

by

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Abstract: This paper is a study of a general class of stochastic games with an atomless measure space of players and an arbitrary time space. The payoffs of the players depend on their own strategy, the trajectory of the system and the function with values being finite dimensional statistics of static profiles. The players' available decisions depend on the trajectories of the system.

The paper deals with relations between static and dynamic equilibria as well as the existence of dynamic equilibria. The results are counterintuitive and much stronger than the results that can be obtained in games with finitely many players. An equivalence theorem is proven between the dynamic equilibrium (according to various definitions) and the family of static equilibria corresponding to it. Theorems on the existence of dynamic equilibria are shown as consequences.

Theoretical results of this paper are illustrated by examples describing exploitation of common ecological systems in which "the tragedy of the commons" appears.

Keywords: games with continuum of players, stochastic games, strong dynamic equilibrium, dynamic equilibrium with respect to utility function, static equilibrium.

1. Introduction

This paper appears in a sequence of author's papers concerning dynamic games with an atomless space of players, Wiszniewska-Matyszkiel (2002a,b, 2003a,b,c). It is a step in building a general theory of such games.

Games with an atomless space of players (alternatively, *games with continuum of players*) (formally defined by Schmeidler, 1973, studied e.g. by Mas-Colell, 1984, Balder, 1995, Wieczorek, 1997, Wiszniewska-Matyszkiel, 2000b), were introduced to model insignificance of a single player. Therefore they are especially suitable for describing exploitation of an ecological system by a large group of users (e.g. global phenomena like greenhouse effect or fishing in open-access oceanic fisheries).

Another natural tool for describing these phenomena are dynamic games with payoffs depending on a finitely dimensional statistic function of profiles and trajectories of the system. In general, we cannot assume that such trajectories are deterministic. Therefore it is natural to extend the results obtained in the quoted papers (especially Wiszniewska-Matyszkiel, 2000a) to the case of stochastic games.

To the best of author's knowledge a general theory of dynamic (deterministic or stochastic) games with a continuum of players has not been developed yet. Besides the previously quoted papers of the author, the papers that can be found in the literature dealing with dynamic games with a continuum of players concern mainly particular cases (e.g. Karatzas, Shubik, Sudderth, 1994, or Wiszniewska-Matyszkiel, 2000a, 2001b), or evolutionary games (e.g. Wieczorek and Wiszniewska, 1999).

The games considered in this paper can be briefly described as follows:

There is an atomless space of players $(\Omega, \Im, \mu)$.

Player's payoffs are equal to instantaneous payoffs discounted and integrated over time

$$\Pi_\omega(D, U, X) = \int_{\mathbb{T}} P_\omega(D(t,x), U(t,x), X(t), t)\Psi(t)dt,$$

where the instantaneous payoffs P_ω depend on player's own strategy, the statistic of a profile, the state of the system and specific time.

The statistic U of a dynamic profile Δ is defined by

$$U(t,x) = \int_\Omega g(\omega, \Delta(\omega,t,x))d\mu(\omega) \text{ for every } t \text{ and } x.$$

The trajectory corresponding to a statistic U at time t, conditional on past realization equal to X, is chosen according to the distribution $\Phi(X, U, t)$; which defines a stochastic process (this encompasses both usual deterministic differential games as well as deterministic or stochastic discrete time multistage games, piecewise-deterministic differential games and many others).

Admissibility of a dynamic strategy D of player ω at a certain trajectory of the system X is defined by

$$D(t,x) \in S_\omega(x,t) \text{ for every } x \text{ and a.e. } t.$$

Formal definition of the game will appear in Section 2.

The paper is organized as follows. In Section 2 the definition of the game is given, in Subsection 2.1 the notions of static and dynamic equilibria are introduced. Notational remarks are grouped in Subsection 2.2. In Section 3 the examined. Section 4 contains existence theorems for dynamic equilibria in the games with a discrete time space, the games with an arbitrary time space and the games with an arbitrary time space and finite number of types of players. Section 5 is devoted to examples describing exploitation of common ecosystems resulting in "the tragedy of the commons".

2. Formulation of the model

Formally, we define a *dynamic game with a measure space of players* as a system

$$\mathfrak{G} = ((\mathbb{X}, \mathcal{X}), x_0, (\mathbb{T}, \mathcal{T}, \lambda), \mathfrak{X}, (\Omega, \Im, \mu), (\mathbb{S}, \mathcal{S}), S, g, P, \Phi, \Psi)$$

with the components called, respectively: state space, initial state, time space, set of trajectories (considered in the model), space of players, space of (static) strategies, correspondence of players' available (static) strategies, pre-statistic function, players' instantaneous payoff function, function of behaviour of the system, and discounting function. All of them are defined below in detail.

The measurable space $(\mathbb{X}, \mathcal{X})$ will be the *space of* (possible) *states* (of the system); the *initial state* is $x_0 \in \mathbb{X}$.

The *time space* is a measure space $(\mathbb{T}, \mathcal{T}, \lambda)$, such that $\mathbb{T} \subset \mathbb{R}_+$ and $\mathbb{T}$ has a minimal element t_0, called *the initial time*. Two obvious examples are $\mathbb{R}_+$ with the Lebesgue measure (and Borel or Lebesgue measurable sets) and $\{0, 1, 2, \dots\}$ with the counting measure (and all subsets); $\mathbb{T}$ may also include both atoms and an atomless part.

The symbol $\mathfrak{X}$ will stand for some set of measurable functions $X : \mathbb{T} \to \mathbb{X}$, and is called a *set of trajectories* (under consideration). We do not, obviously, have to take all the measurable functions. It may be natural to restrict the attention to e.g. absolutely continuous or piecewise absolutely continuous functions in the case of a continuous time space.

Every function $X \in \mathfrak{X}$ such that $X(t_0) = x_0$ will be called a *trajectory of the system starting from* x_0; $X(t)$ will denote the state of the system at time t for the trajectory X.

The *players* are assumed to form an atomless measure space $(\Omega, \Im, \mu)$ with finite measure μ.

A measurable space $(\mathbb{S}, \mathcal{S})$ will be the *space of* (static) *strategies*. We assume that the set $\mathbb{S}$ is topologized with a Hausdorff topology. All topological assumptions about objects defined on $\mathbb{S}$ refer to this topology. The Borel σ-field of $\mathbb{S}$ is not assumed to coincide with the σ-field $\mathcal{S}$.

A nonempty-valued correspondence $S : \Omega \times \mathbb{X} \times \mathbb{T} \multimap \mathbb{S}$ is a *correspondence of available (static) players' strategies.* The function $S(\omega, \cdot, \cdot)$ will be denoted by S_ω. The set $S_\omega(x, t)$ is understood as the *set of* (static) *strategies available to player* ω *at* time t *and* state x. Every $d \in S_\omega(x, t)$ is an *individual static strategy available to player* ω *at* t *and* x. Any $\mathcal{T} \otimes \mathcal{X}$-measurable function $D : \mathbb{T} \times \mathbb{X} \to \mathbb{S}$ such that $D(t, x) \in S_\omega(x, t)$ for a.e. t and every x is a *dynamic strategy available*

The set of all $\Im$-measurable functions $\delta : \Omega \to \mathbb{S}$ will be denoted by Σ and the set of all $\mathcal{T} \otimes \mathcal{X}$-measurable functions $D : \mathbb{T} \times \mathbb{X} \to \mathbb{S}$ by $\mathfrak{S}$.

Any function $\delta \in \Sigma$ such that for almost every $\omega, \delta(\omega) \in S_\omega(x,t)$ will be called a *static profile available at* t *and* x; $\delta(\omega)$ is the *player* ω*'s strategy at* (profile) δ.

To construct the next element of the game we need the integrably bounded, $\Im \otimes \mathcal{S}$-measurable functions $g_1, \dots, g_m : \Omega \times \mathbb{S} \to \mathbb{R}$. The function

$$g = \begin{bmatrix} g_1 \\ \vdots \\ g_m \end{bmatrix}$$

will be called a *pre-statistic function.*

For a static profile δ, the symbol u_δ denotes the vector

$$\left[\int_\Omega g_i(\omega, \delta(\omega)) d\mu(\omega) \right]_{i=1}^m,$$

and it is called a *statistic of* (the profile) δ. We shall also abbreviate

$$\left[\int_\Omega g_i(\omega, \delta(\omega)) d\mu(\omega) \right]_{i=1}^m$$

by $\int_\Omega g(\omega, \delta(\omega)) d\mu(\omega)$ (this applies also to integrals of all vector-valued functions and correspondences with values contained in $\mathbb{R}^m$).

The statistic of a profile contains all information about a static profile necessary both to fully describe the behaviour of the system and instantaneous payoff of a player given his own strategy.

For time t and state x, the set $Y(x,t) = \int_\Omega g(\omega, S_\omega(x,t)) d\mu(\omega)$ (if the integrand is a multivalued correspondence, the symbol $\int$ denotes the Aumann, 1965, integral) is the *set of statistics available at* t *and* x, so the correspondence $Y : \mathbb{X} \times \mathbb{T} \multimap \mathbb{R}^m$ may be called a *correspondence of available profile statistics.* The space $\mathbb{R}^m$ meaning the *space of profile statistics*, will be denoted by $\mathbb{Y}$. The elements of $\mathbb{Y}$ will be sometimes called *control variables* (they are control variables of Ω treated as a single decision maker). The set of all $\mathcal{T} \otimes \mathcal{X}$-measurable functions $U : \mathbb{T} \times \mathbb{X} \to \mathbb{Y}$ will be denoted by $\mathfrak{Y}$, and its elements will be called *control functions* (they are control functions of Ω treated as a single decision maker). They are naturally related to dynamic profiles defined in the sequel.

The next element of the game is a function $P : \Omega \times \mathbb{S} \times \mathbb{Y} \times \mathbb{X} \times \mathbb{T} \to \mathbb{R} \cup \{-\infty\}$, called a *players' instantaneous payoff function.* The function $P(\omega, \cdot, \cdot, \cdot, \cdot)$, denoted by P_ω, is an *individual instantaneous payoff function* (*instantaneous payoff* for short) *of player* ω.

A function $\Phi : \mathfrak{X} \times \mathfrak{Y} \times \mathbb{T} \to M_1(\mathbb{X})$ with $\Phi(X, U, t_0)$ being equal to the

X starting from x_0, is called a *function of the behaviour of the system* and its meaning will be explained in the sequel. The symbol $M_1(\mathbb{X})$ denotes the set of probability measures on $(\mathbb{X}, \mathcal{X})$ (equivalently, we can consider random variables with values in $\mathbb{X}$, since only the distribution will matter).

To make the game realistic, we assume that the behaviour of the system at time t is defined by the states of the system and the values of profile statistics for earlier moments only, i.e. Φ is such that for every t, if trajectories X_1, X_2 coincide for $s < t$ and control functions U_1, U_2 coincide for $s < t$ and all x, then $\Phi(X_1, U_1, t) = \Phi(X_2, U_2, t)$.

Moreover, the behaviour of the system does not depend on irrelevant alternatives, only on the actual trajectory i.e. for every trajectory X and control functions U_1, U_2 such that $U_1(t, X(t)) = U_2(t, X(t))$ for a.e. t we have $\Phi(X, U_1, t) = \Phi(X, U_2, t)$ for all t.

Let U be a control function. For this U, the function Φ defines a stochastic process: for a trajectory coinciding with X for $s < t$, the state of the system at time t is chosen according to the distribution $\Phi(X, U, t)$. Every trajectory X starting from x_0 and fulfilling $\lambda(\{t \mid X(t) \notin \operatorname{supp}\Phi(X, U, t)\}) = 0$ (where the symbol supp stands for the support of a distribution), will be called *corresponding to* U. This means that $\Phi(X, U, t)$ defines the expectation of the state at time t for the control function U conditional on the earlier trajectory coinciding with X.

We assume that for every control function there exists at least one corresponding trajectory.

REMARK 1. *This general definition of the behaviour of the system encompasses:*

1. Deterministic discrete time models with state defined by difference equations e.g. $X(0) = x_0$, $X(t+1) = X(t) + \phi(X(t), U(t, X(t)))$ *or, more generally,* $X(t+1) = X(0) + \sum_{i=0}^{t} \phi_i(X(i), U(i, X(i)))$.

2. Deterministic continuous time models with state defined by differential equations e.g. $X(0) = x_0$, $\dot{X}(t) = \phi(X(t), U(t, X(t)))$ *i.e.* $X(t) = X(0) + \int_0^t \phi(X(s), U(t, X(s)))ds$ *(then, obviously,* $\mathfrak{X}$ *is a subset of absolutely continuous functions). This includes also the delayed differential equations.*

3. Stochastic discrete time models with state defined by difference equations, like random walk, e.g. $X(0) = x_0$, $X(t+1) = X(t) + \phi(X(t), U(t, X(t))) + \xi_t$ *or, more generally,* $X(t+1) = X(0) + \sum_{i=0}^{t} \phi_i(X(i), U(i, X(i))) + \xi_t$, *where* ξ_t *are any distributions.*

4. Piecewise deterministic continuous time models with state defined piecewise by differential equations with stochastic jumps, e.g. $X(0) = x_0$, $\dot{X}(t) = \phi(X(t), U(t, X(t)))$ *for* $t \notin \mathbb{N}$ *and* $X(n) = \lim_{t \to n^-} X(t) + \xi_n$ *for* $n \in \mathbb{N}$ *i.e.* $X(t) = X(0) + \int_0^t \phi(X(s), U(s, X(s)))ds + \xi_t \cdot \chi_{\mathbb{N}\setminus\{0\}}(t)$ *(where* $\chi_{\mathbb{N}\setminus\{0\}}$ *is the characteristic function of positive integers), where* ξ_t *are any distributions. In this case* $\mathfrak{X}$ *is a subset of functions absolutely continuous in interval* $[n, n+1)$

In these four cases Φ obviously defines a distribution on the set of trajectories (alternative definition of a stochastic process). Nevertheless, such formulation does not encompass stochastic processes defined by nontrivial stochastic differential equations.

A control function U is *admissible* if for almost every t and every x, $U(t,x)$ is available at t and x (i.e. for almost every t and every x, $U(t,x) \in Y(x,t)$). If there exists an admissible control function U such that a trajectory X corresponds to it, then this X will be called *admissible*.

The next element of the model is an integrable function $\Psi : \mathbb{T} \to \mathbb{R}_+ \setminus \{0\}$, called a *discounting function*.

A $\Im \otimes \mathcal{T} \otimes \mathcal{X}$-measurable function $\Delta : \Omega \times \mathbb{T} \times \mathbb{X} \to \mathbb{S}$ is called a *dynamic profile* if for almost every t and every x the function $\Delta(\cdot, t, x)$ is a static profile available at time t and state x and the function $U_\Delta : \mathbb{T} \times \mathbb{X} \to \mathbb{Y}$ defined by $U_\Delta(t,x) = u_{\Delta(\cdot,t,x)}$ for all t, x, called the *statistic of* (the dynamic profile) Δ, is an admissible control function. Every trajectory X corresponding to U_Δ is called *corresponding to* (the dynamic profile) Δ. The set of all dynamic profiles is denoted by Σ.

The *payoff* function of a player ω is a function $\Pi_\omega : \mathfrak{S} \times \mathfrak{Y} \times \mathfrak{X} \to \overline{\mathbb{R}}$ (it is a function of ω's own strategy at a profile, the statistic of this profile and the actual trajectory). The payoff is equal to instantaneous payoffs discounted and integrated over time:

$$\Pi_\omega(D,U,X) = \int_{\mathbb{T}} P_\omega(D(t, X(t), U(t, X(t)), X(t), t)\Psi(t) d\lambda(t).$$

A game $G_{t,x}$ with the same space of players as that of $\mathfrak{G}$, player's ω payoff function equal to his instantaneous payoff function with two last arguments fixed at x and t ($P_\omega(\cdot,\cdot,x,t)$) and player's ω strategy set equal to $S_\omega(x,t)$, will be called a *static game at time t and state x corresponding to $\mathfrak{G}$.*

2.1. Static and dynamic equilibria

In this section we are going to cope with two kinds of equilibria: stochastic dynamic equilibrium (in a dynamic game) and static equilibria (in static games corresponding to the dynamic game).

A *Nash* (or *Cournot–Nash*) *equilibrium* is such a profile that almost no player has an incentive to change his strategy, unless the remaining players have changed theirs.

Because all "external" information about a profile, important from the point of view of any player is stored in the profile's statistic function, this definition can be reformulated: an equilibrium is profile such that almost no player has an incentive to change his strategy, unless the statistic of the profile has changed.

Definition 1. *A* static equilibrium at time t and state x *is an equilibrium in* $G_{t,x}$, *i.e. a static profile* δ *such that*

$$\text{for a.e. } \omega, \quad \delta(\omega) \in \operatorname{Argmax}_{d \in S_\omega(x,t)} P_\omega(d, u_\delta, x, t)$$

(i.e. for a.e. ω *and every* $d \in S_\omega(x,t)$, $P_\omega(\delta(\omega), u_\delta, x, t) \geq P_\omega(d, u_\delta, x, t)$*).*

At first sight this definition may seem different from the verbal description of Nash equilibrium in a static game. Formally, a static equilibrium profile δ should fulfill for a.e. ω the condition $\delta(\omega) \in \operatorname{Argmax}_{d \in S_\omega(x,t)} P_\omega(d, u_{[d,\delta_{-\omega}]}, x, t)$, where $[d, \delta_{-\omega}]$ denotes the profile δ' such that $\delta'(\omega) = \delta(\omega)$ and $\delta'(\nu) = \delta(\nu)$ for $\nu \neq \omega$. Since in our game $u_\delta = u_{[d,\delta_{-\omega}]}$, we can use this simplified definition.

Definition 2. *The* static best response set of player ω to $u \in \mathbb{Y}$ *(representing the statistic of a profile)* at time t and state x *is defined by*

$$B_\omega(u, x, t) = \operatorname{Argmax}_{d \in S_\omega(x,t)} P_\omega(d, u, x, t)$$

and the statistic of the static best response set to u at time t and state x *is defined by*

$$\overline{B}(u, x, t) = \int_\Omega g(\operatorname{Gr}(B_\cdot(u, x, t))) d\mu(\omega),$$

where the symbol Gr *denotes the graph of a correspondence and* $B_\cdot(u, x, t)$ *stands for* $B_\omega(u, x, t)$ *treated as a correspondence of* ω.

The static best responses define a correspondence called *static best response correspondence* and the statistics of the static best responses define *statistic of the static best response correspondence.*

Static equilibrium profile δ may be equivalently defined by the condition

$$u_\delta \in \overline{B}(u_\delta, x, t),$$

i.e. δ is a profile, whose statistic is a fixed point of the correspondence $\overline{B}(\cdot, x, t)$.

There may be various concepts of dynamic equilibria. In this paper we shall consider two definitions:

Definition 3. *A* dynamic equilibrium with respect to a utility function $F : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ *is a dynamic profile* Δ *such that*

$$\begin{aligned} &\text{for a.e. } \omega, \quad \Delta(\omega, \cdot, \cdot) \in \operatorname{Argmax}\{\mathrm{E}(F(\Pi_\omega(D, U_\Delta, X))) \mid \\ &D \in \mathfrak{S}, D(t, x) \in S_\omega(x, t) \text{ for every } x \text{ and a.e. } t\}, \end{aligned}$$

where E *stands for the expectation taken over the set of all trajectories corresponding to* U_Δ.

In the theory of choice under uncertainty the maximized function is called von Neumann–Morgenstern utility or expected utility while the function F – Bernoulli utility

Definition 4. *A* strong dynamic equilibrium *is a dynamic profile* Δ *such that*

$$\begin{aligned}&\text{for a.e. } \omega, \text{ for every trajectory } X \text{ corresponding to } U_\Delta\\ &\Delta(\omega,\cdot,\cdot)\\ &\in \operatorname{Argmax}\{\Pi_\omega(D,U_\Delta,X) \mid D \in \mathfrak{S}, D(t,x)\in S_\omega(x,t) \text{ for every } x \text{ and a.e. } t\}.\end{aligned}$$

Obviously, a strong dynamic equilibrium is also an equilibrium with respect to every increasing utility function. The opposite implication is true if every trajectory is of positive probability (which is fulfilled whenever $\mathbb{X}$ is countable and $\mathbb{T}$ finite).

Since the definition of the strong dynamic equilibrium sets very strong conditions, we usually cannot expect existence. It is not the case in this paper.

Definition 5. *The* set of player ω's dynamic best responses to U *is equal to*

$$\mathbf{B}_\omega(U) = \bigcap_{X\in\mathfrak{X} \text{ corresponding to } U} \mathbf{B}_\omega^X(U),$$

where

$$\begin{aligned}&\mathbf{B}_\omega^X(U)\\ &= \operatorname{Argmax}\{\Pi_\omega(D,U,X) \mid D\in\mathfrak{S}, D(t,x)\in S_\omega(x,t) \text{ for a.e. } t \text{ and every } x\}\end{aligned}$$

is the set of player ω's dynamic best responses to U *at a trajectory* X.

The statistic of the dynamic best response set to U *is equal to*

$$\overline{\mathbf{B}}(U) = \{U_\Delta \mid \Delta\in\Sigma, \Delta(\omega,\cdot,\cdot)\in\mathbf{B}_\omega(U) \text{ for a.e. } \omega\}.$$

A strong dynamic equilibrium is, equivalently, a dynamic profile Δ whose statistic U_Δ is a fixed point of the correspondence $\overline{\mathbf{B}}$.

2.2. Notational remarks

In order to avoid ambiguity, we group here some definitions and notational conventions.

To simplify the notation we shall identify a distribution concentrated at a point with this point.

The symbol $\operatorname{diag}\mathbb{X}$ will denote the diagonal in $\mathbb{X}^2$: $\operatorname{diag}\mathbb{X} = \{(x,x) \mid x\in\mathbb{X}\}$.

For a set $\mathbb{X}$ the symbol $\mathrm{Id}_\mathbb{X}$ denotes the identity function on $\mathbb{X}$.

A measurable space $(\mathbb{X},\mathcal{X})$ is called a *subspace of a measurable space* $(\mathbb{Y},\mathcal{Y})$ if $\mathbb{X}\subset\mathbb{Y}$ and $\mathcal{X} = \{W\cap\mathbb{X} \mid W\in\mathcal{Y}\}$.

A measurable space $(\mathbb{X},\mathcal{X})$ is called a *measurable image of* (a measurable space) $(\mathbb{Y},\mathcal{Y})$ if there exists a measurable function $f:\mathbb{X}\xrightarrow{\text{onto}}\mathbb{Y}$.

If $(\mathbb{X},\mathcal{X},\lambda)$ is a measure space, then $\overline{\mathcal{X}}$ will denote the completion of $\mathcal{X}$ with

If $\mathcal{X}$ is a family of subsets of a set $\mathbb{X}$, then a subset of $\mathbb{X}$ is called $\mathcal{X}$-*analytic* if it can be obtained by a Souslin $\mathcal{A}$-operation performed on $\mathcal{X}$ (see e.g. Kuratowski, 1996, or Saks, 1937; this family contains e.g. continuous or measurable images of measurable sets, projections of measurable sets and inverse images of measurable sets by a function with measurable graph). The family of all $\mathcal{X}$-analytic sets will be denoted by $\mathcal{A}(\mathcal{X})$.

If $(\mathbb{X}, \mathcal{X})$ is a measurable space, then a function $f : \mathbb{X} \to \overline{\mathbb{R}}$ is called $\mathcal{X}$-*analytically measurable* if inverse images of all Borel subsets of $\overline{\mathbb{R}}$ are $\mathcal{X}$-analytic.

If f_1 and f_2 are any functions , then the symbol (f_1, f_2) denotes the function defined on the product of their domains by $(f_1, f_2)(x, y) = (f_1(x), f_2(y))$.

The symbol $f_1 \times f_2$ for functions (correspondences) f_1 and f_2 with the same domains will denote a function (correspondence) with the same domain defined by $(f_1 \times f_2)(x) = (f_1(x), f_2(x))$ (in the case of correspondences $(f_1 \times f_2)(x) = f_1(x) \times f_2(x)$). If $\operatorname{dom} f_1 = \mathbb{X} \times \mathbb{Y}$ and $\operatorname{dom} f_2 = \mathbb{X}$, then $\operatorname{dom} f_1 \times f_2 = \operatorname{dom} f_1$ and $(f_1 \times f_2)(x, y) = (f_1(x, y), f_2(x))$ (in the case of correspondences $(f_1 \times f_2)(x, y) = f_1(x, y) \times f_2(x)$). Analogously, can we define $f_1 \times f_2$ for functions or correspondences in similar cases (e.g. domains $\mathbb{X} \times \mathbb{Y}$ and $\mathbb{Y} \times \mathbb{X}$).

Let $\mathbb{X}$ be an arbitrary set. We can define a convex structure on $\mathbb{X}$ e.g. by an operation of taking a convex hull of a set (in the case of a linear space it is the usual convex hull), otherwise there are various notions of abstract convexity e.g. based on the relation of "lying between" (see e.g. Wieczorek, 1992). The operation of taking a convex hull defines, in particular, convexity of a set and quasi-concavity of a function. The only properties that we need are those guaranteeing that the maximum of a quasi-concave function over a convex set is attained at a convex set, while the maximum of a strictly quasi-concave function over a convex set is attained at at most one point.

2.3. Assumptions

Now we shall consider a static game $G_{t,x}$ with fixed time t and state x.

We shall state the following assumptions about $G_{t,x}$, necessary in the sequel (this set of assumption is sufficient for the existence of an equilibrium in $G_{t,x}$ by Theorem 3.1 of Wiszniewska-Matyszkiel, 2000b).

A1. The space of strategies $\mathbb{S}$ is such that $\operatorname{diag} \mathbb{S}$ is $\mathcal{S} \otimes \mathcal{S}$-measurable and $\mathbb{S}$ is a measurable image of a measurable space $(\mathbb{Z}, \mathcal{Z})$ being an analytic subspace of a separable compact topological space $\mathbb{W}$ (with the σ-field of Borel subsets $\mathcal{B}(\mathbb{W})$).

This assumption is obviously fulfilled by Polish spaces, or, more generally, Souslin spaces, but it does not require introduction of any specific topology on the space $\mathbb{S}$: it is measure-theoretic.

A2. For almost every ω the set $S_\omega(x, t)$ is nonempty and compact.

A3. The function $P_\omega(\cdot, \cdot, x, t)$ is upper semicontinuous on $S_\omega(x, t) \times Y(x, t)$

A4. The graph of $S_{\cdot}(x,t)$ is $\overline{\Im}\otimes\mathcal{S}$ -analytic.

A5. The function $P_\omega(d,\cdot,x,t)$ is continuous on $Y(x,t)$ for almost every ω and every $d\in S_\omega(x,t)$.

A6. For every $u\in Y(x,t)$ the function $P_{\cdot}(\cdot,u,x,t)|_{\mathrm{Gr}(S_{\cdot}(x,t))}$ is $\overline{\Im}\otimes\mathcal{S}$ - analytically measurable.

A7. The functions g is measurable, integrably bounded and such that $g(\omega,\cdot)$ is continuous on $S_\omega(x,t)$ for almost every ω.

3. Relations between static and dynamic equilibria

There are some obvious relations between dynamic and static best response sets, and between dynamic and static equilibria. The dynamic objects are in the dynamic game $\mathfrak{G}$, while the static ones in static games $G_{t,x}$ corresponding to it.

THEOREM 1. a) *For an admissible control function U and a dynamic strategy D we have the following implication:*

If for every trajectory X corresponding to U and a.e. t, the vector $D(t,X(t))$ is a static best response of player ω to $U(t,X(t))$ in the static game at time t and state of the system $X(t)$, then the function D is ω's dynamic best response to U.

b) *If Δ is a dynamic profile and for every trajectory X corresponding to Δ the static profiles $\Delta(\cdot,t,X(t))$ are for almost every t static equilibria at time t and state of the system $X(t)$, then Δ is a strong dynamic equilibrium.*

c) *Let $\mathbb{S}$ fulfill A1. Assume that for a.e. t and every u, x, the function $P_\omega(\cdot,u,x,t)$ is upper semicontinuous, the function P_ω is $\mathcal{S}\otimes\mathcal{B}(\mathbb{Y})\otimes\mathcal{X}\otimes\overline{\mathcal{T}}$-analytically measurable and the correspondence S_ω has an $\mathcal{X}\otimes\overline{\mathcal{T}}\otimes\mathcal{S}$-analytic graph and compact values. If U is an admissible control function and D a dynamic best response of player ω to U at a trajectory X corresponding to U such that the payoff $\Pi_\omega(D,U,X)$ is finite, then for a.e. t the vector $D(t,X(t))$ is a static best response of player ω to $U(t,X(t))$ in the static game at time t and state of the system $X(t)$.*

d) *Let $\mathbb{S}$ fulfill A1. Assume that for a.e. t and every u, x, the function $P_\omega(\cdot,u,x,t)$ is upper semicontinuous, the function P_ω is $\mathcal{S}\otimes\mathcal{B}(\mathbb{Y})\otimes\mathcal{X}\otimes\overline{\mathcal{T}}$-analytically measurable and the correspondence S_ω has an $\mathcal{X}\otimes\overline{\mathcal{T}}\otimes\mathcal{S}$-analytic graph and compact values. If U is an admissible control function and D a dynamic best response to U such that the payoff $\Pi_\omega(D,U,X)$ is finite for every trajectory X corresponding to U, then for a.e. t and every trajectory X corresponding to U, the vector $D(t,X(t))$ is a static best response of player ω to $U(t,X(t))$ in the static game at time t and state of the system $X(t)$.*

e) *Let $\mathbb{S}$ fulfill* A1. *Assume that for a.e. ω, t and every u, x, the function $P_\omega(\cdot,u,x,t)$ is upper semicontinuous, for a.e. ω the function P_ω is $\mathcal{S}\otimes\mathcal{B}(\mathbb{Y})\otimes\mathcal{X}\otimes\overline{\mathcal{T}}$ -analytically measurable and the correspondence S_ω has an $\mathcal{X}\otimes\overline{\mathcal{T}}\otimes\mathcal{S}$-*

trajectory X *corresponding to* Δ *the payoff* $\Pi_\omega(\Delta(\omega,\cdot,\cdot),U_\Delta,X)$ *is finite, then every dynamic profile* Δ *being a dynamic equilibrium with respect to an increasing utility function, is such that for almost every trajectory* X *corresponding to it, static profiles* $\Delta(\cdot,t,X(t))$ *are for almost every* t *static equilibria at time* t *and state of the system* $X(t)$.

f) *Let* $\mathbb{S}$ *fulfill* A1. *Assume that for a.e.* ω, t *and every* u, x, *the function* $P_\omega(\cdot,u,x,t)$ *is upper semicontinuous, for a.e.* ω *the function* P_ω *is* $\mathcal{S}\otimes\mathcal{B}(\mathbb{Y})\otimes\mathcal{X}\otimes\overline{\mathcal{T}}$*-analytically measurable and the correspondence* S_ω *has an* $\mathcal{X}\otimes\overline{\mathcal{T}}\otimes\mathcal{S}$*-analytic graph and compact values. Every strong dynamic equilibrium* Δ *such that for almost every player* ω *and every trajectory* X *corresponding to* Δ *the payoff* $\Pi_\omega(\Delta(\omega,\cdot,\cdot),U_\Delta,X)$ *is finite, fulfils the following condition: for every trajectory* X *corresponding to* Δ *and for almost every* t, *static profiles* $\Delta(\cdot,t,X(t))$ *are static equilibria at time* t *and state of the system* $X(t)$.

Theorem 1 allows to reduce a problem of finding a dynamic equilibrium or best response to solution of a parametrized family of static problems.

In the proof we shall need the following lemma:

LEMMA 2. a) *Let us assume that* $\mathbb{S}$ *fulfils A1, while* $\mathbb{X}$ *is countable with* $\mathcal{X}=2^{\mathbb{X}}$, *or* $\mathbb{T}$ *is countable with* $\mathcal{T}=2^{\mathbb{T}}$, $\lambda(t)>0$ *for every* t *and there exists a complete measure* ξ *on* $\mathcal{X}$. *Moreover, assume that for a.e.* ω, t *and every* u, x, *the function* $P_\omega(\cdot,u,x,t)$ *is upper semicontinuous, for a.e.* ω *the function* P_ω *is* $\mathcal{S}\otimes\mathcal{B}(\mathbb{Y})\otimes\mathcal{X}\otimes\overline{\mathcal{T}}$*-analytically measurable and the correspondence* S_ω *has an* $\mathcal{X}\otimes\overline{\mathcal{T}}\otimes\mathcal{S}$ *-analytic graph and compact values. Let* U *be a control function. There exists a function* $D\in\mathfrak{S}$ *with* $D(t,x)\in B_\omega(U(t,x),x,t)$ *for a.e.* t *and every* x.

b) *Let us assume that* $\mathbb{S}$ *fulfils A1 and there exists a function* $D\in\mathfrak{S}$ *being a dynamic best response to a control function* U. *Moreover, assume that for a.e.* ω, t *and every* u, x, *the function* $P_\omega(\cdot,u,x,t)$ *is upper semicontinuous, for a.e.* ω *the function* P_ω *is* $\mathcal{S}\otimes\mathcal{B}(\mathbb{Y})\otimes\mathcal{X}\otimes\overline{\mathcal{T}}$*-analytically measurable and the correspondence* S_ω *has an* $\mathcal{X}\otimes\overline{\mathcal{T}}\otimes\mathcal{S}$*-analytic graph and compact values. Let* X *be a trajectory. There exists a dynamic strategy of player* ω, $D'\in\mathfrak{S}$ *such that* $D'(t,X(t))\in B_\omega(U(t,X(t)),X(t),t)$ *for a.e.* t.

Proof. (of Lemma 2)

a) Let ξ be a measure on $\mathcal{X}$ such that $\xi(x)>0$ for every x.

We define a correspondence Γ by $\Gamma(t,x)=\operatorname{Argmax}_{d\in S_\omega(x,t)} P_\omega(d,U(t,x),x,t)$. The graph of the correspondence S_ω is $\mathcal{X}\otimes\overline{\mathcal{T}}\otimes\mathcal{S}$-analytic and the function $P_\omega\circ(\mathrm{Id}_{\mathbb{S}},(U\times\mathrm{Id}_{\mathbb{X}}\times\mathrm{Id}_{\mathbb{T}}))$ is $\mathcal{S}\otimes\overline{\mathcal{T}}\otimes\mathcal{X}$-analytically measurable and for almost every t and every x the values of Γ are nonempty. Therefore, by Lemma 4.1 from Wiszniewska-Matyszkiel (2002a) the graph of Γ is $\overline{\mathcal{T}}\otimes\mathcal{X}\otimes\mathcal{S}$ -analytic and there exists a $\mathcal{T}\otimes\mathcal{X}$-measurable a.e. selection from Γ.

This completes the part of the proof for the case when $\mathbb{X}$ is countable. If by Leese's Theorem 5.5 (Leese, 1978) there exists a $\mathcal{T} \otimes \mathcal{X}$-measurable selection from Γ.

b) We define a correspondence Γ by

$$\Gamma(t) = \mathrm{Argmax}_{d \in S_\omega(X(t),t)} P_\omega(d, U(t), X(t), t).$$

The graph of the correspondence $S_\omega \circ (X \times \mathrm{Id}_{\mathbb{T}})$ is $\overline{\mathcal{T}} \otimes \mathcal{S}$-analytic and the function $P_\omega \circ (\mathrm{Id}_{\mathbb{S}}, (U \times X \times \mathrm{Id}_{\mathbb{T}}))$ is $\mathcal{S} \otimes \overline{\mathcal{T}}$-analytically measurable and almost all values of Γ are nonempty. Therefore, by Lemma 4.1 from Wiszniewska-Matyszkiel (2002a) the graph of Γ is $\overline{\mathcal{T}} \otimes \mathcal{S}$-analytic and there exists a $\mathcal{T}$-measurable a.e. selection from Γ. Let us denote it by f. We define

$$D'(t,x) = \begin{cases} f(t) & \text{if } x = X(t), \\ D(t,x) & \text{if } x \neq X(t). \end{cases}$$

Such a function D' is in $\mathfrak{S}$, it fulfils $D'(t,x) \in S_\omega(x,t)$ for a.e. t and every x, therefore it is a dynamic strategy available to player ω. Moreover, $D'(t, X(t)) \in B_\omega(U(t, X(t)), X(t), t)$ for a.e. t. ■

Proof. (of Theorem 1)

a) The players are negligible (of measure 0), therefore one player's strategy affects neither the statistic nor the behaviour of the system: i.e. for all the dynamic profiles Δ and Δ' such that $\Delta(\nu) = \Delta'(\nu)$ for every $\nu \neq \omega$, we have $U_\Delta = U_{\Delta'}$ and $X^\Delta = X^{\Delta'}$. Therefore, by changing one player's strategy D we do not change neither the statistic U of the profile nor the trajectory corresponding to it. This means that optimization can be done at any time independently, with $U(t, X(t))$ and $X(t)$ being parameters. Hence a dynamic profile whose static profiles are optimal in static games is dynamically optimal.

b) Let Δ be a strong dynamic equilibrium. Then for every trajectory X corresponding to Δ and a.e. ω, the function $\Delta(\omega, \cdot, \cdot)$ is ω's dynamic best response to U_Δ.

By a), the latter condition is implied by the following: for every trajectory X corresponding to Δ, a.e. ω, t, the static strategy $\Delta(\omega, t, X(t))$ is ω's static best response to $U_\Delta(t, X(t))$ at time t and state of the system $X(t)$, which is equivalent to: for every trajectory X corresponding to Δ, for almost every t, $\Delta(\cdot, t, X(t))$ is an equilibrium in the static game at the time t and the state of the system $X(t)$.

c) Let us take a dynamic strategy D being a dynamic best response of player ω to the control function U at a trajectory X corresponding to U, such that $\Pi_\omega(D, U, X)$ is finite. If there exists a function $D' \in \mathfrak{S}$ with $D'(t,x) \in S_\omega(x,t)$ for a.e. t and every x and $D'(t, X(t)) \in B_\omega(U(t, X(t)), X(t), t)$ for a.e. t, then the static strategies $D(t, X(t))$ are for a.e. t, static best responses to $U(t, X(t))$ at time t and state $X(t)$, since otherwise we could increase the payoff by changing the dynamic strategy on the graph of X. Such a function D' exists

d) Every dynamic best response to U is a dynamic best response to U at each trajectory X corresponding to it. Therefore, for every trajectory X corresponding to U and almost every t, we have $D(t, X(t)) \in B_\omega(U(t, X(t)), X(t), t)$.

e) Let Δ be a dynamic equilibrium with respect to an increasing utility function, such that for a.e. trajectory X corresponding to Δ, for a.e. player ω the payoff $\Pi_\omega(\Delta(\omega, \cdot, \cdot), U_\Delta, X)$ is finite. Then, the function $\Delta(\omega, \cdot, \cdot)$ is ω's dynamic best response to U_Δ at almost every trajectory X corresponding to U_Δ.
Since the payoff of almost every player is finite, by c) the latter statement implies the following: for almost every trajectory X corresponding to U_Δ, a.e. ω, t, $\Delta(\omega, t, X(t))$ is ω's static best response to $U_\Delta(t, X(t))$ at t and $X(t)$, therefore for a.e. t, $\Delta(\cdot, t, X(t))$ is an equilibrium in the static game at the time t and the state of the system $X(t)$.

f) Let Δ be a strong dynamic equilibrium, such that for every trajectory X corresponding to Δ, for a.e. player ω the payoff $\Pi_\omega(\Delta(\omega, \cdot, \cdot), U_\Delta, X)$ is finite. Then, for a.e. ω the function $\Delta(\omega, \cdot, \cdot)$ is ω's dynamic best response to U_Δ.
Since the payoff of almost every player is finite, by d) the latter statement implies the following: for every trajectory X corresponding to U_Δ, a.e. ω, t, $\Delta(\omega, t, X(t))$ is ω's static best response to $U_\Delta(t, X(t))$ at t and $X(t)$, therefore for a.e. t, $\Delta(\cdot, t, X(t))$ is an equilibrium in the static game at the time t and the state of the system $X(t)$. ■

It is worth emphasizing once again that an equilibrium does not depend on the discounting function. This result is counterintuitive when compared to the results that can be obtained in games with finitely many players. In such games discount rate always matters. In the case in some sense opposite to the one considered in our paper, with only one owner of an exhaustible resource, marginal utility obtained from exploitation of the resource must grow at the rate equal to the discount rate or, equivalently, user costs resulting from depletion of the resource grow at this rate. This property is known as the *Hotelling rule* since it was first proven in Hotelling (1931). Nowadays, the analysis became more complicated (see e.g. Kuuluvainen and Tahvonen, 1996), but in games with finitely many players, extraction rate depends on the discount rate, at least to some extent.

In fact, the result obtained in this paper is not contradictory to the known standard results for games with a finite number of players.

To illustrate this fact we can compare games with increasing finite number of players and such that the joint influence on the system is normalized. This does not mean e.g. introduction of additional fishermen into the same fishery, but only treating the decision making processes as becoming more and more decentralized. In the simplest and the most abstract situation, the same fishery fleet can be regarded as one decision maker (which corresponds to a single player game). Then we decompose the set of players into disjoint subsets and the decisions of the others (e.g. we realize that each country's fleet makes their decisions themselves), and finally, after a few steps we start to treat the decision making problems as they really are (e.g. each owner or captain of a small ship decides himself). What we can expect, is that in such games the intensity of exploitation of the ecosystem (i.e. the extraction rate) grows with the number of players. Such an analysis was made in e.g. Wiszniewska-Matyszkiel (2003b,c). It was proven that the extraction rate always increases as the number of players increases and it tends to the extraction rate in a continuum of players counterpart of the game (this rate is independent from the discount rate) as the number of players tends to infinity.

The following facts are simple consequences of Theorem 1:

COROLLARY 3. *Let $\mathbb{X}$ be countable with $\mathcal{X} = 2^{\mathbb{X}}$. If the function P_ω is upper semicontinuous in d and $\mathcal{S}\otimes\mathcal{B}(\mathbb{Y})\otimes\mathcal{X}\otimes\overline{\mathcal{T}}$-analytically measurable, the graph of the correspondence S_ω is $\mathcal{X}\otimes\overline{\mathcal{T}}\otimes\mathcal{S}$-analytic and the values of S_ω are compact, then for every measurable function U there exists ω's dynamic best response to U.*

Proof. By Theorem 1 the function D from Lemma 2 a) is an ω's dynamic best response to U. ■

COROLLARY 4. *Assume that $\mathbb{X}$ is countable with $\mathcal{X} = 2^{\mathbb{X}}$ and there is a convex structure on $\mathbb{S}$ (e.g. given by a family of sets called convex sets as in e.g. Wieczorek, 1992). If the function P_ω is upper semicontinuous and strictly quasiconcave in d and $\mathcal{S}\otimes\mathcal{B}(\mathbb{Y})\otimes\mathcal{X}\otimes\overline{\mathcal{T}}$-analytically measurable, the values of the correspondence S_ω are compact convex, and the graph of S_ω is $\mathcal{X}\otimes\overline{\mathcal{T}}\otimes\mathcal{S}$-analytic, then for every measurable function U such that the set of admissible payoffs of player ω at this U is bounded from above and differs from the singleton $\{-\infty\}$, then for all the functions D_1 and D_2 being dynamic best responses to U, every trajectory X corresponding to U, and a.e. t we have $D_1(t, X(t)) = D_2(t, X(t))$, and the set of dynamic best responses to U is nonempty.*

Proof. The static best response sets are singletons for every fixed values of x, t and u. Let D_1 and D_2 be dynamic best responses to U, and X a trajectory corresponding to U. By Theorem 1, the value of every dynamic best response to U are for a.e. t in the corresponding static best response set, so for a.e. t we have $D_1(t, X(t)) = D_2(t, X(t))$. The set of dynamic best responses to U is nonempty by Corollary 3. ■

Theorem 1 allows also to prove strong results on the existence of dynamic

4. Existence of dynamic equilibria

Let the symbol $\mathrm{SE}(x,t)$ denote the set of static equilibria at time t and state x and let $\overline{\mathrm{SE}}(x,t)$ denote the set of their statistics.

THEOREM 5. *Assume that $\mathbb{T}$ and $\mathbb{X}$ are countable with $\mathcal{T}=2^{\mathbb{T}}$ and $\mathcal{X}=2^{\mathbb{X}}$, and that $(\mathbb{S},\mathcal{S})$ fulfils A1. If for every x and a.e. t assumptions A2–A7 are fulfilled in $G_{t,x}$, then there exists a strong dynamic equilibrium.*

Proof. By Theorem 3.1 from Wiszniewska-Matyszkiel (2000b), there exist static equilibria for all possible states and times.

Let us choose a function $\Delta:\Omega\times\mathbb{T}\times\mathbb{X}\to\mathbb{S}$ such that $\Delta(\cdot,t,x)\in\mathrm{SE}(x,t)$ for all t and x. The function Δ is $\overline{\Im}\otimes\mathcal{X}\otimes\mathcal{T}$-measurable and $\Delta(\omega,t,x)\in S_\omega(x,t)$ for a.e. ω and every x, t. Therefore, Δ is a dynamic profile whose static profiles are static equilibria. By Theorem 1, it is a dynamic equilibrium. ∎

Even if the time space is not discrete, a true existence theorem can be stated, but only in a special case, with additional assumptions on $\mathbb{S}$, Φ, P and S.

From now on we shall assume that there is a convex structure on $\mathbb{S}$.

THEOREM 6. *Let $\mathbb{X}$ be countable with $\mathcal{X}=2^{\mathbb{X}}$. Assume that the function P is constant in u (the statistic). If the function P is $\overline{\Im}\otimes\mathcal{S}\otimes\mathcal{B}(\mathbb{R}^m)\otimes\mathcal{X}\otimes\overline{\mathcal{T}}$-analytically measurable, for a.e. ω, t and every u, x the function $P_\omega(\cdot,u,x,t)$ is strictly quasi-concave, for a.e. t and every x the game $G_{t,x}$ fulfils assumptions A1, A2, A3, A5 and A7, the graph of the correspondence S is $\overline{\Im}\otimes\mathcal{X}\otimes\overline{\mathcal{T}}\otimes\mathcal{S}$-analytic and for a.e. ω, t and every x the set $S_\omega(x,t)$ is convex, then there exists a strong dynamic equilibrium Δ and such that for a.e. t and every x the static profile $\Delta(\cdot,t,x)$ is a static equilibrium in the corresponding game $G_{t,x}$.*

To prove Theorem 6 we shall need the following sequence of lemmata.

LEMMA 7. *If the correspondence $\overline{B}$ has a $\mathcal{B}(\mathbb{Y})\otimes\overline{\mathcal{X}\otimes\mathcal{T}}\otimes\mathcal{B}(\mathbb{Y})$-analytic graph, then the correspondence of statistics of static equilibria $\overline{\mathrm{SE}}$ has an $\overline{\mathcal{X}\otimes\mathcal{T}}\otimes\mathcal{B}(\mathbb{Y})$-analytic graph.*

LEMMA 8. *Let $\mathbb{X}$ be countable with $\mathcal{X}=2^{\mathbb{X}}$. If the correspondence $\overline{B}$ is single valued for every u, x and a.e. t and has a $\mathcal{B}(\mathbb{Y})\otimes\overline{\mathcal{X}\otimes\mathcal{T}}\otimes\mathcal{B}(\mathbb{Y})$-analytic graph and for every x and a.e. t there exist static equilibria in games $G_{t,x}$, then there exists a strong dynamic equilibrium Δ in $\mathfrak{S}$ and such that for almost every t and every x the static profile $\Delta(\cdot,t,x)$ is a static equilibrium in $G_{t,x}$.*

LEMMA 9. *Let $\mathbb{X}$ be countable with $\mathcal{X}=2^{\mathbb{X}}$. If the function P is $\overline{\Im}\otimes\mathcal{S}\otimes\mathcal{B}(\mathbb{R}^m)\otimes\mathcal{X}\otimes\overline{\mathcal{T}}$-analytically measurable, for a.e. ω, t and every u, x the function $P_\omega(\cdot,u,x,t)$ is strictly quasi-concave, for a.e. t and every x the game $G_{t,x}$ fulfils assumptions A1, A2, A3, A5 and A7, the graph of the correspondence S is $\overline{\Im}\otimes\mathcal{X}\otimes\overline{\mathcal{T}}\otimes\mathcal{S}$-analytic and for a.e. ω, t and every x the set $S_\omega(x,t)$ is convex,*

$\overline{\mathcal{T}} \otimes \mathcal{S}$-analytic graph, equal to B for every u, x and a.e. ω, t. The statistic $\overline{B'}$ for every u, x and a.e. t equals the correspondence $\overline{B}$ and $\overline{B'}$ treated as a function is $\overline{\mathcal{B}(\mathbb{Y}) \otimes \mathcal{X} \otimes \mathcal{T}}$-measurable (therefore its graph is $\overline{\mathcal{B}(\mathbb{Y}) \otimes \mathcal{X} \otimes \mathcal{T}} \otimes \mathcal{B}(\mathbb{Y})$-measurable), where the completion is with respect to the product measure of the Lebesgue measure on $\mathcal{B}(\mathbb{Y})$, and a measure ξ on $\mathcal{X}$ such that $\xi(x) > 0$ for all $x \in \mathbb{X}$, and λ on $\mathcal{T}$.

Proof. (of Lemma 7)

Let us examine the graph of the correspondence of statistics of static equilibria. We have:

$$\begin{aligned}
&\mathrm{Gr}\,\overline{\mathrm{SE}} = \{(x,t,u) \in \mathbb{X} \times \mathbb{T} \times \mathbb{Y} \mid u = \overline{B}(u,x,t)\} \\
&= \mathrm{Proj}_{\mathbb{X}\times\mathbb{T}\times\mathbb{Y}}(\{(x,t,u,v) \in \mathbb{X} \times \mathbb{T} \times \mathbb{Y} \times \mathbb{Y} \mid v = \overline{B}(u,x,t)\} \\
&\cap \{(x,t,u,v) \in \mathbb{X} \times \mathbb{T} \times \mathbb{Y} \times \mathbb{Y} \mid v = u\}) \\
&= \mathrm{Proj}_{\mathbb{X}\times\mathbb{T}\times\mathbb{Y}}(\mathrm{Gr}\,\overleftrightarrow{B} \cap (\mathbb{X} \times \mathbb{T} \times \mathrm{diag}_{\mathbb{Y}}))
\end{aligned}$$

where $\overleftrightarrow{B}(x,t,u)$ denotes $\overline{B}(u,x,t)$.

Since the graph of the correspondence $\overline{B}$ is $\mathcal{B}(\mathbb{Y}) \otimes \overline{\mathcal{X} \otimes \mathcal{T}} \otimes \mathcal{B}(\mathbb{Y})$-analytic and the diagonal $\mathrm{diag}_{\mathbb{Y}}$ is in $\mathcal{B}(\mathbb{Y}) \otimes \mathcal{B}(\mathbb{Y})$, by theorem of Marczewski and Ryll-Nardzewski (1953), the graph of the correspondence $\overline{\mathrm{SE}}$ is $\overline{\mathcal{X} \otimes \mathcal{T}} \otimes \mathcal{B}(\mathbb{Y})$-analytic. ■

Proof. (of Lemma 8)

Let ξ be a measure on $\mathcal{X}$ such that $\xi(x) > 0$ for all $x \in \mathbb{X}$.

By Lemma 7, the graph of the correspondence of statistics of static equilibria $\overline{\mathrm{SE}}$ is $\overline{\mathcal{X} \otimes \mathcal{T}} \otimes \mathcal{B}(\mathbb{Y})$-analytic. Moreover, for every x and almost every t the set $\overline{\mathrm{SE}}(x,t)$ is nonempty. Therefore there exists an $\overline{\mathcal{X} \otimes \mathcal{T}}$-measurable a.e. selection U (Leese, 1978, Theorem 5.5). By Proposition 3.2 from Wiszniewska-Matyszkiel (2000b), there exists an $\mathcal{X} \otimes \mathcal{T}$-measurable function $\overline{U}$ almost everywhere equal to U.

By Lemma 4.1 from Wiszniewska-Matyszkiel (2002a), there exists an $\Im \otimes \mathcal{T} \otimes \mathcal{X}$-measurable function $\Delta : \Omega \times \mathbb{T} \times \mathbb{X} \to \mathbb{S}$ such that for a.e. ω, t and x the static strategy $\Delta(\omega,t,x)$ is in $B_\omega(\overline{U}(t,x),x,t)$.

For almost every t and x we have $\overline{B}(\overline{U}(t,x),x,t) = \{\overline{U}(t,x)\}$, therefore $U_\Delta(t,x) = \overline{U}(t,x)$. For these t and x the static profile $\Delta(\cdot,t,x)$ is a static equilibrium. But since all elements of $\mathbb{X}$ are of positive measure, $\Delta(\cdot,t,x)$ is a static equilibrium for a.e. t and every x. Since Δ is a dynamic profile, whose static profiles are for a.e. t and every x static equilibria, by Theorem 1, it is a strong dynamic equilibrium. ■

Proof. (of Lemma 9)

The static best response sets $B_\omega(u,x,t)$ are singletons except those for ω in

Let us take any $\bar{d} \in \mathbb{S}$. We define a correspondence B' by

$$B'_\omega(u,x,t) = \begin{cases} B_\omega(u,x,t) & \text{if } \omega \in \setminus\Omega_0,\ t \in \setminus\mathbb{T}_0, \\ \bar{d} & \text{otherwise.} \end{cases}$$

Both the correspondence B' and its statistic $\overline{B'}$ are single valued. Let us note that whenever B has an $\overline{\Im} \otimes \mathcal{B}(\mathbb{Y}) \otimes \mathcal{X} \otimes \overline{\mathcal{T}} \otimes \mathcal{S}$-analytic graph, B' has also an $\overline{\Im} \otimes \mathcal{B}(\mathbb{Y}) \otimes \mathcal{X} \otimes \overline{\mathcal{T}} \otimes \mathcal{S}$-analytic graph and for $t \notin \mathbb{T}_0$ the statistics of $B_\cdot(u,x,t)$ and $B'_\cdot(u,x,t)$ are equal.

Since B' and $\overline{B'}$ are single valued, we can treat them as functions.

By Lemma 4.1 from Wiszniewska-Matyszkiel (2002a), the graph of the function B' is $\overline{\Im \otimes \mathcal{B}(\mathbb{Y}) \otimes \mathcal{X} \otimes \mathcal{T}} \otimes \mathcal{S}$-analytic. Since the space $(\mathbb{S}, \mathcal{S})$ fulfils assumption A1 and the diagonal diag $\mathbb{Y}$ is $\mathcal{B}(\mathbb{Y}) \otimes \mathcal{B}(\mathbb{Y})$-measurable, by Lemma 4.2 of Wiszniewska-Matyszkiel (2002a), the graph of the function $g \circ (\mathrm{Id}_\Omega \times B')$ is $\overline{\Im \otimes \mathcal{B}(\mathbb{Y}) \otimes \mathcal{X} \otimes \mathcal{T}} \otimes \mathcal{B}(\mathbb{R}^m)$-analytic.

The inverse images of $\mathcal{B}(\mathbb{R}^m)$-measurable sets by this function are $\overline{\Im \otimes \mathcal{B}(\mathbb{Y}) \otimes \mathcal{X} \otimes \mathcal{T}}$-analytic, since they are projections of $\overline{\Im \otimes \mathcal{B}(\mathbb{Y}) \otimes \mathcal{X} \otimes \mathcal{T}} \otimes \mathcal{B}(\mathbb{R}^m)$-analytic sets (by theorem of Marczewski and Ryll-Nardzewski, 1953). Therefore they are $\overline{\Im \otimes \mathcal{B}(\mathbb{Y}) \otimes \mathcal{X} \otimes \mathcal{T}}$-measurable (by theorem of Saks, 1937).

Hence the function $g \circ (\mathrm{Id}_\Omega \times B')$ is $\overline{\Im \otimes \mathcal{B}(\mathbb{Y}) \otimes \mathcal{X} \otimes \mathcal{T}}$-measurable, which implies that the function $\overline{B'}$ is $\overline{\mathcal{B}(\mathbb{Y}) \otimes \mathcal{X} \otimes \mathcal{T}}$-measurable by the general Fubini theorem (as an integral of a function measurable with respect to the completion of the product σ-field) and its graph is $\overline{\mathcal{B}(\mathbb{Y}) \otimes \mathcal{X} \otimes \mathcal{T}} \otimes \mathcal{B}(\mathbb{Y})$-measurable (since diag $\mathbb{Y}$ is measurable). ■

Proof. (of Theorem 6)

At first we show that it can be assumed without loss of generality that static best response sets are singletons, therefore B, as well as $\overline{B}$, are functions:
Note that a dynamic profile Δ is an a.e. selection from B if and only if it is an a.e. selection from B' defined in the proof of Lemma 9, for almost every ω, t equal to B. Therefore we can work with B and $\overline{B}$, or B' and $\overline{B'}$, equivalently.

Since the function P does not depend on u, neither B nor $\overline{B}$ depends on u.

Let ξ be a measure on $\mathcal{X}$ such that $\xi(\{x\}) > 0$ for every x.

By Lemma 9, the function $\overline{B}$ is $\overline{\mathcal{B}(\mathbb{Y}) \otimes \mathcal{X} \otimes \mathcal{T}}$-measurable. Since it does not depend on u, the inverse image of any $\mathcal{B}(\mathbb{Y})$-measurable set has the form $\mathbb{Y} \times A$ for some $A \in \overline{\mathcal{X} \otimes \mathcal{T}}$, which implies that it is $\mathcal{B}(\mathbb{Y}) \otimes \overline{\mathcal{X} \otimes \mathcal{T}}$-measurable and since every $x \in \mathbb{X}$ is of positive measure, $\mathcal{B}(\mathbb{Y}) \otimes \mathcal{X} \otimes \overline{\mathcal{T}}$-measurable. Therefore the function $\overline{B}$ is $\mathcal{B}(\mathbb{Y}) \otimes \mathcal{X} \otimes \overline{\mathcal{T}}$-measurable. Since diag $\mathbb{Y}$ is in $\mathcal{B}(\mathbb{Y}) \otimes \mathcal{B}(\mathbb{Y})$, the graph of $\overline{B}$ is $\mathcal{B}(\mathbb{Y}) \otimes \mathcal{X} \otimes \overline{\mathcal{T}} \otimes \mathcal{B}(\mathbb{Y})$-measurable.

Therefore, by Lemma 7, the correspondence of statistics of static equilibria has an $\mathcal{X} \otimes \overline{\mathcal{T}} \otimes \mathcal{B}(\mathbb{Y})$-analytic graph. Moreover, it has a.e. nonempty values (since there exist static equilibria: the analyticity assumptions in this lemma are stronger than assumptions A4 and A6 – analyticity assumptions used in The-in a game with an atomless space of players, and the remaining assumptions are assumed to be fulfilled).

By Lemma 8, there exists a dynamic profile Δ being a strong dynamic equilibrium with almost every static profile being a static equilibrium. ■

4.1. Existence of dynamic equilibria in games with finite number of types of players

Now we shall consider a game in which the players can be divided into k disjoint sets $\Omega_1, \ldots, \Omega_k$ of positive measure and such that $\bigcup_{i=1}^{k} \Omega_i = \Omega$. The players from the same Ω_i are identical i.e. there exist functions P^i, g^i and correspondences S^i (we may call them a *payoff function*, a *pre-statistic function* and an *available strategy correspondence of type* i) such that $P_\omega(d,u,x,t) = P^i(d,u,x,t)$, $S_\omega(x,t) = S^i(x,t)$ and $g(\omega,d) = g^i(d)$ for all $\omega \in \Omega_i$, u, x and t.

THEOREM 10. *Let $\mathbb{X}$ be countable with $\mathcal{X} = 2^{\mathbb{X}}$. If for a.e. t and every x the game $G_{t,x}$ fulfils assumptions A1, A2, A3, A5 and A7, every function P^i is $\mathcal{S}\otimes\mathcal{B}(\mathbb{R}^m)\otimes\mathcal{X}\otimes\overline{\mathcal{T}}$-analytically measurable and constant with respect to statistic, for a.e. t and every i, u, x the function $P^i(\cdot,u,x,t)$ is quasi-concave, the graph of the correspondence S^i is $\mathcal{X}\otimes\overline{\mathcal{T}}\otimes\mathcal{S}$-analytic, for every i, x and a.e. t the set $S^i(x,t)$ is convex, and for every i the type's i pre-statistic function g^i preserves convexity (i.e. the image of a convex set is convex, e.g. g is affine), then there exists a strong dynamic equilibrium Δ and for a.e. t and every x the static profile $\Delta(\cdot,t,x)$ is a static equilibrium in $G_{t,x}$.*

Proof. We define a vector valued function $\widetilde{g} : \Omega \times \mathbb{S} \to \mathbb{R}^{m\cdot k}$, decomposed as $\widetilde{g} = [\widetilde{g}^i]_{i=1}^k$, where the vector functions $\widetilde{g}^i : \Omega \times \mathbb{S} \to \mathbb{R}^m$ are given by

$$\widetilde{g}^i(\omega,d) \stackrel{\text{def}}{=} [\widetilde{g}_j(\omega,d)]_{j=(i-1)\cdot m+1}^{i\cdot m} = \begin{cases} g^i(d) & \text{if } \omega \in \Omega_i, \\ 0 & \text{if not,} \end{cases}$$

(after integration along a profile, the function $\widetilde{g}^i$ defines the statistic of decisions of players of type i).

Let us denote $\{q \in \mathbb{R}^{mk} \mid q \in \int_\Omega \widetilde{g}(\{\omega\} \times S_\omega(x,t))d\mu(\omega),\ x \in \mathbb{X},\ t \in \mathbb{T}\}$ by $\mathbb{Q}$ and by $\mathfrak{Q}$ the set of all $\mathcal{T}\otimes\mathcal{X}$-measurable functions $Q : \mathbb{T}\times\mathbb{X} \to \mathbb{Q}$. Every $q \in \mathbb{Q}$ will be written as $[q^i]_{i=1}^k$, where q^i denotes the vector $[q_j]_{j=(i-1)\cdot m+1}^{i\cdot m}$. Every function with values in $\mathbb{Q}$ will be written analogously.

Let us define a correspondence $\widetilde{\overline{B}} : \mathbb{Q}\times\mathbb{X}\times\mathbb{T} \multimap \mathbb{Q}$ by

$$\widetilde{\overline{B}}(q,x,t) = \int_\Omega \widetilde{g}(\{\omega\} \times B_\omega(q^1 + \ldots + q^k, x, t))d\mu(\omega).$$

Note that by definition of $\widetilde{\overline{B}}$, the vector $q^1 + \ldots + q^k$ for q being a fixed point

Let us consider a modified game

$$\widetilde{\mathfrak{G}} = ((\mathbb{X}, \mathcal{X}), x_0, (\mathbb{T}, \mathcal{T}, \lambda), \mathfrak{X}, (\Omega, \Im, \mu), (\mathbb{S}, \mathcal{S}), S, \widetilde{g}, \widetilde{P}, \widetilde{\Phi}, \Psi)$$

with $\widetilde{P}_\omega(d, q, x, t) = P_\omega(d, q^1 + \cdots + q^k, x, t)$ and $\widetilde{\Phi}(X, Q, t) = \Phi(X, Q^1 + \cdots + Q^k, t)$. Let us denote the static best response correspondence in $\widetilde{\mathfrak{G}}$ by $\widetilde{B}$. Obviously, the correspondence $\overline{\widetilde{B}}$, defined before, is the correspondence of statistics of the static best responses in $\widetilde{\mathfrak{G}}$.

By Lemma 4.1 from Wiszniewska-Matyszkiel (2002a), there exists an $\Im \otimes \mathcal{B}(\mathbb{Q}) \otimes \mathcal{X} \otimes \mathcal{T}$-measurable function F being an a.e. selection from $\widetilde{B}$.

Take any function $Q \in \mathfrak{Q}$. A function $\overline{Q} : \mathbb{T} \times \mathbb{X} \to \mathbb{Q}$ defined by

$$\overline{Q}(t, x) = \int_\Omega \widetilde{g}(\omega, F(\omega, Q(t, x), x, t)) d\mu(\omega)$$

is $\overline{\mathcal{X} \otimes \mathcal{T}}$-measurable and it is an a.e. selection from the correspondence of fixed points of $\overline{\widetilde{B}} \circ (Q \times (\mathrm{Id}_\mathbb{X} \times \mathrm{Id}_\mathbb{T}))$ (since the payoff does not depend on the statistic).

Since for a.e. t and every u, x, i, the function $P^i(\cdot, u, x, t)$ is quasi-concave and upper semicontinuous, for a.e. t and every x, i, the set $S^i(x, t)$ is convex and compact, and for every i, the function g^i preserves convexity, the set $\widetilde{g}(\{\omega\} \times B_\omega(q^1 + \ldots + q^k, x, t))$ is nonempty and convex (for a.e. t and every q, x), therefore for $\omega \in \Omega_i$ and for a.e. t we have $\frac{\overline{Q}^i(t,x)}{\mu(\Omega_i)} \in \widetilde{g}^i(\{\omega\} \times B_\omega(\overline{Q}(t, x)^1 + \ldots + \overline{Q}(t, x)^k, x, t))$.

Since the correspondence $(i, t, x) \mapsto (\widetilde{g}^i)^{-1}(\{\frac{\overline{Q}^i(t,x)}{\mu(\Omega_i)}\})$ has a measurable graph, it admits a measurable a.e. selection $J = (J^1, \ldots, J^k)$.

Its coordinate functions fulfill the condition $J^i(t, x) \in (\widetilde{g}^i)^{-1}(\{\frac{\overline{Q}^i(t,x)}{\mu(\Omega_i)}\})$ for almost every t and x (and since $\mathbb{X}$ is countable, for almost every t and every x). Let $\Delta : \Omega \times \mathbb{T} \times \mathbb{X} \to \mathbb{S}$ fulfill $\Delta(\omega, t, x) = J^i(t, x)$ for all x, t and $\omega \in \Omega_i$. This function Δ is a strong dynamic equilibrium in $\mathfrak{G}$ and for a.e. t and every x the static profile $\Delta(\cdot, t, x)$ is a static equilibrium in $G_{t,x}$. ■

5. Examples

The examples presented in this section are devoted to stochastic games describing exploitation of common ecosystems by large groups of users. Some deterministic models of this problem are contained in Wiszniewska-Matyszkiel (2000a, 2001b). They show a substantial difference between equilibria in dynamic games with a continuum of players and their counterparts with finitely many players. An ample review on games modelling the exploitation of common

EXAMPLE 1. *Discrete time fishery with exogenous disasters*

Let us consider on oceanic fishery open to all fishermen of one species of fish. They fish every season and sell fish at the market. The cost of catching one unit (e.g. shoal) of fish depends on the amount of fish in the environment and the equipment specific fisherman possesses, while the price of fish depends only on the amount at the market. Catching during spawning is forbidden (which makes the discretization of time natural). The rate of population growth, if there were no disasters, would be $0 < r < 1$, but disasters or epidemics happen with a certain probability q independent on the amount of fish – then one unit perishes.

Formally:

The time space is $\mathbb{N}$ with the counting measure.

The space of players $(\Omega, \Im, \mu)$ is the unit interval with the Lebesgue measure.

The space of states $\mathbb{X} = \mathbb{R}_+$ and $x_0 > 0$.

Players' sets of available static strategies are $S_\omega(x,t) = [0, m_\omega \cdot x]$.

Players payoff functions have the form $P_\omega(d,u,x,t) = (p(u) - c_\omega(x)) \cdot d$, where the function $p : \mathbb{Y} \to \mathbb{R}_+$ reflects the price of fish and is nonincreasing and continuous, while $c : \Omega \times \mathbb{X} \to \mathbb{R}_+$ reflects the cost of catching a unit of fish and is strictly decreasing in the second argument. Obviously, the functions m and $c(\cdot, x)$ (given any fixed x ; c treated as the function of its first argument) are assumed to be $\Im$-measurable. Moreover, we assume that there exist: a low state x' such that $p(0) < c_\omega(x')$ for all ω (in this state there is so little fish that catching one unit costs more than the possible highest market price) and high state x'' such that $p(M \cdot x) > c_\omega(x'')$, where $M = \int_\Omega m_\omega d\mu(\omega)$ (in x'' there is so much fish that catching costs less than the lowest possible market price).

The pre-statistic function is $g(\omega, d) = d$.

The function of the behaviour of the system is defined by $\Phi(X,U,0) = x_0$ (see notational remarks) for every trajectory X starting from x_0, and $\Phi(X,U,t) = \max(0, (1+r)X(t-1) - U(t-1) + \zeta)$ for $t > 0$, where ζ is a distribution describing ecological disasters or epidemics: $P(\zeta = -1) = q \in (0,1)$ and $P(\zeta = 0) = 1 - q$.

The discounting function is $\Psi(t) = (1+\xi)^{-t}$.

PROPOSITION 11. a) *Every strong dynamic equilibrium fulfils the following condition:*

$$\text{for a.e. } \omega \text{ and every } x,t, \quad \Delta(\omega,t,x) = \begin{cases} m_\omega \cdot x & \text{if } x > x'', \\ 0 & \text{if } x < x'. \end{cases}$$

b) *Let Δ be a strong dynamic equilibrium. If $M > r$ and $x'' < \frac{1}{1+r}$, then almost every trajectory corresponding to Δ tends to* 0 *(the population of fish will extinct with probability* 1*).*

Proof. a) Since the player's instantaneous payoff function is strictly increasing in d for $x > x''$, strictly decreasing for $x \le x'$, and the sets of available strategies for each of these x: $m_\omega \cdot x$ for $x > x''$ and 0 for $x < x'$. Since payoffs are finite, by Theorem 1, every strong dynamic equilibrium consists of static equilibria.

b) Let us take an arbitrary trajectory X corresponding to Δ. By a), it fulfils $X(t+1) = (1+r-M) \cdot X(t) + \zeta \leq (1+r-M) \cdot X(t)$ whenever $X(t) > x''$, so that even without a disaster after finitely many steps we have $X(\bar{t}) \leq x''$. Whatever the profile is, $X(\bar{t}+1) \leq (1+r) \cdot X(\bar{t}) \leq 1$. Obviously, for all $t > \bar{t}$ we have $X(t) \leq 1$. So if a disaster happens at any moment $t > \bar{t}$, the subsequent states will be 0. A disaster happens at some $t > \bar{t}$ with probability one. ■

EXAMPLE 2. *Continuous time forest with seasonal disasters*

This example describes exploitation of a forest being the only basis of existence of the players. The state is measured by volume of wood. The rate of growth is $r \in (0,1)$, but every summer a fire can destroy a part of wood, therefore the trajectories are piecewise continuous with stochastic jumps in integers. Every player's instantaneous payoff function is strictly increasing in his own strategy with $-\infty$ for individual extraction equal to 0 (starvation).

Formally:

The time space is $\mathbb{R}_+$ with the Lebesgue measure.

The space of players $(\Omega, \Im, \mu)$ is the unit interval with the Lebesgue measure.

The space of states $\mathbb{X} = \mathbb{R}_+$ and $x_0 > 0$.

Players' sets of available static strategies are $S_\omega(x,t) = [0, m_\omega \cdot x]$ for a measurable function m. We assume that $M = \int_\Omega m_\omega d\mu(\omega)$ is finite.

Players payoff functions have the form

$$P_\omega(d,u,x,t) = \begin{cases} \ln d - u^2 + x & \text{if } d > 0, \\ -\infty & \text{if } d = 0. \end{cases}$$

The pre-statistic function is $g(\omega, d) = d$.

The function of the behaviour of the system is defined by

$$\Phi(X,U,t) = \begin{cases} x_0 & t = 0, \\ \max(0, X([t]) + \int_{[t]}^{t} rX(t) - U(t) d\lambda(t)) & t \notin \mathbb{N}, \\ \max(0, \lim_{s \to t^-} X(s) + \zeta) & t \in \mathbb{N} \setminus \{0\}, \end{cases}$$

(see notational remarks in Subsection 2.2), where ζ is a distribution describing the loss caused by fire: $P(\zeta = 0) = p < 1$, $P(\zeta > 0) = 0$ and $P(\zeta \leq -1) = q \in (0,1)$.

The discounting function is $\Psi(t) = (1+\xi)^{-t}$.

PROPOSITION 12. a) *The formula $\Delta(\omega,t,x) = m_\omega \cdot x$ defines a strong dynamic equilibrium.*

b) *If U is an admissible control function and X a trajectory corresponding to U such that $X(t) > 0$ for every t, then every function D being a dynamic best response to U at X fulfils $D(t, X(t)) = m_\omega \cdot X(t)$*

c) *If $M \geq r$, then no dynamic profile such that almost every player gets finite payoff with positive probability is a dynamic equilibrium with respect to an increasing function F.*

d) *If $M \geq r$, then every dynamic profile Δ yielding the destruction of the system in finite time with probability one (i.e. for almost every trajectory X corresponding to Δ there exists $t_X \in \mathbb{R}$ such that $X(t_X) = 0$) is a strong dynamic equilibrium.*

Proof. Both a) and b) are straightforward consequences of Theorem 1.

c) Let Δ be a dynamic equilibrium such that the set of players who get finite payoffs with positive probability is of measure 1. The only static best response of every player to $U(t, x)$ at time t and state x is $d = m_\omega \cdot x$, therefore the statistic of the resulting profile is

$$U_\Delta(t, x) = M \cdot x \geq r \cdot x.$$

Hence, every corresponding trajectory X fulfils $X(t) \leq x_0 \cdot e^{-(r-M)t} \leq x_0$ and is strictly nonincreasing. A fire destroying at least one unit happens $[x_0] + 1$ times over the infinite time interval with probability one. Let $\bar{t}$ denote a time such that it happened $[x_0] + 1$ times before $\bar{t}$. We have $X(t) = 0$ for $t \geq \bar{t}$, therefore $S_\omega(X(t), t) = \{0\}$ for $t \geq \bar{t}$, so for $t \geq \bar{t}$ every dynamic profile fulfils $\Delta(\omega, t, X(t)) \equiv 0$ for a.e. ω. Hence, for $t \geq \bar{t}$ the instantaneous payoff is equal to $-\infty$. Therefore almost every player's payoff is equal to $-\infty$.

That means that there exists no equilibrium with respect to an increasing function F with finite payoff of almost every player.

d) Almost every trajectory X corresponding to Δ fulfils the condition $X(t) = 0$ for $t > t_X$ for some $t_X \in \mathbb{R}$. Since the only possible strategy of player ω for $t \in (t_X, +\infty)$ and $x = 0$ is $d = 0$, his only possible payoff at the control function U_Δ is $-\infty$, whatever admissible dynamic strategy he chooses. Therefore $\mathbf{B}^X_\omega(U_\Delta)$ is the set of all dynamic strategies, which implies that it contains $\Delta(\omega, \cdot, \cdot)$. ■

6. Conclusions

In this paper stochastic games with a continuum of players were examined. These games model situations with random response of the system to players' decisions and such that a single player's decision has insignificant influence on the global parameters. Because of this insignificance, there is an equivalence between equilibria in the dynamic game and those in static games corresponding to it.

This equivalence reduces the problem of finding an equilibrium in a dynamic game to solution of a parametrized family of static problems. Moreover, it makes proving certain existence results possible. These results are strong and

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