Control and Cybernetics

vol. **32** (2003) No. 1

General elections modelled with infinitely many voters*

by

Maria Ekes

Warsaw School of Economics and Institute of Computer Science
Polish Academy of Sciences

Abstract: The paper examines a model of general elections with electorate composed of infinitely many voters classified into a finite number of types. We focus on the case of elections among two candidates, we give a full characterization of equilibria in such models and we classify equilibria with respect to their stability.

Keywords: general elections, equilibrium, preference-indiffer relation.

1. Introduction

In the present paper we examine a model of general elections with electorate composed of infinitely many voters classified into a finite number of types. We focus on the case of elections among two candidates, which takes place for example in the second round of presidential elections in Poland. We can equivalently consider referenda where two options are presented to a society and citizens are to choose one of them. Consideration of two options or two candidates allows to omit problems connected with various concepts of solutions in voting games (see e.g. Nurmi, 1995). We focus on the notion of equilibrium, give a full characterization of equilibria in such models and classify equilibria with respect to their stability. We do not consider here the problems concerning behaviour of candidates, like the choice of their position in the issue or policy space over which they compete (see e.g. Brams, 1978, Kramer, 1978, McKelvey, Ordeshook, Ungar, 1980, Owen, Shapley, 1989). The former is obvious if we model a referendum while when describing presidential elections it is rather assumed that all candidates have already chosen their policies and each voter shall vote for the most favourable candidate.

Some concepts in this paper are related to those in Myerson, Weber (1993), but the models are essentially different – we assume that individual preferences depend not only on the final result of the elections but also on the voter's individual choice and we consider infinitely many voters, therefore no individual has influence on the outcome of the elections. There arises, threrefore, the question why, in this case, the voters do participate in elections at all? Yet, this question is not specific for our model, but it applies to real elections as well. The question is known and has been extensively discussed in the literature, see e.g. Schuessler (2000).

Problems related to those considered in the present paper have also been studied by Gambarelli (1999), Hołubiec (1999), Mercik (1999).

2. Description of the model

We deal with the model of general elections (such as referendum or presidential elections) in which the electorate has to choose, by voting, one of a fixed number of options, possibly one of them being abstention. Formally, the electorate is choosing an element of the set $K = \{1, \dots, k\}$ or $K = \{0, 1, \dots, K\}$ if abstention, denoted by 0, is allowed and taken into consideration. The electorate generates a distribution $(P_1, \dots, P_k)$ or $(P_0, P_1, \dots, P_k)$ on K and the *winner* is the unique element of $\{1, \dots, k\}$ with the largest corresponding P_j; if there is no such unique element, we say that the elections end up with a *draw* and we denote this result by D, so the set of all a outcomes is $\mathcal{O} = \{D, 1, \dots, k\}$. Since the electorate may generate any distribution, we must see it as infinite; this point of view is very convenient and not leading to contradiction. As usual, members of the electorate should have some preferences (formally, preference-indifference relations, in this paper assumed to be pre-ordering relations), which do not apply only to the results of the elections but also to their individual behaviour, so that each member of the electorate has a pre-ordering relation on the set being the product of the set of all options and the set of all outcomes, i.e. on $K \times \mathcal{O}$. Obviously, this set has $k(k+1)$ or $(k+1)^2$ elements, depending on whether the abstention is permitted or not. The number of possible pre-ordering relations is then very large; even for $k = 2$ there are 4683 pre-ordering relations in the case without abstention and 7087261 relations in the case with abstention allowed (this is counted by a recursive formula, saying that $[m, l]$, the number of preference relations on an m-element set having l equivalence classes is equal to $l[m-1, l] + l[m-1, l-1]$; here we are interested in eventually calculating $\sum_{l=1}^{6} [6, l]$ in the case without abstention and $\sum_{l=1}^{9} [9, l]$ in the case with abstention). Most of these relations are strange and contradictory in the common sense, so even if dealing with all preference-indifference relations present among the electorate were technically possible, the results obtained would be hardly readable. Therefore, in the present paper we assume that only few "reasonable" preference-indifference relations are represented in the electorate. Hence, the whole electorate is divided into n populations, differing in their preferences; the size of the i-th population $(i = 1, \dots, n)$, having a preference-indifference relation and by $\sim_i$ the indifference relation, both generated by $\succsim_i$). So the i-th population generates in the course of elections a distribution p^i on K. Formally, p^i is an element of the standard simplex of dimension k or $k-1$, depending on the case (this simplex is denoted by $\Delta_{|K|}$). Consequently, a sequence of distributions of the decisions of all respective types, $\mathbf{p} = (p^1, \dots, p^n)$, which is a sequence of n elements of $\Delta_{|K|}$, generates a distribution of votes in the whole electorate. For the j-th option to be chosen ($j = 1, \dots, k$ if the abstention is not allowed and $j = 0, 1, \dots, k$ if the abstention is allowed), we have then $P_j = Q^{-1} \cdot \sum_{i=1}^{n} q_i p_j^i$, where Q denotes $\sum_{i=1}^{n} q_i$. We say that the j-th option is winning at the elections if $P_j > P_l$ for all $l = 1, \dots, k$, $l \neq j$. If there exist at least two different options j and j' such that $P_j = P_{j'} = \max_{l=1,\dots,k} P_l$ then the elections end up with a draw. Observe that each sequence of distributions of the voters' decisions $\mathbf{p}$ uniquely determines the outcome of the elections, denoted by $x_{\mathbf{p}} \in \mathcal{O}$. We say that the sequence of distributions $\mathbf{p}$ is at *equilibrium* whenever, for $i = 1, \dots, n$ and $j \in \{m \in K \mid p_m^i > 0\}$ the following condition

$$(j, x_{\mathbf{p}}) \succsim_i (l, x_{\mathbf{p}})$$

holds for all $l \in K$, which informally means that no voters could improve their satisfaction by changing their individual decision on how to vote.

In the remainder of the paper we focus on the case where $|K| = 2$, i.e. there are two options (in the sequel called candidates) and the abstention is not allowed. We present full classification of equilibria in the case of some specific types of voters present in the electorate and we consider also the characterization of equilibria with respect to their stability.

3. Equilibria in the case of voting for one of two candidates

Consider the case of voting for one of two candidates, who are denoted here by A and B for convenience. Each voter has to decide which candidate to vote for; abstention is not allowed. Therefore the set K has the form $K = \{A, B\}$. The set of outcomes is then $\mathcal{O} = \{D, A, B\}$, where D denotes draw, A denotes that A is the winner of elections and B denotes that B is the winner. A draw is not possible in practice (note that we deal with an infinite set of voters), but we still take it into consideration, since in the analysis of equilibrium we find out that there exists an equilibrium at which the elections end up with a draw. In this setting there are six pairs consisting of an individual decision and an outcome of the elections. We enumerate them in the following way:

1. vote for A & A wins;
2. vote for A & a draw;
3. vote for A & B wins;
4. vote for B & A wins; (3.1)
5. vote for B & a draw;
6. vote for B & B wins;

We restrict ourselves to the analysis of a situation, where there are only eight different types of voters. It seems that these types are similar to the profiles of preferences existing in any real electorate. All preference relations considered here are strict preferences. Voters of the first four types are the supporters of A, while voters of the last four types are the supporters of B. Differences between them concern their preferences in some specific situations. The following tables represent the preferences of voters of all types, for each type in order from the most preferred decision-outcome pair down to the least preferred one.

SUPPORTERS OF THE CANDIDATE A			
Nonconformist	Pragmatist	Moderate opportunist	Opportunist
TYPE 1	TYPE 2	TYPE 3	TYPE 4
1	**1**	**1**	**1**
2	**4**	**2**	**6**
3	**2**	**6**	**2**
4	**5**	**5**	**5**
5	**3**	**3**	**3**
6	**6**	**4**	**4**

Table 3.1.a

SUPPORTERS OF THE CANDIDATE B			
Nonconformist	Pragmatist	Moderate opportunist	Opportunist
TYPE 5	TYPE 6	TYPE 7	TYPE 8
6	**6**	**6**	**6**
5	**3**	**5**	**1**
4	**5**	**1**	**5**
3	**2**	**2**	**2**
2	**4**	**4**	**4**
1	**1**	**3**	**3**

Table 3.1.b

Supporters of A prefer the pair **1**, that is voting for the candidate A and his winning, to all remaining pairs. Similarly, supporters of B prefer the pair **6** to all others. They differ in their preferences concerning other possibilities. For example, a pragmatist prefers to vote for the opponent (opposing party) if it is related to the victory of his favourite, nonconformist always votes for his candidate and opportunist prefers to be on the winning side (with the top dog), although he prefers the victory of his candidate. Assume that the numbers $q_1, \ldots, q_8$, $q_i \geq 0$, for $i = 1, \ldots, 8$, describe the size of the population of voters respective types of voters in the electorate, in this case we have $\sum_{i=1}^{8} q_i = 1$. Let $\mathbf{p} = (p^1, \ldots, p^8)$ be a sequence of distributions of decisions of the voters of all types, i.e. $p^i = (p^i_A, p^i_B)$, $p^i_A + p^i_B = 1$, p^i_A, $p^i_B \geq 0$ for $i = 1, \ldots, 8$. Denote by P_A the fraction of voices won by the candidate A, which is the number $P_A = Q^{-1} \cdot \sum_{i=1}^{8} q_i p^i_A$ and, similarly, $P_B = Q^{-1} \cdot \sum_{i=1}^{8} q_i p^i_B$, where $Q = \sum_{i=1}^{8} q_i$. Therefore, one of the following cases may be the *result* of the elections:

$$\begin{array}{ll} (A) & P_A > P_B, \text{ i.e. candidate } A \text{ wins the elections;} \\ (B) & P_A < P_B, \text{ i.e. candidate } B \text{ wins the elections;} \\ (D) & P_A = P_B, \text{ i.e. a draw occurs.} \end{array} \tag{3.2}$$

The definition of equilibrium reduces in this model to the following conditions: for $i = 1, \ldots, 8$ we have

$$p^i_B = 0 \text{ or } (B, x_{\mathbf{p}}) \succsim_i (A, x_{\mathbf{p}})$$

and

$$p^i_A = 0 \text{ or } (A, x_{\mathbf{p}}) \succsim_i (B, x_{\mathbf{p}}).$$

Notice that pairs $(A, x_{\mathbf{p}})$ and $(B, x_{\mathbf{p}})$ denote one of the pairs **1**,…,**6** given in (3.1) (e.g. (A, A) is the pair number **1**, (A, D) is **2**, (B, B) is **6** etc.).

In order to find equilibria we need to check what are decisions of the voters of type i ($i = 1, \ldots, 8$), voting for A or B, in the cases $(A), (B), (D)$, respectively. Consider the case (A). If A wins, then voters of the first four types, voting for A stick to their choice, while those, who voted for B would change their decision, since (A, A) is the most preferred pair for them. It follows that the distribution of decisions of these types of voters in this case is $p^1_A = p^2_A = p^3_A = p^4_A = 1$. Voters of the type fifth and sixth voting for A would change their decision, while those voting for B stick to their choice, since the pair **1** is the least preferred by them. And finally voters of the type 7 and 8 voting for A would not change their decision, while those voting for B should change it, because the pair number **1** is better for them that the pair number **4**. The distribution of decisions of the remaining types of voters will be given by following: $p^5_B = p^6_B = p^7_A = p^8_A = 1$. If we consider the remaining cases in the same way, we get the following proposition:

PROPOSITION 1 *A sequence of distributions* $\mathbf{p}$ *is at equilibrium if and only if the three conditions are satisfied:*
Case (A) *occurs at the distribution* $p^1_A = p^2_A = p^3_A = p^4_A = p^5_B = p^6_B = p^7_A = p^8_A = 1$;
Case (B) *occurs at the distribution* $p^1_A = p^2_A = p^3_B = p^4_B = p^5_B = p^6_B = p^7_B = p^8_B = 1$;
Case (D) *occurs at the distribution* $p^1_A = p^2_A = p^3_A = p^4_A = p^5_B = p^6_B = p^7_B =$ $p^8_B = 1$

After some transformations, using the formulas for P_A and P_B and (3.2), we obtain the following conditions, describing the size of the populations of voters in the electorate:

$$
\begin{aligned}
&(A') \quad q_1 + q_2 + q_3 + q_4 + q_7 + q_8 > q_5 + q_6;\\
&(B') \quad q_3 + q_4 + q_5 + q_6 + q_7 + q_8 > q_1 + q_2;\\
&(D') \quad q_1 + q_2 + q_3 + q_4 = q_5 + q_6 + q_7 + q_8.
\end{aligned}
$$

If the voters of the first and second type (or the voters of the fifth and sixth type) jointly constitute not less than half of the electorate (i. e. $q_1 + q_2 \geq \frac{1}{2}\sum_{i=1}^{8} q_i$ or $q_5 + q_6 \geq \frac{1}{2}\sum_{i=1}^{8} q_i$), then there exist at most two different equilibria; if one of the inequalities is strict, then there exists only one equilibrium. Otherwise, i.e. for $q_1 + q_2 < \frac{1}{2}\sum_{i=1}^{8} q_i$ and $q_5 + q_6 < \frac{1}{2}\sum_{i=1}^{8} q_i$, there may occur one of two or three equilibria (for example, assuming that q_i describe the distribution of the voters of respective types in the electorate, for $q_1 = q_2 = q_4 = q_5 = q_6 = q_7 = \frac{1}{10}$, $q_3 = q_8 = \frac{2}{10}$ all the conditions (A'), (B') and (D') hold, while for $q_1 = \frac{5}{24}$, $q_2 = \frac{8}{24}$, $q_3 = q_6 = q_8 = \frac{2}{24}$, $q_4 = \frac{3}{24}$, $q_5 = q_7 = \frac{1}{24}$ only the condition (A') holds).

Supporters of the candidate A	
Type *I*	Type *II*
Nonconformist/ Pragmatist	(Moderate) opportunist
$1 \succ_I 2 \succ_I 5 \succ_I 6$	$1 \succ_{II} 2 \succ_{II} 5 \succ_{II} 3 \succ_{II} 4$
$1 \succ_I 4 \succ_I 5$	$1 \succ_{II} 6 \succ_{II} 5$
$2 \succ_I 3 \succ_I 6$	
$(4 \succ_I 2)$ or $(3 \succ_I 4)$	
$(3 \succ_I 4)$ or $(5 \succ_I 3)$	

Table 3.2a

Supporters of the candidate B	
Type *III*	Type *IV*
Nonconformist/ Pragmatist	(Moderate) opportunist
$6 \succ_{III} 5 \succ_{III} 2 \succ_{III} 1$	$6 \succ_{IV} 5 \succ_{IV} 2 \succ_{IV} 4 \succ_{IV} 3$
$6 \succ_{III} 3 \succ_{III} 2$	$6 \succ_{IV} 1 \succ_{IV} 2$
$5 \succ_{III} 4 \succ_{III} 1$	
$(3 \succ_{III} 5)$ or $(4 \succ_{III} 3)$	
$(4 \succ_{III} 3)$ or $(2 \succ_{III} 4)$	

Table 3.2b

An analysis of the results presented above leads to the conclusion that the strategic consequences of the behaviour of voters of different types (e.g. type 1 and 2, or type 3 and 4) are identical, which suggest a possibility and usefulness of an aggregation of types. After this operation we obtain four new types of voters, arising from the aggregation of the former types 1 and 2, types 3 and 4, types 5 and 6, and types 7 and 8. Indeed, the behaviour of voters of new aggregate types is the same in each of the possible states of equilibrium. Therefore, we consider a new classification of voters into the types I, II, III and IV. Their preferences are assumed to exclude indifference between different options and fulfill the conditions listed in the tables below.

Although the preferences of voters of each new type are not determined uniquely in this case (hence these are not exactly the types of voters in the meaning described in the previous section), this inaccuracy is not significant, since the behaviour of all voters of each new "type" at any equilibrium is the same. It is easy to check that the conditions for the preferences of the type I hold only for the voters of the previous types 1 and 2. Similarly, type II corresponds to the previous types 3 and 4, type III corresponds to the types 5 and 6 and finally, type IV corresponds to the types 7 and 8. Denote by q_I, q_{II}, q_{III}, q_{IV} the sizes of populations of respective types and by $\mathbf{p} = (p^I, p^{II}, p^{III}, p^{IV})$ the sequence of distributions of decisions of voters. Hence, we have $P_A = Q^{-1} \cdot \sum_{i=I}^{IV} q_i p^i_A$ and $P_B = Q^{-1} \cdot \sum_{i=I}^{IV} q_i p^i_B$, where $Q = \sum_{i=I}^{IV} q_i$. The result of the elections is then given also by (3.2). As before, we have the following proposition concerning equilibria of this case:

PROPOSITION 2 *A sequence of distributions* $\mathbf{p}$ *is at equilibrium if and only if the three conditions hold:*
Case (A) *occurs at the distribution* $p^I_A = p^{II}_A = p^{III}_B = p^{IV}_A = 1$;
Case (B) *occurs at the distribution* $p^I_A = p^{II}_B = p^{III}_B = p^{IV}_B = 1$;
Case (D) *occurs at the distribution* $p^I_A = p^{II}_A = p^{III}_B = p^{IV}_B = 1$.

We obtain the following conditions for the sizes of the populations of voters of different types (in the same way as before):

$$\begin{array}{ll}(A') & q_I + q_{II} + q_{IV} > q_{III};\\ (B') & q_{II} + q_{III} + q_{IV} > q_I;\\ (D') & q_I + q_{II} = q_{III} + q_{IV}.\end{array}$$

The conclusions are that if the supporters of A of the type I or the supporters of B of the type III constitute not less than a half of the whole electorate, then there exist at most two different equilibria; if the voters of the type I or the voters of the type III constitute a majority of the electorate then there exists only one equilibrium. Otherwise, one of two or even three different equilibria may be obtained. The new classification is more economic from the strategic analysis point of view, and it preserves all the results obtained in the previous

4. Classification of equilibria

One can observe that equilibria occurring for conditions (A) or (B) are essentially different from equilibrium occurring at the condition (D) (see 3.2). If an equilibrium distribution is slightly perturbed, because of the voters' incidental mistakes, then in both cases (A) and (B) the outcome of the elections will remain unchanged, if the perturbation is small enough. In the last case (D), an arbitrarily small perturbation of the equilibrium distribution may cause a change in the elections' outcome. We define the notion of a stable sequence of distributions.

Definition. *We say that a sequence of distributions* $\mathbf{p} \in (\Delta_k)^n$ *is stable if there exists* $\varepsilon > 0$ *such that for any sequence of distributions* $\tilde{\mathbf{p}} \in (\Delta_k)^n$ *fulfilling the inequality* $\|\tilde{\mathbf{p}} - \mathbf{p}\| < \varepsilon$, *the elections's outcome remains unchanged (*$\|\cdot\|$ *denotes the Euclidean norm).*

Consider the equilibrium $\mathbf{p} = (p^I, p^{II}, p^{III}, p^{IV})$ occurring at (A); in this case we have: $p_A^I = p_A^{II} = p_B^{III} = p_A^{IV} = 1$. Assume that the distribution of decisions of the voters of type I changes in the following way: $\tilde{p}_A^I = 1 - \varepsilon$, $\tilde{p}_B^I = \varepsilon$. Then we have:

$$\tilde{P}_A = Q^{-1} \cdot (q_I(1-\varepsilon) + q_{II} + q_{IV}),$$
$$\tilde{P}_B = Q^{-1} \cdot (q_{III} + \varepsilon q_I).$$

Since $\mathbf{p}$ is at equilibrium at (A), the following inequality holds:

$$q_I + q_{II} + q_{IV} > q_{III}. \tag{4.1}$$

Should the outcome of the elections remain unchanged, there must be $\tilde{P}_A > \tilde{P}_B$, that is

$$q_I(1-\varepsilon) + q_{II} + q_{IV} > q_{III} + \varepsilon q_I.$$

We transform the last inequality to the following one:

$$q_I + q_{II} + q_{IV} > q_{III} + 2\varepsilon q_I. \tag{4.2}$$

Since the condition (4.1) holds, for $\varepsilon < \frac{1}{2}(q_I)^{-1}(q_I + q_{II} + q_{IV} - q_{III})$ the condition (4.2) also holds. We obtain the same conclusion in case (B). If we deal with the case (D), each perturbation at equilibrium may cause a change of the elections' outcome. For example, assume that $p_A^I = p_A^{II} = p_B^{III} = p_B^{IV} = 1$, which means that $\mathbf{p}$ is at equilibrium in case (D), and consider a new distribution, the same as before, that is $\tilde{p}_A^I = 1 - \varepsilon$, $\tilde{p}_B^I = \varepsilon$. We have

$$q_I + q_{II} = q_{III} + q_{IV}.$$

At the new distribution we have

$$\tilde{P}_A = Q^{-1} \cdot (q_I(1-\varepsilon) + q_{II})$$

Therefore, $\tilde{P}_B > \tilde{P}_A$ (since $\tilde{P}_A < P_A$, $\tilde{P}_B > P_B$ and $P_A = P_B$), that is – the outcome is not a draw in this case. This change does not depend on how small ε is and it occurs for any $\varepsilon > 0$.

Basing on these considerations we conclude that equilibria occurring at (A) and (B) are stable, while the equilibrium at (D) is not stable. This result means that although there exist an equilibrium resulting in a draw, it is not stable and it is rather not possible that it occurs in reality, since an arbitrary small perturbation of voters' decisions can alter the elections' outcome. Hence, in reality we may expect that there occur equilibria of the cases (A) or (B).

5. Concluding remarks

The model described in this paper can be extended in many ways. Some possibilities of extending the model are: to permit the abstention from voting, to enlarge the set of candidates, to take into consideration more types of voters and to replace strict preferences by preference-indifference relations.

An example of some extension is to introduce two additional types of voters, existing in reality, denoted by V and VI (see tables 3.2a,b). Voters of these types are moderately interested in the outcome of the elections. The type V is characterized by the relation

$$\mathbf{1} \sim_V \mathbf{6} \succ_V \mathbf{2} \sim_V \mathbf{5} \succ_V \mathbf{3} \sim_V \mathbf{4},$$

and the type VI – by the relation

$$\mathbf{1} \sim_{VI} \mathbf{2} \sim_{VI} \mathbf{3} \sim_{VI} \mathbf{4} \sim_{VI} \mathbf{5} \sim_{VI} \mathbf{6}.$$

Voters of type V do not support any candidate, they only care about the outcome to be consistent with their choice. Voters of type VI do not care at all about the candidates and the outcome of elections, they are completely indifferent to any aspect of elections. In this case the conditions describing the sizes of populations of respective types of voters for different outcomes are

$$\begin{array}{ll}
(A'') & q_I + q_{II} + q_{IV} + q_V + q_{VI} p_A^{VI} > q_{III} + q_{VI} p_B^{VI}; \\
(B'') & q_{II} + q_{III} + q_{IV} + q_V + q_{VI} p_B^{VI} > q_I + q_{VI} p_B^{VI}; \\
(D'') & q_I + q_{II} + q_V p_A^V + q_{VI} p_A^{VI} = q_{III} + q_{IV} + q_V p_B^V + q_{VI} p_B^{VI}.
\end{array}$$

Since voters of the new types are indifferent to some pairs: individual decision-elections' outcome, they do not undertake the same decisions within respective types (as it was in the case of strict preferences). This is the reason why we do not have here unique conditions for the sizes of populations of respective types of voters, not depending on the distributions of voters' decisions.

Inclusion of abstention in the model must influence the definition of the outcome of elections, e.g. by making the outcome of the elections dependent assume that if the number P_0, denoting the fraction of the voters in the whole electorate who decide to abstain, exceeds a given treshold, then the elections will not be decisive (there will be no winner). The outcome, denoted by D and called a draw, can also describe this situation. It is easy to notice that the results concerning stability of equilibira leading to a draw may be different in this setting. Obviously, if we allow abstention in the model, we must take it into account when defining preferences of the voters.

This model is related to the *modified simple games* (see Wieczorek, 1996, also Ekes, 1999), which are games with continuum of players classified into a finite number of types. A model of elections can be represented as an auxiliary simple game, modified by considering players preference-indifference relations instead of their payoff functions.

There is a possibility in the considered model the *bandwagon effect* or the *underdog effect*. The first effect occurs if voters become more inclined to vote for a given candidate if his standing in preelection polls improves, the second one if, conversely, the voters become more inclined to vote for a candidate whose standing in polls is becoming worse (see Brams, 1976, Myerson, Weber, 1993). Occurrence of these effects depends on the composition of the electorate. For example, the first effect may occur if the population of voters of the type II and IV (opportunists; see tables 3.2a,b) is large. In order to examine the underdog effect we should consider a new type of voters, who do not support any candidate but are more inclined to accept one of them. The analysis of these effects would be possible after some modifications of the model, since taking into account the results of preelection polls needs applying some dynamics.

Acknowledgements

I am grateful to Andrzej Wieczorek for many useful comments. I wish also to thank anonymous referees whose remarks led to many improvements in the paper.

References

BRAMS, S.J. (1976) *Paradoxes in Politics. An Introduction to the Nonobvious in Political Science.* The Free Press, New York.

BRAMS, S.J. (1978) *The Presidential Election Game.* Yale University Press, New Haven and London.

EKES, M. (1999) *Models of Economy with Infinitely Many Consumers and Producers Classified into a Finite Number of Types.* Thesis (in Polish), Warsaw School of Economics.

GAMBARELLI, G. (1999) Maximax Apportionments. *Group Decision and Negotiation*, **8** (6), 441–461.

HOŁUBIEC, J.W. (1999) Prognozowanie wyników wyborów (Forecasting elec-

Przykłady zastosowań., ed. by Honorata Sosnowska, Wydawnictwo Naukowe Scholar, Warszawa, 81–102.

KRAMER, G.H. (1978) Existence of Electoral Equilibrium. In: *Game Theory and Political Science*, ed. by Peter Ordeshook, New York University Press, New York, 375–389.

MCKELVEY, R.D., ORDESHOOK, P.C. and UNGAR, P. (1980) Conditions for Voting Equilibria in Continuous Voter Distributions. *SIAM Journal of Applied Mathematics.* Vol. **39**, 161–168.

MERCIK, J.W. (1999) Modelowanie zachowań parlamentarnych – Sejm II kadencji (Modeling of parliamentary behaviour. The second term of the Diet; in Polish). In: *Grupowe podejmowanie decyzji. Elementy teorii. Przykłady zastosowań.*, ed. by Honorata Sosnowska, Wydawnictwo Naukowe Scholar, Warszawa, 123–138.

MYERSON, R.B. and WEBER, R.J. (1993) A Theory of Voting Equilibria. *American Political Science Review*, vol. **87**, 102–114.

NURMI, H. (1995) On the Difficulty of Making Social Choices. *Theory and Decisions*, **38**, 99–119.

OWEN, G. and SHAPLEY, L.S. (1989) Optimal Location of Candidates in Ideological Space. *International Journal of Game Theory*, vol **18**, 339–356.

SCHUESSLER, A.A. (2000) *A Logic of Expressive Choice.* Princeton University Press, Princeton, New Jersey.

WIECZOREK A. (1996) Elementary Large Games and an Application to Economies with Many Agents, report **805**, Institute of Computer Science, Polish Academy of Sciences, Warszawa.