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Nonlinear boundary control of coupled Burgers' equations

by

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Abstract: This paper is concerned with adaptive stabilization of two coupled viscous Burgers' equations by nonlinear boundary controllers. Under the existence of bounded deterministic disturbances, the adaptive controllers are constructed by the concept of high-gain nonlinear output feedback and the estimation mechanism of the unknown parameters. In the controlled system the global stability and the convergence of the system states to zero will be guaranteed. It is shown that the theory can be generalized to the systems with higher-order nonlinearity.

Keywords: coupled Burgers' equation, global stabilization, adaptive regulation, nonlinear boundary control.

1. Introduction

The Burgers' equation is a simplified fluid flow model which nonetheless exhibits some of the important aspects of turbulence. It is often referred to as an approximation to the one-dimensional Navier–Stokes equations. It is also referred to a model of traffic flow (Farlow, 1982, Haberman, 1977). The Burgers' equation is a natural first step towards developing methods for control of flows. While many recent papers (Burns, Kang, 1991, Byrnes, et al., 1998, Henry, 1981, Ito, Yan 1998, Temam, 1997, Van Ly et al., 1997) have investigated local stabilization and global analysis of attractors, the problem of global asymptotic stabilization has been investigated in (Kobayashi, 2001, Krstic, 1999, 2000a, 2000b, 2000c).

One of the most important applications of feedback is to achieve regulation and servoaction, that is, to obtain a stable closed-loop system that rejects a given class of external disturbances and tracks a given class of reference signals with zero asymptotic error. The advantage of the adaptive control is that good control performance can be automatically achieved even in the presence of varclasses of distributed parameter systems (Kobayashi, 1987, 1988, 1996, 1997, 2000c, Logemann, Martensson, 1992, 1997, Luo et al., 1999, Böhm et al., 1998, Wen, Balas, 1989).

This paper is concerned with adaptive stabilization of two coupled viscous Burgers' equations by nonlinear boundary controllers. Under the existence of bounded deterministic disturbances, the adaptive controllers are constructed by the concept of high-gain nonlinear output feedback and the estimation mechanism of the unknown parameters. In the controlled system the global stability and the convergence of the system states to zero will be guaranteed. It is shown that the theory can be generalized to the systems with higher-order nonlinearity.

2. System description

Consider the viscous Burgers' system

$$\left.\begin{aligned} u_t(x,t) &= \epsilon_1 u_{xx}(x,t) - a_1 u(x,t) u_x(x,t) + \rho[w(x,t) - u(x,t)], \\ &\quad x \in (0,1),\ t > 0 \\ w_t(x,t) &= \epsilon_2 w_{xx}(x,t) - a_2 w(x,t) w_x(x,t) + \rho[u(x,t) - w(x,t)], \\ &\quad x \in (0,1),\ t > 0 \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} &u_x(0,t) + b_1 u(0,t) = -f_1(t) - \theta^T v(t), \quad u_x(1,t) = f_2(t) \\ &w_x(0,t) + b_2 w(0,t) = -g_1(t), \quad w_x(1,t) = g_2(t) \end{aligned}\right\} \tag{2}$$

$$y(t) = [u(0,t), u(1,t), w(0,t), w(1,t)]^T, \tag{3}$$

where ϵ_1, ϵ_2 and ρ are positive constants, a_1, a_2, b_1 and b_2 are constants, $f_1(t)$, $f_2(t)$, $g_1(t)$ and $g_2(t)$ are inputs and $y(t)$ is the output. We assume that the disturbance vector function $v(t)$ bounded and known, but θ is the l-dimensional unknown constant vector. For example, we shall consider $v(t)$ such that

$$v(t) = \begin{bmatrix} 1 \\ |\sin t| \\ \cos 2t \end{bmatrix}.$$

It should be noted that the elements of $v(t)$ are not necessarily assumed to satisfy a linear, time-invariant, finite-dimensional differential equation. We can consider signals such as a periodic rectangular pulse.

The objective of adaptive control design is to construct the control input f_1, f_2, g_1, g_2 such that the closed-loop system will be globally stable when the system parameters $\epsilon_1, \epsilon_2, \rho, a_1, a_2, b_1, b_2$ and θ are unkonwn.

The proof of existence and uniqueness of solutions is nontrivial for nonlinear partial differential equations, especially if the boundary conditions are nonlinear too. The well posedness of the closed-loop system has been considered for the Burger's equation (Liu, Krstic, 2000a, 2000b, 2000c). In the paper we shall

3. Regulator design (non-adaptive case)

In this section we consider non-adaptive regulator design for the system (1)-(3) in the case where $\epsilon_1, \epsilon_2, \rho, a_1, a_2, b_1, b_2$ and θ are known.

We start from the Lyapunov function

$$V(t) = \frac{1}{2}\int_0^1 [u^2(x,t) + w^2(x,t)]dx. \tag{4}$$

The time derivative of $V(t)$ along the solution of system (1)-(3) is

$$\begin{aligned}
\dot{V}(t) &= \int_0^1 (uu_t + ww_t)dx \\
&= \int_0^1 [u(\epsilon_1 u_{xx} - a_1 uu_x) + \rho u(w-u) + w(\epsilon_2 w_{xx} - a_2 ww_x) + \rho w(u-w)]dx \\
&= \epsilon_1 u(1,t)u_x(1,t) - \epsilon_1 u(0,t)u_x(0,t) - \epsilon_1 \int_0^1 u_x^2 dx - \frac{a_1}{3}\int_0^1 (u^3)_x dx \\
&+ \epsilon_2 w(1,t)w_x(1,t) - \epsilon_2 w(0,t)w_x(0,t) - \epsilon_2 \int_0^1 w_x^2 dx - \frac{a_2}{3}\int_0^1 (w^3)_x dx \\
&- \rho\int_0^1 u^2 - \rho\int_0^1 w^2 dx + 2\rho\int_0^1 uw dx \\
&= \epsilon_1 f_2(t)u(1,t) + \epsilon_1 b_1 u^2(0,t) + \epsilon_1 f_1(t)u(0,t) + \epsilon_1 \theta^T v(t)u(0,t) \\
&- \epsilon_1 \int_0^1 u_x^2 dx - \frac{a_1}{3}u^3(1,t) + \frac{a_1}{3}u^3(0,t) - \rho\int_0^1 u^2 + 2\rho\int_0^1 uw dx \\
&+ \epsilon_2 g_2(t)w(1,t) + \epsilon_2 b_2 w^2(0,t) + \epsilon_2 g_1(t)w(0,t) \\
&- \epsilon_2 \int_0^1 w_x^2 dx - \frac{a_2}{3}w^3(1,t) + \frac{a_2}{3}w^3(0,t) - \rho\int_0^1 w^2.
\end{aligned}$$

Here, because

$$u(x) = u(0) + \int_0^x u_x dx,$$

it holds that

$$u^2(x) \le 2u^2(0) + 2\left(\int_0^x u_x dx\right)^2 \le 2u^2(0) + 2\int_0^1 u_x^2 dx.$$

Thus

$$\int_0^1 u^2 dx \le 2u^2(0) + 2\int_0^1 u_x^2 dx. \tag{5}$$

Using this relation and

we obtain

$$\dot{V}(t) \leq -\frac{\epsilon_1}{2}\int_0^1 u^2 dx + \epsilon_1 u(1,t)\left[f_2(t) - \frac{a_1}{3\epsilon_1}u^2(1,t)\right]$$
$$+\epsilon_1 u(0,t)\left[f_1(t) + \theta^T v(t) + (b_1+1)u(0,t) + \frac{a_1}{3\epsilon_1}u^2(0,t)\right]$$
$$-\frac{\epsilon_2}{2}\int_0^1 w^2 dx + \epsilon_2 w(1,t)\left[g_2(t) - \frac{a_2}{3\epsilon_2}w^2(1,t)\right]$$
$$+\epsilon_2 w(0,t)\left[g_1(t) + (b_2+1)w(0,t) + \frac{a_2}{3\epsilon_2}w^2(0,t)\right]. \tag{6}$$

If we apply a control law

$$\left.\begin{array}{l} f_1(t) = -\kappa_1 u(0,t) - \theta^T v(t) - (b_1+1)u(0,t) - \frac{a_1}{3\epsilon_1}u^2(0,t),\ \kappa_1 \geq 0, \\ f_2(t) = -\kappa_2 u(1,t) + \frac{a_1}{3\epsilon_1}u^2(1,t),\quad \kappa_2 \geq 0, \\ g_1(t) = -\kappa_3 w(0,t) - (b_2+1)w(0,t) - \frac{a_2}{3\epsilon_2}w^2(0,t),\quad \kappa_3 \geq 0, \\ g_2(t) = -\kappa_4 w(1,t) + \frac{a_2}{3\epsilon_2}w^2(1,t),\quad \kappa_4 \geq 0, \end{array}\right\} \tag{7}$$

then we obtain

$$\dot{V}(t) \leq -\frac{\epsilon_1}{2}\int_0^1 u^2 dx - \frac{\epsilon_2}{2}\int_0^1 w^2 dx = -min\{\epsilon_1, \epsilon_2\}V(t). \tag{8}$$

This implies that $V(t)$ will be bounded and exponentially converge to zero. The equilibrium $u(x) \equiv 0$ and $w(x) \equiv 0$ is globally exponentially stable in $L^2(0,1)$.

The following theorem holds:

THEOREM 1 *The controller (7) globally exponentially stabilizes the system (1)–(3) in L^2 sense.*

In the case where the boundary conditions are

$$\left.\begin{array}{ll} u_x(0,t) + b_1 u(0,t) = -f_1(t) - {\theta_1}^T v_1(t), & u_x(1,t) = f_2(t) \\ w_x(0,t) + b_2 w(0,t) = -g_1(t) - {\theta_1}^T v_2(t), & w_x(1,t) = g_2(t) \end{array}\right\} \tag{9}$$

we can exponentially stabilize the system by a similar control law.

However, it follows from the relation (6) that the system with boundary conditions

$$\left.\begin{array}{ll} u_x(0,t) + b_1 u(0,t) = -f_1(t) - {\theta_1}^T v_1(t), & u(1,t) = 0 \\ w(0,t) = 0, \quad w(1,t) = 0 & \end{array}\right\} \tag{10}$$

4. Adaptive regulator design

In this section we construct an adaptive regulator for the system (1)-(3) in the case where $\epsilon_1, \epsilon_2, \rho, a_1, a_2, b_1, b_2$ and θ are unknown.

Firstly, in place of the controller (7), we shall consider an adaptive controller

$$\left.\begin{aligned}
f_1(t) &= -k_1(t)u(0,t) - k_2(t)u^3(0,t) - \widehat{\theta}(t)^T v(t) - \alpha_1(t)u^2(0,t),\\
f_2(t) &= -k_3(t)[u(1,t) + u^3(1,t)] - \alpha_2(t)u^2(1,t),\\
g_1(t) &= -k_4(t)w(0,t) - k_5(t)w^3(0,t) - \alpha_3(t)w^2(0,t),\\
g_2(t) &= -k_6(t)[w(1,t) + w^3(1,t)] - \alpha_4(t)w^2(1,t),
\end{aligned}\right\} \quad (11)$$

where

$$\left.\begin{aligned}
&\dot{k}_1(t) = r_1 u^2(0,t),\ k_1(0) > 0,\ r_1 > 0,\\
&\dot{k}_2(t) = r_2 u^4(0,t),\ k_2(0) > 0,\ r_2 > 0,\\
&\dot{k}_3(t) = r_3[u^2(1,t) + u^4(1,t)],\ k_3(0) > 0,\ r_3 > 0,\\
&\dot{k}_4(t) = r_4 w^2(0,t),\ k_4(0) > 0,\ r_4 > 0,\\
&\dot{k}_5(t) = r_5 w^4(0,t),\ k_5(0) > 0,\ r_5 > 0,\\
&\dot{k}_6(t) = r_6[w^2(1,t) + w^4(1,t)],\ k_6(0) > 0,\ r_6 > 0,\\
&\dot{\widehat{\theta}}(t) = Pu(0,t)v(t),\quad P : \text{positive definite matrix}\\
&\dot{\alpha}_1(t) = q_1 u^3(0,t),\ q_1 > 0,\\
&\dot{\alpha}_2(t) = q_2 u^3(1,t),\ q_2 > 0,\\
&\dot{\alpha}_3(t) = q_3 w^3(0,t),\ q_3 > 0,\\
&\dot{\alpha}_4(t) = q_4 w^3(1,t),\ q_4 > 0.
\end{aligned}\right\} \quad (12)$$

Then, from (6), the time derivative of $V(t)$ becomes

$$\begin{aligned}
\dot{V}(t) \le& -\frac{\epsilon_1}{2}\int_0^1 u^2 dx - \epsilon_1[k_1(t) - (b_1+1)]u^2(0,t) - \epsilon_1 k_2(t)u^4(0,t)\\
&- \epsilon_1[\widehat{\theta}(t) - \theta]^T v(t)u(0,t) - \epsilon_1\left[\alpha_1(t) - \frac{a_1}{3\epsilon_1}\right]u^3(0,t)\\
&- \epsilon_1 k_3(t)[u^2(1,t) + u^4(1,t)] - \epsilon_1\left[\alpha_2(t) + \frac{a_1}{3\epsilon_1}\right]u^3(1,t).\\
&- \frac{\epsilon_2}{2}\int_0^1 w^2 dx - \epsilon_2[k_4(t) - (b_2+1)]w^2(0,t) - \epsilon_2\ k_5(t)w^4(0,t)\\
&- \epsilon_2\left[\alpha_3(t) - \frac{a_2}{3\epsilon_2}\right]w^3(0,t)\\
&- \epsilon_2 k_6(t)[w^2(1,t) + w^4(1,t)] - \epsilon_2\left[\alpha_4(t) + \frac{a_2}{3\epsilon_2}\right]w^3(1,t).
\end{aligned} \quad (13)$$

Here we introduce another non-negative function $E(t)$ by

$$E(t) = V(t) + \ldots$$

$$+\frac{\epsilon_2}{2r_4}[k_4(t)-(b_2+1)]^2+\frac{\epsilon_2}{2r_5}k_5^2(t)+\frac{\epsilon_2}{2r_6}k_6^2(t)$$
$$+\frac{\epsilon_1}{2}[\widehat{\theta}(t)-\theta]^T P^{-1}[\widehat{\theta}(t)-\theta]+\frac{\epsilon_1}{2q_1}\left[\alpha_1(t)-\frac{a_1}{3\epsilon_1}\right]^2+\frac{\epsilon_1}{2q_2}\left[\alpha_2(t)+\frac{a_1}{3\epsilon_1}\right]^2$$
$$+\frac{\epsilon_2}{2q_3}\left[\alpha_3(t)-\frac{a_2}{3\epsilon_2}\right]^2+\frac{\epsilon_2}{2q_4}\left[\alpha_4(t)+\frac{a_2}{3\epsilon_2}\right]^2. \tag{14}$$

Using (12), (13), we can estimate the time derivative of $E(t)$

$$\dot{E}(t)\leq -min\{\epsilon_1,\epsilon_2\}V(t). \tag{15}$$

It follows from this that $E(t)\leq E(0)$, and then $k_i(t)<\infty, i=1,2,\cdots,6$, $\|\widehat{\theta}(t)\|<\infty$, $|\alpha_j(t)|<\infty$, $j=1,2,3,4$ for any $t>0$. Thus, by (12) we obtain

$$\left.\begin{array}{l} u(0,t)\in L^2(0,\infty)\cap L^4(0,\infty),\ u(1,t)\in L^2(0,\infty)\cap L^4(0,\infty),\\ w(0,t)\in L^2(0,\infty)\cap L^4(0,\infty),\ w(1,t)\in L^2(0,\infty)\cap L^4(0,\infty). \end{array}\right\} \tag{16}$$

Next we shall show the convergence of $V(t)$. Put $\epsilon_{min}=min\{\epsilon_1,\epsilon_2\}$. From (13), using the Gronwall lemma (Curtain, Zwart, 1995), we have

$$V(t)\leq exp(-\epsilon_{min}t)V(0)$$
$$-\epsilon_1\int_0^t exp[-\epsilon_{min}(t-\tau)]\left\{[k_1(\tau)-(b_1+1)]u^2(0,\tau)\right.$$
$$\left.+k_2(\tau)u^4(0,\tau)+[\widehat{\theta}(\tau)-\theta]^T v(\tau)u(0,\tau)+\left[\alpha_1(\tau)-\frac{a_1}{3\epsilon_1}\right]u^3(0,\tau)\right\}d\tau$$
$$-\epsilon_1\int_0^t exp[-\epsilon_{min}(t-\tau)]\left\{k_3(\tau)[u^2(1,\tau)+u^4(1,\tau)]\right.$$
$$\left.+\left[\alpha_2(\tau)+\frac{a_1}{3\epsilon_1}\right]u^3(1,\tau)\right\}d\tau$$
$$-\epsilon_2\int_0^t exp[-\epsilon_{min}(t-\tau)]\left\{[k_4(\tau)-(b_2+1)]w^2(0,\tau)\right.$$
$$\left.+k_5(\tau)w^4(0,\tau)+\left[\alpha_3(\tau)-\frac{a_2}{3\epsilon_2}\right]w^3(0,\tau)\right\}d\tau$$
$$-\epsilon_2\int_0^t exp[-\epsilon_{min}(t-\tau)]\left\{k_6(\tau)[w^2(1,\tau)+w^4(1,\tau)]\right.$$
$$\left.+\left[\alpha_4(\tau)+\frac{a_2}{3\epsilon_2}\right]w^3(1,\tau)\right\}d\tau$$
$$\leq exp(-\epsilon_{min}t)V(0)+\epsilon_1 C_{1,max}\int_0^t exp[-\epsilon_{min}(t-\tau)]\left\{u^2(0,\tau)+u^4(0,\tau)\right.$$
$$\left.+|u(0,\tau)|+|u^3(0,\tau)|+u^2(1,\tau)+u^4(1,\tau)+|u^3(1,\tau)|\right\}d\tau,$$
$$+exp(-\epsilon_{min}t)V(0)+\epsilon_2 C_{2,max}\int_0^t exp[-\epsilon_{min}(t-\tau)]\left\{w^2(0,\tau)+w^4(0,\tau)\right.$$

where we take the supremum on $t \geq 0$ and

$$C_{1,max} = max\left\{sup|k_1(t)-(b_1+1)|, \sup|k_2(t)|, \sup|[\widehat{\theta}(t)-\theta]^T v(t)|\right.,$$
$$\left. sup\left|\alpha_1(t)-\frac{a_1}{3\epsilon_1}\right|, \sup|k_3(t)|, \sup|\alpha_2(t)+\frac{a_1}{3\epsilon_1}|\right\},$$
$$C_{2,max} = max\left\{sup|k_4(t)-(b_2+1)|, \sup|k_5(t)|, ,\right.$$
$$\left. \sup\left|\alpha_3(t)-\frac{a_2}{3\epsilon_2}\right|, \sup|k_6(t)|, \sup\left|\alpha_4(t)+\frac{a_2}{3\epsilon_2}\right|\right\}.$$

We shall estimate each integral terms. First

$$\begin{aligned}
&\int_0^t exp[-\epsilon_{min}(t-\tau)]u^2(\tau)d\tau \\
&\leq \int_0^{\frac{t}{2}} exp[-\epsilon_{min}(t-\tau)]u^2(\tau)d\tau + \int_{\frac{t}{2}}^t exp[-\epsilon_{min}(t-\tau)]u^2(\tau)d\tau \\
&\leq \int_{\frac{t}{2}}^t exp(-\epsilon_{min}\tau)u^2(t-\tau)d\tau + \int_{\frac{t}{2}}^t exp[-\epsilon_{min}(t-\tau)]u^2(\tau)d\tau \\
&\leq exp\left(-\frac{\epsilon_{min}}{2}t\right)\int_{\frac{t}{2}}^t u^2(t-\tau)d\tau \\
&+ max_{\frac{t}{2}\leq\tau\leq t}\left(exp[-\epsilon_{min}(t-\tau)]\right)\int_{\frac{t}{2}}^t u^2(\tau)d\tau \\
&\leq exp\left(-\frac{\epsilon_{min}}{2}t\right)\int_0^\infty u^2(\tau)d\tau + \int_{\frac{t}{2}}^\infty u^2(\tau)d\tau.
\end{aligned} \tag{18}$$

When $u(t) \in L^2(0,\infty)$,

$$\int_0^t exp[-\epsilon_{min}(t-\tau)]u^2(\tau)d\tau \to 0 \text{ as } t\to\infty. \tag{19}$$

In a similar way for $u(t) \in L^4(0,\infty)$, we obtain

$$\int_0^t exp[-\epsilon_{min}(t-\tau)]u^4(\tau)d\tau \to 0 \text{ as } t\to\infty. \tag{20}$$

Next, using the Cauchy-Schwartz inequality we can have the following relations

$$\begin{aligned}
&\int_0^t exp[-\epsilon_{min}(t-\tau)]|u(\tau)|d\tau \\
&\leq \left[\int_0^t exp[-\epsilon_{min}(t-\tau)]d\tau\right]^{\frac{1}{2}}\left[\int_0^t exp[-\epsilon_{min}(t-\tau)]u^2(\tau)d\tau\right]^{\frac{1}{2}} \\
&\leq \left(\frac{1}{\quad}\right)^{\frac{1}{2}}\left[\int^t exp[-\epsilon_{min}(t-\tau)]u^2(\tau)d\tau\right]^{\frac{1}{2}}
\end{aligned} \tag{21}$$

and

$$\int_0^t exp[-\epsilon_{min}(t-\tau)]|u^3(\tau)|d\tau$$
$$\leq \left[\int_0^t exp[-\epsilon_{min}(t-\tau)]u^2(\tau)d\tau\right]^{\frac{1}{2}} \left[\int_0^t exp[-\epsilon_{min}(t-\tau)]u^4(\tau)d\tau\right]^{\frac{1}{2}}. \quad (22)$$

For $u(t) \in L^2(0,\infty) \cap L^4(0,\infty)$ it holds that

$$\int_0^t exp[-\epsilon_{min}(t-\tau)]|u(\tau)|d\tau \to 0 \text{ as } t \to \infty, \quad (23)$$
$$\int_0^t exp[-\epsilon_{min}(t-\tau)]|u^3(\tau)|d\tau \to 0 \text{ as } t \to \infty. \quad (24)$$

From (17), (19), (20), (23) and (24) we can show that

$$\int_0^1 [u^2(x,t) + w^2(x,t)]dx \to 0 \text{ as } t \to \infty. \quad (25)$$

The following theorem holds:

THEOREM 2 *If the adaptive control law (11), (12) is applied to the system (1)–(3), then the resulting closed-loop system will be globally stable, the equilibrium $u(x) \equiv 0$ and $w(x) \equiv 0$ will be regulated to zero in $L^2(0,1)$ and the signals $k_i(t)$ $(i = 1,2,\cdots,6)$, $\widehat{\theta}(t)$, $\alpha_j(t)$ $(j = 1,2,3,4)$ will be bounded for any $t \geq 0$.*

REMARK 1 *In order to globally stabilize the system (1)–(3), we can also apply the adaptive control law such that $\alpha(t) = \alpha_1(t) = -\alpha_2(t)$ and*

$$\dot{\alpha}(t) = q[u^3(0,t) - u^3(1,t)], \; q > 0$$

in the control law (11), (12).

5. System with mixed boundary conditions

In this section we shall consider the Burgers' system (1) with the following boundary conditions

$$\left.\begin{array}{l} u_x(0,t) + b_1 u(1,t) = -f_1(t) - \theta^T v(t), \quad u_x(1,t) = f_2(t) \\ w_x(0,t) + b_2 w(1,t) = -g_1(t), \quad w_x(1,t) = g_2(t) \end{array}\right\} \quad (26)$$

The time derivative of $V(t) = \frac{1}{2}\int_0^1 (u^2 + w^2)dx$ along the solution of system (1), (26) is

$$\dot{V}(t) = \int_0^1 (uu_t + ww_t)dx$$

$$= \int_0^1 [u(\epsilon_1 u_{xx} - a_1 u u_x) + \rho u(w-u) + w(\epsilon_2 w_{xx} - a_2 w w_x) + \rho w(u-w)]dx$$
$$= \epsilon_1 u(1,t)u_x(1,t) - \epsilon_1 u(0,t)u_x(0,t) - \epsilon_1 \int_0^1 u_x^2 dx - \frac{a_1}{3}\int_0^1 (u^3)_x dx$$
$$+ \epsilon_2 w(1,t)w_x(1,t) - \epsilon_2 w(0,t)w_x(0,t) - \epsilon_2 \int_0^1 w_x^2 dx - \frac{a_2}{3}\int_0^1 (w^3)_x dx$$
$$- \rho \int_0^1 u^2 - \rho \int_0^1 w^2 dx + 2\rho \int_0^1 uw dx$$
$$= \epsilon_1 f_2(t)u(1,t) + \epsilon_1 b_1 u(0,t)u(1,t) + \epsilon_1 f_1(t)u(0,t) + \epsilon_1 \theta^T v(t)u(0,t)$$
$$- \epsilon_1 \int_0^1 u_x^2 dx - \frac{a_1}{3}u^3(1,t) + \frac{a_1}{3}u^3(0,t) - \rho\int_0^1 u^2 + 2\rho \int_0^1 uw dx$$
$$+ \epsilon_2 g_2(t)w(1,t) + \epsilon_2 b_2 w(0,t)w(1,t) + \epsilon_2 g_1(t)w(0,t)$$
$$- \epsilon_2 \int_0^1 w_x^2 dx - \frac{a_2}{3}w^3(1,t) + \frac{a_2}{3}w^3(0,t) - \rho \int_0^1 w^2. \quad (27)$$

Here, since

$$u(1) = u(0) + \int_0^1 u_x dx \le u(0) + \sqrt{\int_0^1 u_x^2 dx},$$
$$u(0)u(1) \le u^2(0) + |u(0)|\sqrt{\int_0^1 u_x^2 dx}$$
$$\le \left(1 + \frac{1}{2\nu^2}\right)u^2(0) + \frac{\nu^2}{2}\int_0^1 u_x^2 dx \quad (28)$$

for any $\nu > 0$.

On the other hand, using the relation

$$u^2(x) = \left[u(0) + \int_0^x u_x dx\right]^2 \le u^2(0) + 2|u(0)|\sqrt{\int_0^1 u_x^2 dx} + \int_0^1 u_x^2 dx$$

we have for $\delta > 0$

$$\int_0^1 u^2 dx \le (1+\delta^2)u^2(0) + \left(1 + \frac{1}{\delta^2}\right)\int_0^1 u_x^2 dx,$$

which implies that

$$\int_0^1 u_x^2 dx \ge \frac{\delta^2}{\delta^2+1}\int_0^1 u^2 dx - \delta^2 u^2(0). \quad (29)$$

From (28)

$$\epsilon_1 b_1 u(0,t)u(1,t) - \epsilon_1 \int_0^1 u_x^2 dx$$

$$\leq \epsilon_1 \left(1 + \frac{1}{2\nu^2}\right) |b_1| u^2(0,t) + \frac{\epsilon_1 |b_1| \nu^2}{2} \int_0^1 u_x^2 dx - \epsilon_1 \int_0^1 u_x^2 dx$$
$$\leq -\epsilon_1 \left(1 - \frac{|b_1|\nu^2}{2}\right) \int_0^1 u_x^2 dx + \epsilon_1 \left(1 + \frac{1}{2\nu^2}\right) |b_1| u^2(0,t). \tag{30}$$

Thus, using (29) and (30) in (27) we obtain

$$\begin{aligned}
\dot{V}(t) \leq & -\frac{\epsilon_1 \delta^2}{\delta^2+1}\left(1 - \frac{|b_1|\nu^2}{2}\right)\int_0^1 u^2 dx + \epsilon_1 u(1,t)\left[f_2(t) - \frac{a_1}{3\epsilon_1} u^2(1,t)\right] \\
& + \epsilon_1 u(0,t)\left[f_1(t) + \theta^T v(t) + \left(1 + \frac{1}{2\nu^2}\right)|b_1| u(0,t)\right. \\
& \left. + \delta^2 \left(1 - \frac{|b_1|\nu^2}{2}\right) u(0,t) + \frac{a_1}{3\epsilon_1} u^2(0,t)\right] \\
& - \frac{\epsilon_2 \delta^2}{\delta^2+1}\left(1 - \frac{|b_2|\mu^2}{2}\right)\int_0^1 w^2 dx + \epsilon_2 w(1,t)\left[g_2(t) - \frac{a_2}{3\epsilon_2} w^2(1,t)\right] \\
& + \epsilon_2 w(0,t)\left[g_1(t) + \left(1 + \frac{1}{2\mu^2}\right)|b_2| w(0,t)\right. \\
& \left. + \delta^2 \left(1 - \frac{|b_2|\mu^2}{2}\right) w(0,t) + \frac{a_2}{3\epsilon_2} w^2(0,t)\right].
\end{aligned} \tag{31}$$

Since there exist $\nu > 0$ and $\mu > 0$ such that $1 - |b_1|\nu^2/2 > 0$, $1 - |b_2|\mu^2/2 > 0$, the adaptive control law (11), (12) can globally stabilize the system (1) with the boundary condition (26).

6. The system with higher-order nonlinearity

In this section we shall show that we can generalize the theory to the following system with higher-order nonlinear terms (Farlow, 1982, Haberman, 1977) for positive integers m and n

$$\left.\begin{aligned}
& u_t(x,t) = \epsilon_1 u_{xx}(x,t) - a_1 u^m(x,t) u_x(x,t) + \rho[w(x,t) - u(x,t)], \\
& w_t(x,t) = \epsilon_2 w_{xx}(x,t) - a_2 w^n(x,t) w_x(x,t) + \rho[u(x,t) - w(x,t)], \\
& x \in (0,1),\ t > 0
\end{aligned}\right\} \tag{32}$$

$$\left.\begin{aligned}
& u_x(0,t) + b_1 u(0,t) = -f_1(t) - \theta^T v(t), \quad u_x(1,t) = f_2(t) \\
& w_x(0,t) + b_2 w(0,t) = -g_1(t), \quad w_x(1,t) = g_2(t).
\end{aligned}\right\} \tag{33}$$

For this system the time derivative of $V(t) = \frac{1}{2}\int_0^1 (u^2 + w^2) dx$ becomes

$$\dot{V}(t) \leq -\frac{\epsilon_1}{2}\int_0^1 u^2 dx + \epsilon_1 u(1,t)\left[f_2(t) - \frac{a_1}{\epsilon_1(m+2)} u^{m+1}(1,t)\right]$$
$$+ \epsilon_1 u(0,t)\left[f_1(t) + \theta^T v(t) + (b_1 + 1)u(0,t) + \frac{a_1}{\quad} u^{m+1}(0,t)\right]$$

$$-\frac{\epsilon_2}{2}\int_0^1 w^2 dx + \epsilon_2 w(1,t)\left[g_2(t) - \frac{a_2}{\epsilon_2(n+2)}w^{n+1}(1,t)\right]$$
$$+\epsilon_2 w(0,t)\left[g_1(t) + (b_2+1)w(0,t) + \frac{a_2}{\epsilon_2(n+2)}w^{n+1}(0,t)\right]. \tag{34}$$

We apply the following adaptive control law to the system

$$\left.\begin{aligned}
f_1(t) &= -k_1(t)u(0,t) - k_2(t)u^{2m+1}(0,t) - \widehat{\theta}(t)^T v(t)\\
&\quad - \alpha_1(t)u^{m+1}(0,t),\\
f_2(t) &= -k_3(t)[u(1,t) + u^{2m+1}(1,t)] - \alpha_2(t)u^{m+1}(1,t),\\
g_1(t) &= -k_4(t)w(0,t) - k_5(t)w^{2n+1}(0,t) - \alpha_3(t)w^{n+1}(0,t),\\
g_2(t) &= -k_6(t)[w(1,t) + w^{2n+1}(1,t)] - \alpha_4(t)w^{n+1}(1,t),
\end{aligned}\right\} \tag{35}$$

where

$$\left.\begin{aligned}
&\dot{k}_1(t) = r_1 u^2(0,t),\ k_1(0) > 0,\ r_1 > 0,\\
&\dot{k}_2(t) = r_2 u^{2(m+1)}(0,t),\ k_2(0) > 0,\ r_2 > 0,\\
&\dot{k}_3(t) = r_3[u^2(1,t) + u^{2(m+1)}(1,t)],\ k_3(0) > 0,\ r_3 > 0,\\
&\dot{k}_4(t) = r_4 w^2(0,t),\ k_4(0) > 0,\ r_4 > 0,\\
&\dot{k}_5(t) = r_5 w^{2(n+1)}(0,t),\ k_5(0) > 0,\ r_5 > 0,\\
&\dot{k}_6(t) = r_6[w^2(1,t) + w^{2(n+1)}(1,t)],\ k_6(0) > 0,\ r_6 > 0,\\
&\dot{\widehat{\theta}}(t) = Pu(0,t)v(t),\quad P : \text{positive definite matrix}\\
&\dot{\alpha}_1(t) = q_1 u^{m+2}(0,t),\ q_1 > 0,\\
&\dot{\alpha}_2(t) = q_2 u^{m+2}(1,t),\ q_2 > 0,\\
&\dot{\alpha}_3(t) = q_3 w^{n+2}(0,t),\ q_3 > 0,\\
&\dot{\alpha}_4(t) = q_4 w^{n+2}(1,t),\ q_4 > 0.
\end{aligned}\right\} \tag{36}$$

Then the time derivative of $V(t)$ can be estimated by

$$\dot{V}(t) \le -\frac{\epsilon_1}{2}\int_0^1 u^2 dx - \epsilon_1[k_1(t) - (b_1+1)]u^2(0,t) - \epsilon_1 k_2(t)u^{2(m+1)}(0,t)$$
$$-\epsilon_1[\widehat{\theta}(t) - \theta]^T v(t)u(0,t) - \epsilon_1\left[\alpha_1(t) - \frac{a_1}{\epsilon_1(m+2)}\right]u^{m+2}(0,t)$$
$$-\epsilon_1 k_3(t)[u^2(1,t) + u^{2(m+1)}(1,t)] - \epsilon_1\left[\alpha_2(t) + \frac{a_1}{\epsilon_1(m+2)}\right]u^{m+2}(1,t).$$
$$-\frac{\epsilon_2}{2}\int_0^1 w^2 dx - \epsilon_2[k_4(t) - (b_2+1)]w^2(0,t) - \epsilon_2\ k_5(t)w^{2(n+1)}(0,t)$$
$$-\epsilon_2\left[\alpha_3(t) - \frac{a_2}{\epsilon_2(n+2)}\right]w^{(n+2)}(0,t)$$
$$-\epsilon_2 k_6(t)[w^2(1,t)+w^{2(n+1)}(1,t)] - \epsilon_2\left[\alpha_4(t)+\frac{a_2}{\epsilon_2(n+2)}\right]w^{n+2}(1,t). \tag{37}$$

Here we introduce another non-negative function $E(t)$ by

$$E(t) = V(t) + \frac{\epsilon_1}{\cdots}[k_1(t) - (b_1+1)]^2 + \frac{\epsilon_1}{\cdots}k_2^2(t) + \frac{\epsilon_1}{\cdots}k_3^2(t)\ \cdots$$

$$
\begin{aligned}
&+\frac{\epsilon_2}{2r_4}[k_4(t)-(b_2+1)]^2+\frac{\epsilon_2}{2r_5}k_5^2(t)+\frac{\epsilon_2}{2r_6}k_6^2(t)\\
&+\frac{\epsilon_1}{2}[\widehat{\theta}(t)-\theta]^T P^{-1}[\widehat{\theta}(t)-\theta]+\frac{\epsilon_1}{2q_1}\left[\alpha_1(t)-\frac{a_1}{\epsilon_1(m+2)}\right]^2\\
&+\frac{\epsilon_1}{2q_2}\left[\alpha_2(t)+\frac{a_1}{\epsilon_1(m+2)}\right]^2\\
&+\frac{\epsilon_2}{2q_3}\left[\alpha_3(t)-\frac{a_2}{\epsilon_2(n+2)}\right]^2+\frac{\epsilon_2}{2q_4}\left[\alpha_4(t)+\frac{a_2}{\epsilon_2(n+2)}\right]^2.
\end{aligned}
\tag{38}
$$

Using (36), (37), we can estimate the time derivative of $E(t)$

$$\dot{E}(t)\le -min\{\epsilon_1,\epsilon_2\}V(t). \tag{39}$$

It follows from the above that $E(t)\le E(0)$, and then $k_i(t)<\infty, i=1,2,\cdots,6$, $\|\widehat{\theta}(t)\|<\infty$, $|\alpha_j(t)|<\infty$, $j=1,2,3,4$ for any $t>0$. Thus, by (36) we obtain

$$
\left.
\begin{aligned}
&u(0,t)\in L^2(0,\infty)\cap L^{2(m+1)}(0,\infty),\\
&u(1,t)\in L^2(0,\infty)\cap L^{2(m+1)}(0,\infty),\\
&w(0,t)\in L^2(0,\infty)\cap L^{2(n+1)}(0,\infty),\\
&w(1,t)\in L^2(0,\infty)\cap L^{2(n+1)}(0,\infty).
\end{aligned}
\right\}
\tag{40}
$$

For $u(t)\in L^2(0,\infty)\cap L^{2(m+1)}(0,\infty)$ we have

$$\int_0^t exp[-\epsilon_{min}(t-\tau)]u^2(\tau)d\tau\to 0 \ \text{ as } \ t\to\infty, \tag{41}$$

$$\int_0^t exp[-\epsilon_{min}(t-\tau)]u^{2(m+1)}(\tau)d\tau\to 0 \ \text{ as } \ t\to\infty. \tag{42}$$

Moreover, from the Cauchy–Schwartz inequality we can have the following relations

$$
\begin{aligned}
&\int_0^t exp[-\epsilon_{min}(t-\tau)]|u(\tau)|d\tau\\
&\le\left(\frac{1}{\epsilon_{min}}\right)^{\frac{1}{2}}\left[\int_0^t exp[-\epsilon_{min}(t-\tau)]u^2(\tau)d\tau\right]^{\frac{1}{2}}
\end{aligned}
\tag{43}
$$

and

$$
\begin{aligned}
&\int_0^t exp[-\epsilon_{min}(t-\tau)]|u^{m+2}(\tau)|d\tau\\
&\le\left[\int_0^t exp[-\epsilon_{min}(t-\tau)]u^2(\tau)d\tau\right]^{\frac{1}{2}}\\
&\times\left[\int_0^t exp[-\epsilon_{min}(t-\tau)]u^{2(m+1)}(\tau)d\tau\right]^{\frac{1}{2}}
\end{aligned}
\tag{44}
$$

Thus for $u(t) \in L^2(0,\infty) \cap L^{2(m+1)}(0,\infty)$ it also holds that

$$\int_0^t exp[-\epsilon_{min}(t-\tau)]|u(\tau)|d\tau \to 0 \ \text{ as } \ t \to \infty. \tag{45}$$

$$\int_0^t exp[-\epsilon_{min}(t-\tau)]|u^{m+2}(\tau)|d\tau \to 0 \ \text{ as } \ t \to \infty. \tag{46}$$

From (37), (40), (41), (42), (45) and (46) we can also obtain that

$$\int_0^1 [u^2(x,t) + w^2(x,t)]dx \to 0 \ \text{ as } \ t \to \infty. \tag{47}$$

The adaptive controller (35), (36) can globally stabilize the system (32), (33).

7. Conclusion

We have investigated the adaptive stabilization of two coupled viscous Burgers' equations by nonlinear boundary controllers. Under the existence of bounded deterministic disturbances, the adaptive controller is constructed by the concept of high-gain nonlinear output feedback and the estimation mechanism of the unknown parameters. In the controlled system the global stability and the convergence of the system states to zero is guaranteed. We have also shown that the theory can be generalized to the systems with higher-order nonlinearity. It should be noted that any finite number of coupled Burgers' equations can be handled with the method of the paper.

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