Control and Cybernetics

vol. **32** (2003) No. 2

Obstacle control problem and the unilateral eigenvalue problem of an elastic pseudoplate

by

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Abstract: This paper concerns an obstacle control problem of an elastic pseudoplate. The state problem is modelled by a semicoercive variational inequality, where the control variable enters the coefficients of linear operator and a linear functional. Moreover, we consider the state eigenvalue problem for a minimal first eigenvalue associated with the vibration of pseudoplate. Existence of an optimal control is verified. Finally, approximate solutions with some convergence analysis are provided.

Keywords: elastic pseudoplate, obstacle control problem, control of variational inequalities, vibrations, state eigenvalue problem, uncertain input data.

Introduction

Obstacle control of elastic structures contains also quasistatic state problems with unilateral boundary conditions, which admit non-trivial virtual rigid body displacements. The simplest example is constitated by a beam unilaterally supported on both ends. We observe that the state problem is modelled by a semicoercive variational inequality. If the structure is fixed on some part of its boundary, then the energy of deformation is coercive and numerous theories from mathematics can be applied to the problem. On the other hand, another interesting case from the point of view of applications is when the body is fixed along some part of its boundary so that the rigid body motions are possible. Hence, the energy of deformation is no longer coercive. However, for several semi-coercive problems it is possible to give conditions on the right hand term (transversal load) in such a way guaranteeing the existence and the uniqueness of solution for the original problem and the corresponding discrete approximations. A semi- coercive elliptic problem with boundary conditions of the Signorini type is solved in this way through a Galerkin schema in (Adly, of semi-coercive variational inequalities involving a monotone (but not strongly monotone) operator, which depends on the control variable. Here we formulate boundary conditions and external forces, which imply the coerciveness of the potential energy over the subset of admissible displacements or over a subspace of the energy space only. Moreover, we restrict ourselves to the cases, when the subspace of rigid virtual displacements have the dimension one, in order to obtain uniqueness of the solution of the state problem.

Here we consider an optimal control problem of an elastic pseudoplate (a plate with small bending rigidity). The bending of the pseudoplate is described by means of shear model: the plate is deformed only by the shear forces (see e.g., Armand, 1972). Firstly, we assume that a homogeneous and isotropic pseudoplate occupying a domain $\Omega \times (-\mathscr{O}, \mathscr{O})$ of the space $\mathbb{R}^3$ is unilaterally supported on the whole boundary. The pseudoplate is loaded by a transversal distributed force $\mathscr{S}(x_1, x_2)$ perpendicular to the plane OX_1X_2. The role of control variables is played by: 1° The thickness of the pseudoplate, 2° The variable distributed load (externally applied pressure). The positive loading is considered down in the direction of Z axis. The cost functionals represent: the resultant of transverse contact forces between the pseudoplate and the rigid inner obstacle or the desired deflection of the pseudoplate. The state problem is modelled by a semi-coercive variational inequality, where the control variables influence the coefficients of the linear, bounded and monotone operator and a linear functional, both defined on a Hilbert space $H^1(\Omega)$. Secondly, we consider the state eigenvalue problem for (the deformation energy being coercive) a minimal first eigenvalue causing the vibrating of a pseudoplate in contact with a boundary obstacle on the space $V(\Omega) \subset H^1(\Omega)$. We assume that the thickness of the pseudoplate is uncertain, being prescribed in some a priori given set (the state eigenvalue problem with some uncertain data) an we employ a method of reliable solution. Here we consider the fundamental eigenfrequency as the functional criterion.

On the basis of the general existence theorems for a class of optimization problems or a reliable solution to the variational inequalities, we prove the existence of at least one solution to each of the problems mentioned above. Finally, we shall propose approximate solution and present some convergence analysis.

1. Setting of the problem

Let the midplane of the pseudoplate occupy a given bounded domain $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. Let $[\sigma_{x_1x_3}, \sigma_{x_2x_3}]$ denote the components of the stress field (shear stresses). We consider an isotropic and homogeneous elastic material. Assuming that the in- plane displacements vanish, we have the following stress-strain relations:

where K is a shear correction factor (a positive constant) and $G = \text{const}$ is the elastic shear modulus. The general forces (shear forces) of the pseudoplate have then the form:

$$V_{x_i x_3} = \int_{-\mathscr{O}}^{\mathscr{O}} \sigma_{x_i x_3} dz = KG\mathscr{O}(\partial v // \partial x_i), \quad i = 1, 2.$$

Hence we obtain the equation (the equilibrium equation of the pseudoplate without any internal obstacles)

$$\partial V_{x_i x_3}/\partial x_1 + \partial V_{x_i x_3}/\partial x_2 + \mathscr{S} = 0 \quad \text{or} \quad \operatorname{div}(KG\mathscr{O}\mathbf{grad}v) = -\mathscr{S}.$$

We denote the standard Sobolev function spaces by $H^k(\Omega)$ $(\equiv W_2^k(\Omega))$, $k = 1, 2$. Let the norm in $H^k(\Omega)$ be denoted by $\|.\|_{H^k(\Omega)}$. In the following, $L_2(\Omega)$ and $L_\infty(\Omega)$ denote the space of Lebesgue-square integrable functions on Ω and the space of essentially bounded functions on Ω, with standard norms $\|.\|_{L_2(\Omega)}$ and $\|.\|_{L_\infty(\Omega)}$, respectively. The inner product in $L_2(\Omega)$ will be denoted by $(.,.)_{L_2(\Omega)}$. If D is a subset in $\mathbb{R}^N$, its boundary is denoted be ∂D and its closure $D \cup \partial D$ by $\bar{D}$.

The transversal displacements (deflections) v belong to the space $V(\Omega) := H^1(\Omega)$. In the following we use the virtual displacement principle to establish a variational formulation of the problem. To this end we introduce the set of admissible deflections in the following way

$$\mathscr{K}(\Omega) := \{v \in V(\Omega) : v \geq 0\text{a.e. on } \Omega_* \text{ and } \mathscr{M}_0 v \geq 0\text{a.e. on } \partial\Omega\},$$

where $\bar{\Omega}_* \subset \Omega$ and $\mathscr{M}_0 v$ is trace of v on $\partial\Omega$, (the trace operator $\mathscr{M}_0 : H^1(\Omega) \to L_2(\partial\Omega)$ is linear and continuous, such that $\mathscr{M}_0 v$, see Fig. 1, is trace of v on $\partial\Omega$ for every v smooth).

For the transversal load $\mathscr{S}$ of the pseudoplate (the control variable), let Ω be decomposed into M disjoint subdomains, i.e.

$$\bar{\Omega} = \bigcup_{k=1}^{M} \bar{\Omega}_k, \quad \Omega_k \cap \Omega_m = \emptyset \text{ if } k \neq m.$$

Consider for the pseudoplate the control space $U(\Omega)$ and the admissible control set $U_{ad}(\Omega)$ as

$$U(\Omega) \equiv C(\bar{\Omega}) \times (\prod_{k=1}^{M} C(\bar{\Omega}_k)) \text{ and } U_{ad}(\Omega) = U_{ad}^{\mathscr{O}}(\Omega) \times U_{ad}^{\mathscr{S}}(\Omega),$$

with $\mathbf{e} = [\mathscr{O}, \mathscr{S}]^T$, where the half-thickness $\mathscr{O}$ belongs to the set

$$U_{ad}^{\mathscr{O}}(\Omega) = \{\mathscr{O} \in C^{(0),1}(\bar{\Omega}) \text{ (i.e. Lipschitz - continuous functions)}:$$

with given positive constants such that $U_{ad}^{\mathscr{O}}(\Omega)$ is non-empty $\mathrm{const}_{\langle 1\rangle} < \mathrm{const}_{\langle 2\rangle}$, $1 \in [\mathrm{const}_{\langle 1\rangle}, \mathrm{const}_{\langle 2\rangle}]$ and

$$U_{ad}^{\mathscr{S}}(\Omega) = \{\mathscr{S} \in L_\infty(\Omega) : \mathscr{S}_0|_{\bar{\Omega}_k} \in C^{(0),1}(\bar{\Omega}_k),$$
$$k = 1, 2, \dots, M,\ \|\mathscr{S} - \mathscr{S}_0\|_{L_\infty(\Omega)} \le \mathrm{const}_{\langle A\rangle},$$
$$\|\partial\mathscr{S}/\partial x_i\|_{L_\infty(\Omega)} \le \mathrm{const}_{\langle B\rangle},\ i = 1, 2\},$$

where $\mathscr{S}_0$ is a given function such that $\mathscr{S}_0|_{\bar{\Omega}_k} \in C^{(0),1}(\bar{\Omega}_k)$, $\|\partial\mathscr{S}_0/\partial x_i\|_{L_\infty(\Omega)} \le \mathrm{const}_{\langle a\rangle}$, $i = 1, 2$ with given positive constants such that $U_{ad}^{\mathscr{S}}(\Omega)$ is a nonempty set.

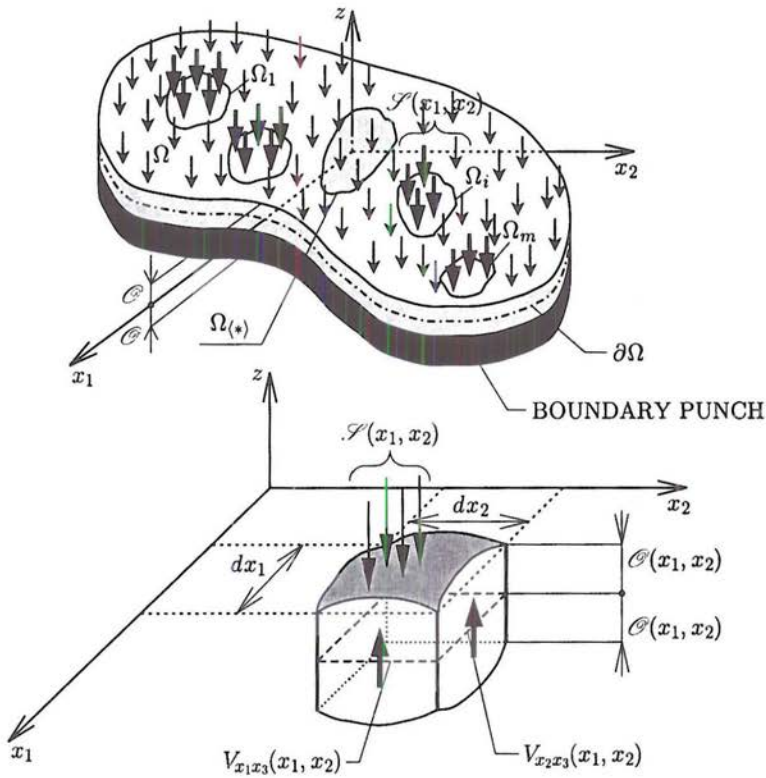

Fig. 1. Pseudoplate loaded by transversal forces

Note that any $\mathscr{S} \in U_{ad}^{\mathscr{S}}(\Omega)$ is a piecewise Lipschitz continuous function which does not differ "too much" from a "central" piecewise Lipschitz continuous

Due to the virtual displacement principle we associate with $\mathscr{O} \in U_{ad}^{\mathscr{O}}(\Omega)$ a bilinear form

$$a(\mathscr{O}, v, z) := \int_{\Omega} KG\mathscr{O}[\mathbf{grad} v \,.\, \mathbf{grad}\, z] d\Omega \text{ for all } v, z \in V(\Omega). \tag{1.1}$$

For a transversal load $\mathscr{S} \in L_{\infty}(\Omega)$, we introduce the linear form on $V(\Omega)$ by the formula (the virtual work of external load)

$$\langle L(\mathscr{S}), v \rangle_{V(\Omega)} = \int_{\Omega} \mathscr{S} v d\Omega. \tag{1.2}$$

It is readily seen that $L(\mathscr{S}) \in V^*(\Omega)$, for $\mathscr{S} \in U_{ad}^{\mathscr{S}}(\Omega)$.

We define on the open Ω the family of the pseudoplate linear operators $\{\mathscr{A}(\mathscr{O})\}$ generated by the bilinear form $a(\mathscr{O}, ., .)$ in the following way

$$\langle \mathscr{A}(\mathscr{O}) v, z \rangle_{V(\Omega)} = a(\mathscr{O}, v, z), \quad \mathscr{O} \in U_{ad}^{\mathscr{O}}(\Omega),\ v, z \in V(\Omega). \tag{1.3}$$

Thus, taking into account (1.1) to (1.3) on the basis of the virtual displacement principle, we introduce the following State Problem:

Find $u(e) \in \mathscr{K}(\Omega)$ such that

$$\langle \mathscr{A}(\mathscr{O}) u(e), v - u(e) \rangle_{V(\Omega)} \geq \langle L(\mathscr{S}), v - u(e) \rangle_{V(\Omega)}, \tag{1.4}$$

holds for given $e \in U_{ad}(\Omega)$ and for all $v \in \mathscr{K}(\Omega)$.

Further, we shall prove that the variational inequality (1.4) has a unique solution $u(e)$ for any $e \in U_{ad}(\Omega)$. On the other hand, for the state variational inequality (1.4), we consider several Optimal Control Problems. First we introduce cost functionals. The simplest will be

$$\mathscr{L}_{\text{DESIRED DEFLECTION}}(., v) = \int_{\Omega} |v - z_{ad}|^2 d\Omega \tag{1.5}$$

where $z_{ad} \in L_2(\Omega)$ is given function.

Let *theta* be any (fixed) function of $H_0^1(\Omega)$ such that $\theta = 1$ on Ω_* a.e. We define (under the condition of active support with non-zero reaction forces on $\partial\Omega$)

$$\mathscr{L}_{\text{TOTAL REACTION}}(e, v) = \int_{\Omega} (KG\mathscr{O} \mathbf{grad}\, v \,.\, \mathbf{grad}\, \theta - \mathscr{S}\theta) d\Omega. \tag{1.6}$$

This functional represents a resultant of transverse reactive forces on the inner obstacle. Let us justify the definition of $\mathscr{L}_{\text{TOTAL REACTION}}$ in detail:

For any $v \in \mathscr{K}(\Omega) \cap H^2(\Omega)$ we decompose domain Ω into the set

$$\mathcal{H}(v) := \{[x_1, x_2] \in \Omega_* : v(x_1, x_2) > 0\},$$

which is open, and its complement, the so-called coincide set: $\mathscr{Z}(v) = \Omega_* \backslash \mathcal{H}(v)$. Obviously, $v = 0$ holds on $\mathscr{Z}(v)$ (in general, the set $\mathscr{Z}(v)$ is not closed). We introduce the following set:

LEMMA 1 *If the solution $u(e)$ of the state problem (1.4) belongs to $H^2(\Omega)$, then one has*

$$\begin{aligned}&\mathscr{L}_{\text{TOTAL REACTION}}(e, u(e)) \\ &= -\int_{\mathscr{Z}(u(e))} (\mathscr{S} + \operatorname{div}(KG\mathscr{O}\,\mathbf{grad}\,u(e)))d\Omega,\end{aligned} \tag{1.7}$$

i.e., it has the same value for all $\theta \in \mathcal{O}_*(\Omega)$.

Proof. Let us show that

$$\mathscr{N}_z(e, u(e)) \equiv -\operatorname{div}(KG\mathscr{O}\,\mathbf{grad}\,u(e)) - \mathscr{S} = 0, \tag{1.8}$$

holds in $\Omega \setminus \mathscr{Z}(u(e))$ a.e.

Consider a point $[x_1^*, x_2^*] \in \mathcal{H}(u(e))$. Then there is a ball $B_\rho([x_1^*, x_2^*]) \subset \mathcal{H}(u(e))$ and a non-negative function $\varphi \in C_0^\infty(B_\rho[x_1^*, x_2^*]))$ such that $\varphi > 0$ on a closed ball $B_{\rho/2}([x_1^*, x_2^*])$ and $u(e) \geq \varphi$ holds in $B_\rho([x_1^*, x_2^*])$. Hence for any $\vartheta \in C_0^\infty(B_{\rho/2}([x_1^*, x_2^*]))$ we may find $\varepsilon > 0$ such that $u(e) + \varepsilon\vartheta \geq \varphi/2$ in $B_{\rho/2}([x_1^*, x_2^*])$. As a consequence, $v \equiv u(e) + \varepsilon\vartheta \in \mathscr{K}(\Omega)$. Now, we substitute this v in the inequality (1.4), we find: $v - u(e) = \varepsilon\vartheta$ and

$$\int_\Omega (KG\mathscr{O}\,\mathbf{grad}\,u(e)\,.\,\mathbf{grad}\,\vartheta)d\Omega \geq \int_\Omega \mathscr{S}\vartheta d\Omega.$$

On the other hand, the opposite inequality follows for $v \equiv u(e) - \varepsilon\vartheta$. This means that we may write

$$\int_\Omega (KG\mathscr{O}\mathbf{grad}\,u(e)\,.\,\mathbf{grad}\,\vartheta - \mathscr{S})d\Omega = 0,$$

for all $\vartheta \in C_0^\infty(B_{\rho/2}([x_1^*, x_2^*]))$. Hence, integrating by parts, we get (1.8) in $\Omega_* \setminus \mathscr{Z}(u(e))$.

Next, consider a point $[x_1^*, x_2^*] \in \Omega \setminus \bar{\Omega}_*$. We may find a ball $B_\rho([x_1^*, x_2^*]) \subset \Omega \setminus \bar{\Omega}_*$ and for any $\vartheta \in C_0^\infty(B_\rho([x_1^*, x_2^*]))$, we substitute $v = u(e) \pm \vartheta$ in (1.4) to find that (1.8) holds in $\Omega \setminus \bar{\Omega}_*$.

Finally, integrating by parts and by virtue of (1.8), we may write

$$\mathscr{L}_{\text{TOTAL REACTION}}(e, u(e)) = \int_\Omega \mathscr{N}_z(e, u(e))\theta d\Omega = \int_{\mathscr{Z}(u(e))} \mathscr{N}_z(e, u(e))d\Omega$$

and the assertion of the lemma follows.

We note that some results and the regularity of solutions to obstacle problems (see Rodriguez, 1987) can justify a conjecture that $u(e) \in H^2(\Omega)$ provided Ω is convex and $\mathscr{O} \in H^2(\Omega_*)$. These assumptions seem to be sufficient to the justification of the functional $\mathscr{L}_{\text{TOTAL REACTION}}$.

Moreover, for $\theta \in \mathcal{O}_*(\Omega)$ one has

$$\mathscr{L}_{\text{TOTAL REACTION}}(e, u(e)) = \int_\Omega \theta d\mu(e, u(e))$$

$$= \int \quad d\mu(e, u(e)) \geq 0 \tag{1.9}$$

where $\mu(e,u(e))$ is a non-negative Radon measure with supp $\mu(e,u(e))\subset\mathscr{Z}(u(e))$. We may rewrite the variational inequality (1.4) (for $v := u(e)+\varphi$, where $\varphi \in \mathcal{O}_*(\Omega)$, $\varphi \geq 0$) in the following form

$$\langle \mathscr{R}(e)u(e),\varphi\rangle_{V(\Omega)} \geq (\mathscr{S},\varphi)_{L_2(\Omega)},$$

where

$$\mathscr{R}(e)v = -\operatorname{div}(KG\mathscr{O}\,\mathbf{grad}\, v), \text{ for any } v \in H^1(\Omega).$$

As a consequence (by the Riesz-Schwartz theorem, Schwartz, 1966) $[\mathscr{R}(e)u(e) - \mathscr{S}]$ is a non-negative distribution on the domain Ω with support contained in $\mathscr{Z}(u(e))$. This means that (1.9) holds. This measure represents the interaction forces between the pseudoplate and the inner obstacle.

LEMMA 2 *The set $\mathscr{K}(\Omega)$ is a closed and convex subset of $V(\Omega)$.*

Proof. Clearly, $0 \in \mathscr{K}(\Omega)$, thus $\mathscr{K}(\Omega)$ is non-empty. The closedness follows from Lebesgue Theorem and convexity is immediate.

In the following, we define the Optimal Control Problems

$$\left\{\begin{array}{l} e_{(*),\text{DESIRED DEFLECTION}} = \operatorname*{Arg\,Min}_{e\in U_{ad}(\Omega)} \mathscr{L}_{\text{DESIRED DEFLECTION}}(e,u(e)), \\ e_{(*),\text{TOTAL REACTION}} = \operatorname*{Arg\,Min}_{e\in U_{ad}(\Omega)} \mathscr{L}_{\text{TOTAL REACTION}}(e,u(e)), \end{array}\right. \tag{1.10}$$

where state function $u(e)$ denotes the solution of the State Problem (1.4).

2. Existence of a solution to the optimal control problem

Let $U(\Omega)$ be a Banach space, $U_{ad}(\Omega) \subset U(\Omega)$ a compact subset, $V(\Omega)$ a Hilbert space equipped with a scalar product $(.,.)_{V(\Omega)}$ and a norm $\|.\|_{V(\Omega)}$, $V^*(\Omega)$ its dual space with a norm $\|.\|_{V^*(\Omega)}$ and let $\langle .,.\rangle_{V(\Omega)}$ denote the dual pairing.

Assume that $|v|_{V(\Omega)}$ is a continuous seminorm in the space $V(\Omega)$, satisfying the following conditions:

$$\text{(M0)}\quad \left\{\begin{array}{l} \text{If we define a subspace } \mathscr{R}(\Omega) = v \in V(\Omega) : |v|_{V(\Omega)} = 0\}\text{and} \\ \quad P_{\mathscr{R}(\Omega)}\text{is the orthogonal projector onto } \mathscr{R}(\Omega) \text{ then} \\ \quad \dim \mathscr{R}(\Omega) < \infty. \\ \text{There exist constants } [M_1 > 0, M_2 > 0]\text{such that} \\ \qquad M_1\|v\|_{V(\Omega)} \leq |v|_{V(\Omega)} + \|\mathrm{P}_{\mathscr{R}(\Omega)}v\|_{V(\Omega)} \leq M_2\|v\|_{V(\Omega)} \\ \quad \text{holds for all } v \in V(\Omega). \end{array}\right.$$

Let $\mathscr{K}(\Omega)$ be a closed convex subset of $V(\Omega)$ such that there exists a functional Φ: $V(\Omega) \to R_1$, satisfying the following conditions

$$\text{(M1)}\quad \left\{ \Phi(v) = 0 \Leftrightarrow v \in \mathscr{K}(\Omega) \Leftrightarrow D\Phi(v,z) = 0 \text{ for all } z \in V(\Omega), \right.$$

and the differential Gâteaux $D\Phi$ is monotone, this means that

$$D\Phi(v+z,z) - D\Phi(v,z) \geq 0 \text{ for any } v, z \in V(\Omega).$$

In the following we assume that

$$\text{(M2)} \quad \mathscr{K}(\Omega) \cap \mathscr{R}(\Omega) \neq \{0\},$$

$f \in V^*(\Omega)$ and a continuous operator $B : U_{ad}(\Omega) \to V^*(\Omega)$ is given such that

$$\text{(M3)} \quad \langle f + Be, p\rangle_{V(\Omega)} < 0 \text{ for any } p \in \mathscr{K}(\Omega) \cap \mathscr{R}(\Omega) \setminus \{0\} \text{ for any } e \in U_{ad}(\Omega).$$

Let $\{\mathscr{A}(e)\}_{e \in U_{ad}(\Omega)}$ be a family of linear operators $\mathscr{A}(e) : \mathscr{K}(\Omega) \to V^*(\Omega)$, which satisfy the following conditions for all $e \in U_{ad}(\Omega)$

$$\text{(A1)} \quad \begin{cases} 1^\circ.\ \langle \mathscr{A}(e)v - \mathscr{A}(e)z, v - z\rangle_{V(\Omega)} \geq \alpha_{\mathscr{A}} |v - z|^2_{V(\Omega)} \\ \quad \text{for any } v, z \in V(\Omega) \text{where the constant } \alpha_{\mathscr{A}} > 0 \text{ is independent of } e, \\ 2^\circ.\ \|e\|_{U(\Omega)} \leq \text{constant}_1, \|v\|_{V(\Omega)} \leq \text{constant}_2 \Rightarrow \|\mathscr{A}(e)v\|_{V^*(\Omega)} \\ \quad \leq \text{ constant}, \\ 3^\circ.\ e_n \in U_{ad}(\Omega), e_n \to e \text{ strongly in } U(\Omega) \Rightarrow \mathscr{A}(e_n)v \to \\ \quad \mathscr{A}(e)v \text{ strongly in } V^*(\Omega) \text{ for any } v \in \mathscr{K}(\Omega). \end{cases}$$

We note that by virtue of the condition (M1), $\mathscr{K}(\Omega)$ is a convex cone with the vertex zero. Indeed, we may write

$$\text{(B1)} \quad \begin{cases} \Phi(v) = \int_0^1 D\Phi(tv, v)dt = (1/2)D\Phi(v,v) \text{ so that } \Phi(tv) = t^2\Phi(v) \\ \text{for any } t > 0, \\ v \in \mathscr{K}(\Omega) \Rightarrow \Phi(v) = 0 \Rightarrow \Phi(tv) = 0 \Rightarrow tv \in \mathscr{K}(\Omega). \end{cases}$$

On the other hand one has, $\Phi(v) \geq 0$ for all $v \in V(\Omega)$.

LEMMA 3 *Let the assumptions (M0), (M1) and (M2), (M3) be satisfied. Then there exist constants $\mathcal{Q}_1 > 0$ and $\mathcal{Q}_2 > 0$, independent of $e \in U_{ad}(\Omega)$ such that*

$$\alpha_{\mathscr{A}} |v|^2_{V(\Omega)} + \Phi(v) - \langle f + Be, v\rangle_{V(\Omega)} \geq \mathcal{Q}_1 \|v\|_{V(\Omega)} - \mathcal{Q}_2 \tag{2.1}$$

holds for all $v \in V(\Omega)$ and $e \in U_{ad}(\Omega)$.

Proof. Let (2.1) not be true, then there are sequences $\{v_n\}_{n\in N}$, $\{e_n\}_{n\in N}$, $\|v_n\|_{V(\Omega)} \to \infty$ and $e_n \to e_0$ strongly in $U(\Omega)$ such that

$$\alpha_{\mathscr{A}} |v_n|^2_{V(\Omega)} + \Phi(v_n) - \langle f + Be_n, v_n\rangle_{V(\Omega)} < (1/n)\|v_n\|_{V(\Omega)} - n. \tag{2.2}$$

We note that the sequence $\{v_n\}_{n\in N}$ cannot be bounded, since then the left-hand side would be bounded from below. It follows from (2.2) that for sufficiently big $n \geq \mathcal{Q}_2$ one has

Due to (B1) and by setting $\mathcal{O}_n = v_n/\|v_n\|_{V(\Omega)}$, we obtain

$$\lim_{n\to\infty} (\alpha_{\mathscr{A}} |\mathcal{O}_n|^2_{V(\Omega)} \|v_n\|_{V(\Omega)} + \Phi(\mathcal{O}_n)\|v_n\|_{V(\Omega)} \\ - \langle f + Be_n, \mathcal{O}_n\rangle_{V(\Omega)} = \mathscr{O}_1 \leq 0, \tag{2.3}$$

choosing a subsequence for $\limsup_{n_k\to\infty}$, if necessary.

Moreover, for a further subsequence we may assume that: $\mathcal{O}_{n_k} \to \mathcal{O}$ weakly in $V(\Omega)$ and $|\mathcal{O}_{n_k}|_{V(\Omega)} \to \mathscr{O}_2$. On the other hand, the estimate (2.3) yields that $\mathscr{O}_2 = 0$. Since $\mathscr{F}(v) = |v|_{V(\Omega)}$ is a weakly lower semicontinuous functional (being convex and continuous), we have

$$|\mathcal{O}|_{V(\Omega)} \leq \liminf_{k\to\infty} |\mathcal{O}_{n_k}|_{V(\Omega)} \to 0,$$

so that $\mathcal{O} \in \mathscr{R}(\Omega)$.

Next, from the weak convergence and due to (M0) it follows that:

$$\|P_{\mathscr{R}(\Omega)}\mathcal{O}_{n_k} - P_{\mathscr{R}(\Omega)}\mathcal{O}\|_{V(\Omega)} \to 0.$$

By virtue of (M0), we may write

$$M_1\|\mathcal{O}_{n_k} - \mathcal{O}\|_{V(\Omega)} \leq |\mathcal{O}_{n_k} - \mathcal{O}|_{V(\Omega)} + \|P_{\mathscr{R}(\Omega)}(\mathcal{O}_{n_k} - \mathcal{O})\|_{V(\Omega)} \to 0.$$

Observe that for subsequence $\{\mathcal{O}_{n_k}\}_{k\in N}$ we may write $\|\mathcal{O}_{n_k}\|_{V(\Omega)} \to \|\mathcal{O}\|_{V(\Omega)}$, $\|\mathcal{O}\|_{V(\Omega)} = 1$.

Then, in view of the above we have

$$\lim_{k\to\infty} (|\mathcal{O}_{n_k}|^2_{V(\Omega)}\|v_{n_k}\|_{V(\Omega)} + \Phi(\mathcal{O}_{n_k})\|v_{n_k}\|_{V(\Omega)} - \langle f + Be_{n_k}, \mathcal{O}_{n_k}\rangle_{V(\Omega)} \\ \geq - \lim_{k\to\infty} \langle f + Be_{n_k}, \mathcal{O}_{n_k}\rangle_{V(\Omega)} = -\langle f + Be, \mathcal{O}\rangle_{V(\Omega)}.$$

Taking into account that: $\lim_{k\to\infty} \Phi(\mathcal{O}_{n_k}) = \Phi(\mathcal{O})$, $\Phi(\mathcal{O}) = 0$ follows from (2.3). This means that $\mathcal{O} \in \mathscr{K}(\Omega) \cap \mathscr{R}(\Omega) \setminus \{0\}$ and $-\langle f + Be, \mathcal{O}\rangle_{V(\Omega)} > 0$ holds by assumption (M3), we arrive at a contradiction with (2.3). ∎

THEOREM 1 *Let the assumptions (M0) to (M3) and (A1) be satisfied. Then there exists a solution $u(e) \in \mathscr{K}(\Omega)$ of the variational inequality*

$$\langle \mathscr{A}(e)u(e), v - u(e)\rangle_{V(\Omega)} \geq \langle f + Be, v - u(e)\rangle_{V(\Omega)} \tag{2.4}$$

for any $v \in \mathscr{K}(\Omega)$ and for any $e \in U_{ad}(\Omega)$.

Here any two solutions differ by an element $p_* \in \mathscr{R}(\Omega)$. If, moreover, $\langle \mathscr{A}(e)v, p\rangle_{V(\Omega)} = 0$ for all $p \in \mathscr{R}(\Omega)$ and $v \in V(\Omega)$, then one has: $\langle f + Be, p_*\rangle_{V(\Omega)} = 0$.

Proof. The linear operator $\mathscr{A}(e)$ is monotone $((A1), 1°)$, bounded $((A1), 2°)$,

$\mathscr{K}(\Omega)\cap Q_\sigma$. Then due to Theorem 8.1, from Lions, (1960) there exists a solution $u_\sigma \in \mathscr{K}_\sigma(\Omega)$ of the following inequality

$$\langle \mathscr{A}(e)u_\sigma, w - u_\sigma\rangle_{V(\Omega)} \geq \langle f + Be, w - u_\sigma\rangle_{V(\Omega)} \text{ for any } w \in \mathscr{K}_\sigma(\Omega). \quad (2.5)$$

By virtue of $((A1), 1°)$ one has (we set $w = 0 \in \mathscr{K}_\sigma(\Omega)$)

$$\alpha_{\mathscr{A}} |u_\sigma|^2_{V(\Omega)} \leq \langle \mathscr{A}(e)u_\sigma, u_\sigma\rangle_{V(\Omega)} \leq \langle f + Be, u_\sigma\rangle_{V(\Omega)}.$$

Next, in view of Lemma 3 we may write

$$0 \geq \alpha_{\mathscr{A}} |u_\sigma|^2_{V(\Omega)} - \langle f + Be, u_\sigma\rangle_{V(\Omega)} \geq \mathcal{Q}_1 \|u_\sigma\|_{V(\Omega)} - \mathcal{Q}_2,$$

so that

$$\|u_\sigma\|_{V(\Omega)} \leq \mathcal{Q} \equiv (\mathcal{Q}_2/\mathcal{Q}_1).$$

Let us choose $\sigma > \mathcal{Q}$ and show that $u_\sigma = u(e)$. Indeed, let $v \in \mathscr{K}(\Omega)$ be arbitrary. There is $t > 0$ such that $w = u_\sigma(1-t) + tv \in \mathscr{K}_\sigma(\Omega)$. So, by (2.5) we conclude that

$$t\langle \mathscr{A}(e)u_\sigma, v - u_\sigma\rangle_{V(\Omega)} \geq t\langle f + Be, v - u_\sigma\rangle_{V(\Omega)}.$$

Consequently, u_σ is a solution $u(e)$.

Next, let u and u_* be two solutions. Taking it into account we may write

$$\begin{cases} \langle \mathscr{A}(e)u_*, u - u_*\rangle_{V(\Omega)} \geq \langle f + Be, u - u_*\rangle_{V(\Omega)}, \\ \langle \mathscr{A}(e)u, u_* - u\rangle_{V(\Omega)} \geq \langle f + Be, u_* - u\rangle_{V(\Omega)}. \end{cases} \quad (2.6)$$

Hence, by addition, we obtain

$$\langle \mathscr{A}(e)u_* - \mathscr{A}(e)u, u - u_*\rangle_{V(\Omega)} \geq 0.$$

Then from $((A1), 1°)$ it follows that

$$|u - u_*|_{V(\Omega)} = 0 \text{ and } u_* - u = p_* \in \mathscr{R}(\Omega).$$

On the other hand, as $\langle \mathscr{A}(e)v, p\rangle_{V(\Omega)} = 0$ for all $p \in \mathscr{R}(\Omega)$ and all $v \in V(\Omega)$, then $\langle f + Be, p_*\rangle_{V(\Omega)} = 0$, due to relation (2.6).

LEMMA 4 *Let the assumptions (M0) to (M3) and (A1) hold. Further, one has*

$$\langle \mathscr{A}(e)v, p\rangle_{V(\Omega)} = 0 \textit{ for any } v \in V(\Omega), p \in \mathscr{R}(\Omega),\ e \in U_{ad}(\Omega), \quad (2.7)$$

and

$$dim\, \mathscr{R}(\Omega) = 1. \quad (2.8)$$

Then there exists a unique solution u(e) of the variational inequality (2.4) for

Proof. Let u and $u_* = u+p$ be two solutions of (2.4), assume that $p \in \mathscr{R}(\Omega) \setminus \{0\}$. Due to the assumption (M2) we can choose a basic element $p_* \in \mathscr{R}(\Omega)$ such that $p_* \in \mathscr{R}(\Omega) \cap \mathscr{K}(\Omega) \setminus \{0\}$. Then one has: $\langle f + Be, p_* \rangle_{V(\Omega)} < 0$ by virtue of (M3). Since we have: $p = Mp_*$, (for some real $M \neq 0$), $\langle f + Be, p \rangle_{V(\Omega)} = M \langle f + Be, p_* \rangle_{V(\Omega)} \neq 0$, which contradicts Theorem 1. ■

THEOREM 2 *Assume that (M0) to (M3) and (A1) hold, moreover, let the family of operators $\{\mathscr{A}(e)\}_{e \in U_{ad}(\Omega)}$ be potential. Further the assumption (2.7) and the relation*

$$\mathscr{A}(e)(v + p) = \mathscr{A}(e)v, \tag{2.9}$$

hold for all $v \in V(\Omega)$, $p \in \mathscr{R}(\Omega)$, $e \in U_{ad}(\Omega)$.

Let there exist a subspace $\mathscr{W}(\Omega) \subset V(\Omega)$ such that $\mathscr{R}(\Omega) \cap \mathscr{W}(\Omega) = \{0\}$ and if $u \in \mathscr{K}(\Omega) \setminus \mathscr{W}(\Omega)$, then there is an element $p_* \in \mathscr{K}(\Omega) \cap \mathscr{R}(\Omega) \setminus \{0\}$ such that $u - p_* \in \mathscr{K}(\Omega) \cap \mathscr{W}(\Omega)$. Then there exists a unique solution $u(e)$ of the variational inequality (2.4) and $u(e) \in \mathscr{K}(\Omega) \cap \mathscr{W}(\Omega)$.

Proof. In view of Theorem 1, there exists a solution u(e) of the state inequality (2.4). Then one has

$$u(e) = \operatorname*{Arg\,Min}_{v \in \mathscr{K}(\Omega)} \mathcal{O}(e, v), \quad e \in U_{ad}(\Omega),$$

where

$$\mathcal{O}(e, v) = \int_0^1 \langle \mathscr{A}(e)(tv), v \rangle_{V(\Omega)} dt - \langle f + Be, v \rangle_{V(\Omega)},$$

is the functional of potential energy, (see e.g. Céa, 1971).

Further, as

$$u(e) \in \mathscr{K}(\Omega) \setminus \mathscr{W}(\Omega),$$
$$[u(e) - p_*] \in \mathscr{K}(\Omega) \cap \mathscr{W}(\Omega), \quad p_* \in \mathscr{K}(\Omega) \cap \mathscr{R}(\Omega) \setminus \{0\},$$

we may write

$$\mathcal{O}(e, [u(e) - p_*]) = \int_0^1 \langle \mathscr{A}(e)(tu(e)), u(e) \rangle_{V(\Omega)} dt - \langle f + Be, u(e) - p_* \rangle_{V(\Omega)}$$
$$= \mathcal{O}(e, u(e)) + \langle f + Be, p_* \rangle_{V(\Omega)} < \mathcal{O}(e, u(e)),$$

for all $e \in U_{ad}(\Omega)$ (due to the relations (2.9), (2.7) and the condition (M3)). As a consequence, any solution u(e) belong to the set: $\mathscr{K}(\Omega) \cap \mathscr{W}(\Omega)$.

In view of Theorem 1 one has: $[u(e) - u_*(e)] \in \mathscr{R}(\Omega)$, if $u(e)$ and $u_*(e)$ are two solutions. Moreover, since $[u(e) - u_*(e)] \in \mathscr{W}(\Omega)$, as well, $[u(e) - u_*(e)] \in$

OPTIMAL CONTROL PROBLEM

Let a cost functional $\mathscr{L} : (U(\Omega) \times V(\Omega)) \to R_1$ be given, which satisfies the following condition:

$$\left\{ \begin{array}{l} \text{If } e_n \to e \text{ strongly in } U(\Omega) \text{ and } v_n \to v \text{ weakly in } V(\Omega), \text{ then} \\ \liminf\limits_{n\to\infty} \mathscr{L}(e_n, v_n) \geq \mathscr{L}(e, v). \end{array} \right. \tag{2.10}$$

We define the following Optimal Control Problem ($\mathscr{P}$): Find

$$e_* = \operatorname*{Arg\,Min}_{e \in U_{ad}(\Omega)} \mathscr{L}(e, u(e)) \tag{2.11}$$

where $u(e)$ denotes the solution of the state variational inequality (2.4).

THEOREM 3 *Let $U_{ad}(\Omega)$ be a compact subset of $U(\Omega)$, the assumptions (M0) to (M3) (A1), and (2.7), (2.9) and (2.10) hold and let at least one of the three following conditions be satisfied*

$$\text{(A2)} \quad \left\{ \begin{array}{l} 1^\circ.\dim \mathscr{R}(\Omega) = 1, \\ 2^\circ.\text{the operators } \mathscr{A}(e) \textit{ are potential and there is a subspace} \\ \mathscr{W}(\Omega) \subset V(\Omega) \text{ such that } \mathscr{R}(\Omega) \cap \mathscr{W}(\Omega) = \{0\} \textit{ and if } u \in \\ \mathscr{K}(\Omega) \setminus \mathscr{W}(\Omega), \text{then there exists a } p_* \in \mathscr{K}(\Omega) \cap \mathscr{R}(\Omega) \setminus \{0\}, \\ [u - p_*] \in \mathscr{K}(\Omega) \cap \mathscr{W}(\Omega), \\ 3^\circ.\text{for any } e \in U_{ad}(\Omega) \text{ there exists at most one solution of the} \\ \text{variational} \\ \text{inequality (2.4).} \end{array} \right.$$

Then there exists at least one solution of the Optimal Control Problem ($\mathscr{P}$).

Proof. By virtue of Lemma 3 one has

$$\begin{aligned} &\alpha_{\mathscr{A}} |u|^2_{V(\Omega)} - \langle f + Be, u \rangle_{V(\Omega)} \geq \mathcal{Q}_1 \|u\|_{V(\Omega)} - \mathcal{Q}_2 \\ &\text{for all } u \in \mathscr{K}(\Omega),\ e \in U_{ad}(\Omega), \end{aligned} \tag{2.12}$$

where the constants $\alpha_{\mathscr{A}}$, $\mathcal{Q}_1$, $\mathcal{Q}_2$ are independent of e and $\Phi(u) = 0$. Then, due to Theorem 1 there exists a solution $u(e)$ of (2.4). On the other hand, the assumptions ((A2), 1°, 2°, 3°) guarantee its uniqueness (in cases ((A2), 1°) or ((A2), 2°) this is a consequence of Lemma 4 and of Theorem 2, respectively).

For a minimizing sequence $\{e_n\}_{n\in N}$, $e_n \in U_{ad}(\Omega)$ (is obviously bounded in $U_{ad}(\Omega)$) we have

$$\lim_{n\to\infty} \mathscr{L}(e_n, u(e_n)) = \inf_{e \in U_{ad}(\Omega)} \mathscr{L}(e, u(e)). \tag{2.13}$$

Let us choose a convergent subsequence $\{e_{n_k}\}_{k\in N}$

Take $v = 0$ in (2.4) with $e = e_{n_k}$. Then, in view of ((A1), 1°) from (2.4) the estimate follows

$$\alpha_{\mathscr{A}} |u(e_{n_k})|^2_{V(\Omega)} \leq \langle \mathscr{A}(e_{n_k})u(e_{n_k}), u(e_{n_k})\rangle_{V(\Omega)} \leq \langle f + Be_{n_k}, u(e_{n_k})\rangle_{V(\Omega)}.$$

Moreover, by taking the estimate (2.12), we obtain

$$\mathcal{Q}_1 \|u(e_{n_k})\|^2_{V(\Omega)} - \mathcal{Q}_2 \leq \alpha_{\mathscr{A}} |u(e_{n_k})|^2_{V(\Omega)} - \langle f + Be_{n_k}, u(e_{n_k})\rangle_{V(\Omega)} \leq 0.$$

This means $\|u(e_{n_k})\|_{V(\Omega)} \leq (\mathcal{Q}_2/\mathcal{Q}_1)$ and we can choose another subsequence $\{u(e_{k_O})\}_{O\in N} \subset \{u(e_{n_k})\}_{k\in N}$, such that

$$u(e_{k_O}) \to u_* \quad \text{weakly in } V(\Omega), \tag{2.15}$$

where $u_* \in \mathscr{K}(\Omega)$ (since $u(e_{k_O}) \in \mathscr{K}(\Omega)$ and $\mathscr{K}(\Omega)$ is weakly closed).

By virtue of ((A1), 2°), we obtain

$$\|A(e_{k_O})u(e_{k_O})\|_{V^*(\Omega)} \leq \text{constant}. \tag{2.16}$$

This means that there exists an element $\mathcal{H}_* \in V^*(\Omega)$ such that a subsequence

$$\mathscr{A}(e_{O_n})u(e_{O_n}) \to \mathcal{H}_* \quad \text{weakly in } V^*(\Omega). \tag{2.17}$$

Due to the monotonicity of $\mathscr{A}(e_{O_n})$ we may write

$$\langle \mathscr{A}(e_{O_n})u(e_{O_n}) - \mathscr{A}(e_{O_n})v, u(e_{O_n}) - v\rangle_{V(\Omega)} \geq 0 \tag{2.18}$$

for any $v \in \mathscr{K}(\Omega)$, $n = 1, 2, \ldots$.

We take $v = u_*$ and $e = e_{O_n}$ in (2.4) then we obtain (due to the convergence of $\{e_{O_n}\}_{n\in N}$ and $\{u(e_{O_n})\}_{n\in N}$)

$$\limsup_{n\to\infty} \langle \mathscr{A}(e_{O_n})u(e_{O_n}), u(e_{O_n}) - u_*\rangle_{V(\Omega)} \leq 0. \tag{2.19}$$

Moreover, by virtue of (2.19) and (2.17), we may write

$$\limsup_{n\to\infty} \langle \mathscr{A}(e_{O_n})u(e_{O_n}), u(e_{O_n})\rangle_{V(\Omega)} \leq \langle \mathcal{H}_*, u_*\rangle_{V(\Omega)}. \tag{2.20}$$

Thus by (2.15), (2.17), (2.18) and (2.20), ((A1), 3°), we conclude that

$$\langle \mathcal{H}_* - \mathscr{A}(e_*)v, u_* - v\rangle_{V(\Omega)} \geq 0 \text{ for any } v \in \mathscr{K}(\Omega). \tag{2.21}$$

Let $v = u_* + t(w - u_*)$, $t \in (0,1)$, $w \in \mathscr{K}(\Omega)$. Then we have

$$\langle \mathcal{H}_* - \mathscr{A}(e_*)(u_* + t(w - u_*)), u_* - w\rangle_{V(\Omega)} \geq 0$$
$$\text{for any } w \in \mathscr{K}(\Omega),\ t \in (0,1). \tag{2.22}$$

Making use of the hemicontinuity ((A1), 2°) and setting again $w = v$ we obtain for $t \to 0_+$

Next we put $v = u_*$ into (2.18). Hence, we get

$$\langle \mathscr{A}(e_{\mathcal{O}_n})u(e_{\mathcal{O}_n}), u(e_{\mathcal{O}_n}) - u_*\rangle_{V(\Omega)} \geq \langle \mathscr{A}(e_{\mathcal{O}_n})u_*, u(e_{\mathcal{O}_n}) - u_*\rangle_{V(\Omega)}.$$

Here the continuity of $\mathscr{A}(.)u_*$ and the weak convergence of $\{u(e_{\mathcal{O}_n}\}_{n\in N}$ imply immediately that

$$\lim_{n\to\infty} \langle \mathscr{A}(e_{\mathcal{O}_n})u_*, u(e_{\mathcal{O}_n}) - u_*\rangle_{V(\Omega)} = 0.$$

Hence

$$\liminf_{n\to\infty} \langle \mathscr{A}(e_{\mathcal{O}_n})u(e_{\mathcal{O}_n}), u(e_{\mathcal{O}_n}) - u_*\rangle_{V(\Omega)} \geq 0.$$

On the other hand, by comparing this with (2.19), we obtain

$$\lim_{\kappa\to\infty} \langle \mathscr{A}(e_{\mathcal{O}_n})u(e_{\mathcal{O}_n}), u(e_{\mathcal{O}_n}) - u_*\rangle_{V(\Omega)} = 0. \tag{2.24}$$

Finally, the relation (2.17), (2.23) and (2.24) enable us to write

$$\langle \mathscr{A}(e_*)u_*, u_* - v\rangle_{V(\Omega)} \leq \lim_{n\to\infty} \langle \mathscr{A}(e_{\mathcal{O}_n})u(e_{\mathcal{O}_n}), u(e_{\mathcal{O}_n}) - v\rangle_{V(\Omega)}, \tag{2.25}$$

for any $v \in \mathscr{K}(\Omega)$.

We are coming now to the conclusion that the element $u_* \in \mathscr{K}(\Omega)$ is a solution of the variational inequality

$$\langle \mathscr{A}(e_*)u_*, u_* - v\rangle_{V(\Omega)} \leq \langle f + Be_*, u_* - v\rangle_{V(\Omega)}, \quad \text{for any } v \in \mathscr{K}(\Omega), \tag{2.26}$$

(in view of (2.25), ((2.4), for $e \equiv e_{\mathcal{O}_n}$) and (2.15) and the continuity of B).

Hence we have proved that

$$u_* = u(e_*), u(e_{\mathcal{O}_n}) \to u(e_*) \text{ weakly in } V(\Omega). \tag{2.27}$$

Furthermore, we observe that (2.10) implies

$$J(e_*) = \mathscr{L}(e_* u(e_*)) \leq \liminf_{n\to\infty} \mathscr{L}(e_{\mathcal{O}_n}, u(e_{\mathcal{O}_n})) = \liminf_{n\to\infty} J(e_{\mathcal{O}_n})$$
$$= \inf_{e\in U_{ad}(\Omega)} J(e).$$

The proof of Theorem 3 is completed. ∎

Now, we will apply Theorem 3 to the proof of existence of solutions to the optimal control problems (1.10). The seminorm $|.|_{V(\Omega)}$ is defined by

$$|v|^2_{V(\Omega)} = \int_\Omega [\mathbf{grad}\, v.\mathbf{grad}\, v] d\Omega. \tag{2.28}$$

Now, we define

$$\langle f + Be, v\rangle_{V(\Omega)} = \langle \mathscr{S}, v\rangle_{L_2(\Omega)},$$

We have $\mathscr{R}(\Omega) = P_0(\Omega) \equiv \{v = \text{ constant } \in R^1\}$, $\dim\mathscr{R}(\Omega) = 1$. In our case we have considered the scalar product in $V(\Omega)$ in the following form

$$(v,z)_{V(\Omega)} := \int_\Omega [\mathbf{grad}v.\mathbf{grad}z]d\Omega + p_0(v)p_0(z), \tag{2.29}$$

where $p_0(v) = v(\Gamma)$ (Here Γ is the line segment in the domain $\bar{\Omega}$). In the sequel we prove the equivalence of norms ((M0), 2°). Due to (2.28) there are positive constants M_0 and M_* such that

$$M_0\|v\|^2_{H^1(\Omega)} \geq |v|^2_{V(\Omega)} + p_0^2(v) \geq M_*\|v\|^2_{H^1(\Omega)} \text{ for all } v \in V(\Omega). \tag{2.30}$$

Moreover, we have an orthogonal decomposition $V(\Omega) = \tilde{\mathcal{O}}(\Omega) \oplus \mathscr{R}(\Omega)$. Set $\mathscr{W}(\Omega) = \{v \in V(\Omega) : p_0^2(v) = 0\}$. Then one has $\tilde{\mathcal{O}}(\Omega) = \mathscr{W}(\Omega)$. Indeed, let $v \in \tilde{\mathcal{O}}(\Omega)$, then we have $(v, o)_{V(\Omega)} = 0$, for $\mathcal{O} \in \mathscr{R}(\Omega)$. Due to the Schwarz inequality, we obtain: $\int_\Omega[\mathbf{grad}\, v.\mathbf{grad}\, z]d\Omega = 0$ and consequently, $p_0(v)p_0(o) = 0$ for every $o \in \mathscr{R}(\Omega)$. Hence, $p_0^2(v) = 0$ and therefore $v \in \mathscr{W}(\Omega)$. Conversely, let $v \in \mathscr{W}(\Omega)$, then $\int_\Omega[\mathbf{grad}\, v.\mathbf{grad}\, o]d\Omega = 0$ yields $(v, o)_{V(\Omega)} = 0$ for every $o \in \mathscr{R}(\Omega)$, and therefore $v \in \tilde{\mathcal{O}}(\Omega)$. Here we have: $\tilde{\mathcal{O}}(\Omega) = H_0^1(\Omega)$.

Next we denote by $\Pi_{\mathscr{W}(\Omega)}$ and $\Pi_{\mathscr{R}(\Omega)}$ the projections in the sense of the scalar product $(.,.)_{V(\Omega)}$. Then. from the estimate (2.30) we get

$$\begin{aligned}&\|\Pi_{\mathscr{W}(\Omega)}v\|^2_{V(\Omega)} \leq \text{constant}_{\mathscr{W}}|\Pi_{\mathscr{W}(\Omega)}v|^2_{H^1(\Omega)} = \text{constant}_{\mathscr{W}}|v - \Pi_{\mathscr{R}(\Omega)}v|^2_{H^1(\Omega)}\\ &= \text{constant}_{\mathscr{W}}|v|^2_{H^1(\Omega)},\\ &\|\Pi_{\mathscr{R}(\Omega)}v\|^2_{V(\Omega)} \leq M_o\|P_{\mathscr{R}(\Omega)}v\|^2_{H^1(\Omega)}.\end{aligned}$$

This means that we may write

$$\begin{aligned}&M_*\|v\|^2_{H^1(\Omega)} \leq \|v\|^2_{V(\Omega)} = (\|\Pi_{\mathscr{W}(\Omega)}v\|^2_{V(\Omega)} + \|\Pi_{\mathscr{R}(\Omega)}v\|^2_{V(\Omega)})\\ &\leq (\text{constant}_{\mathscr{W}}|v|^2_{H^1(\Omega)} + M_o\|P_{\mathscr{R}(\Omega)}v\|^2_{H^1(\Omega)}),\end{aligned}$$

which yields the left-hand side of $((M_0), 2°)$. Simultaneously, the right-hand side is obvious.

Let us define

$$\Phi(v) = \int_{\partial\Omega}([v^-])^2 dS, \tag{2.31}$$

where $([a]^-) \stackrel{df}{=} \min(0, a)$ is the negative part of the member a.

Then one has

$$D\Phi(v,z) = 2\int_{\partial\Omega}([v^-])z dS.$$

This means that the conditions (M1) are satisfied (we have $([a]^- - [b]^-)(a-b) \geq$

Let the loading be represented by the functional (1.2) (the virtual work of external loads). Assume that

$$\langle L(\mathscr{S}), p\rangle_{V(\Omega)} < 0 \text{ for all } p \in P_0(\Omega) \setminus \{0\},\ p > 0.$$

Then the condition (M3) is satisfied.

Moreover, we have

$$\begin{cases} \langle \mathscr{A}(\mathscr{O})v, p\rangle_{V(\Omega)} = a(\mathscr{O}, v, p) = 0, \\ (\text{by (1.1) for all } v \in V(\Omega),\ p \in \mathscr{R}(\Omega),\ \mathscr{O} \in U^{\mathscr{O}}_{ad}(\Omega)), \\ \mathscr{A}(\mathscr{O})(v+p) = \mathscr{A}(\mathscr{O})v + \mathscr{A}(\mathscr{O})p = \mathscr{A}(\mathscr{O})v. \end{cases} \tag{2.32}$$

Due to (1.3) and (1.1) we may write ($\mathscr{N}_{x_i x_i}(v) = \partial v/\partial x_i,\ i = 1,2$)

$$\begin{aligned} \langle \mathscr{A}(\mathscr{O})(v-z), v-z\rangle_{V(\Omega)} &\geq KG\text{const}_{(1)\mathscr{O}} \sum_{i=1}^{2} \int_\Omega [\mathscr{N}_{x_i x_i}(v-z)]^2 d\Omega \\ &= KG\text{const}_{(1)\mathscr{O}} |v-z|^2_{V(\Omega)}, \end{aligned} \tag{2.33}$$

for any $v, z \in V(\Omega)$ and for any $\mathscr{O} \in U^{\mathscr{O}}_{ad}(\Omega)$, and

$$\begin{aligned} |\langle \mathscr{A}(\mathscr{O}_n)v - \mathscr{A}(\mathscr{O})v, z\rangle_{V(\Omega)}| &= \Big| \int_\Omega KG(\mathscr{O}_n - \mathscr{O})[\mathbf{grad}\, v . \mathbf{grad}\, z] d\Omega \Big| \\ &\leq \text{constant } \|\mathscr{O}_n - \mathscr{O}\|_{C(\bar\Omega)} \|v\|_{V(\Omega)} \|z\|_{V(\Omega)}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} |\langle \mathscr{A}(\mathscr{O})v, w\rangle_{V(\Omega)} - \mathscr{A}(\mathscr{O})z, w\rangle_{V(\Omega)}| &= \Big| \int_\Omega KG\mathscr{O}(\mathbf{grad}(v-z) . \mathbf{grad} w) d\Omega \Big| \\ &\leq KG\text{const}_{(2)\mathscr{O}} \|v-z\|_{V(\Omega)} \|w\|_{V(\Omega)}. \end{aligned}$$

This means that

$$\begin{cases} \|A(\mathscr{O}_n)v - \mathscr{A}(\mathscr{O})v\|_{V^*(\Omega)} \leq \text{constant}(\|\mathscr{O}_n - \mathscr{O}\|_{C(\bar\Omega)}) \|v\|_{V(\Omega)}, \\ \|\mathscr{A}(\mathscr{O})v\|_{V^*(\Omega)} \leq \text{constant}. \end{cases} \tag{2.34}$$

The verification of the conditions (M0), (M1) and (M2), (A1) is now completed. As a consequence of Lemma 4 this yields the existence and uniqueness of a solution u(e) of the state variational inequality (1.4) for any $e \in U_{ad}(\Omega)$.

LEMMA 5 *Any of the functionals ((1.5), (1.6)) satisfies the assumption (2.10).*

Proof. Let $e_n \in U_{ad}(\Omega)$, $e_n \to e$ strongly in $U(\Omega)$ and $v_n \to v$ weakly in $V(\Omega)$. By Rellich Theorem, $v_n \to v$ strongly in $L_2(\Omega)$, so that

$$\begin{aligned} &|\mathscr{L}_{\text{DESIRED DEFLECTION}}(.,v_n) - \mathscr{L}_{\text{DESIRED DEFLECTION}}(.,v)| \\ &\leq \int_\Omega |(v_n - z_{ad})^2 - (v - z_{ad})^2| d\Omega \leq \|v_n + v - 2z_{ad}\|_{L_2(\Omega)} \|v_n - v\|_{L_2(\Omega)} \end{aligned}$$

and so

$$\lim_{n\to\infty} \mathscr{L}_{\text{DESIRED DEFLECTION}}(.,v_n) = \mathscr{L}_{\text{DESIRED DEFLECTION}}(.,v).$$

Moreover, we may write (consider a fixed $\theta \in \mathcal{O}_*(\Omega)$ in (1.6))

$$\mathscr{L}_{\text{TOTAL REACTION}}(e_n,v_n) = \mathscr{L}_{\text{TOTAL REACTION}}(e,v_n) + \mathscr{M}_n, \tag{2.35}$$

where

$$\begin{aligned} |\mathscr{M}_n| &= \left| \int_\Omega [(KG(\mathscr{O}_n - \mathscr{O})\mathbf{grad}v_n.\mathbf{grad}\theta) - (\mathscr{S}_n - \mathscr{S})\theta] d\Omega \right| \\ &\leq KG\text{constant } [\|\mathscr{O}_n - \mathscr{O}\|_{L_\infty(\Omega)} \|v\|_{H^1(\Omega)} + \|\mathscr{S}_n - \mathscr{S}\|_{L_\infty(\Omega)}] \to 0, \end{aligned} \tag{2.36}$$

since the norms $\|v_n\|_{H^1(\Omega)}$ are bounded.

We can conclude that

$$\lim_{n\to\infty} \mathscr{L}_{\text{TOTAL REACTION}}(e,v_n) = \mathscr{L}_{\text{TOTAL REACTION}}(e,v). \tag{2.37}$$

Thus, due to (2.35) to (2.37), we arrive at

$$\lim_{n\to\infty} \mathscr{L}_{\text{TOTAL REACTION}}(e_n,v_n) = \mathscr{L}_{\text{TOTAL REACTION}}(e,v).$$

Next, on the basis of the Arzelá-Ascoli Theorem, the compactness of the sets $U_{ad}^{\mathscr{O}}(\Omega)$ and $U_{ad}^{\mathscr{S}}(\Omega)$ follows in the space $C(\bar{\Omega})$ and $(\prod_{i=1}^{M} C(\bar{\Omega}_i))$. Then, $U_{ad}(\Omega)$ is compact in $U(\Omega)$.

Altogether, all assumptions of Theorem 3 are fulfilled by Lemmas 1 to 4. As a consequence each of the Optimal Control Problems (1.10) has at least one solution.

3. Approximate optimal control

In the following, we assume that the domain Ω has a polygonal boundary $\partial\Omega$. Let us consider a regular family of triangularizations $\{\mathscr{T}_{h_n}\}$, $h_n \to 0^+$ of the domain Ω, which are consistent with the partitions: $\bar{\Omega} = \cup_{i=1}^{M} \bar{\Omega}_i$. We introduce the finite-dimensional space of piecewise linear functions P_1 (i.e. all the finite elements of all the triangulation are affine-equivalent to a single reference finite element, there exists $\theta_0 > 0$ such that $\theta \geq \theta_0$ and $h_n \to 0_+$)

$$\mathcal{H}_{h_n}(\Omega) = \{v_{h_n} \in C(\bar{\Omega}) : v_{h_n}|_T \in P_1(T) \text{ for all triangles } T \in \mathscr{T}_{h_n}\}$$

and the following sets

$$\begin{cases} V_{h_n}(\Omega) = \mathcal{H}_{h_n}(\Omega) \cap V(\Omega) \text{ (each triangulation } \mathscr{T}_{h_n} \text{ will be associated} \\ \text{with a finite-dimensional space of piecewise linear functions),} \\ U_{ad,h_n}^{\mathscr{O}}(\Omega) = U_{ad}^{\mathscr{O}}(\Omega) \cap \mathcal{H}_{h_n}(\Omega), \quad U_{ad,h_n}^{\mathscr{S}}(\Omega) = U_{ad}^{\mathscr{S}}(\Omega) \cap \mathcal{H}_{h_n}(\Omega). \end{cases}$$

Assume that $\mathscr{S}_0 \in \mathcal{H}_{h_0}(\Omega)$ for some triangulation $\mathscr{T}_{h_0}$. Hence, we have to assume that the triangulations $\mathscr{T}_{h_n}$ are consistent also with the boundaries $\partial\Omega_k$, $k = 1, 2, \ldots, M$, which play a role in the definition of $U^{\mathscr{S}}_{ad,h}(\Omega)$. Then we define

$$U_{ad,\langle h\rangle}(\Omega) = U^{\mathscr{O}}_{ad,h_n}(\Omega) \times U^{\mathscr{S}}_{ad,h_n}(\Omega)$$

and consider approximate control $\mathrm{e}_h = \langle \mathscr{O}_h, \mathscr{S}_h\rangle^T \in U_{ad,h}(\Omega)$.

Here, with each triangulation $\mathscr{T}_{h_n}$ we also associate the following subset of $V_{h_n}(\Omega)$

$$\mathscr{K}_{h_n}(\Omega) = \{v_{h_n} \in V_{h_n}(\Omega) : v_{h_n}(A) \geq 0 \text{ for all nodes } \mathrm{A} \in \Sigma^*_h \cup \Sigma_h \cap \partial\Omega \\ = \Sigma^*_h \cup (\Sigma_h - \Sigma^0_h)\},$$

where Σ^*_h denotes the set of all vertices of triangles $T \in \mathscr{T}_h$, $T \subset \bar{\Omega}_*$, and $\Sigma^0_h = \{A \in \Sigma_h, A \notin \partial\Omega\}$, $\Sigma_h = \{A \in \bar{\Omega},\ A \text{ is a vertex of } T \in \mathscr{T}_h\}$.

We suppose that we are given a sequence $\{h_n\}_{n\in N}$ converging to zero and a family $\{V_{h_n}(\Omega)\}_{h_n}$ of closed subspaces of $V(\Omega)$. We are also given a family $\{\mathscr{K}_{h_n}(\Omega)\}_{n\in N}$ of closed convex non-empty subsets of $V(\Omega)$ with $\mathscr{K}_{h_n}(\Omega) \subset V_{h_n}(\Omega)$ for any h_n such that $\{\mathscr{K}_{h_n}(\Omega)\}_{n\in N}$ satisfies the following conditions (we introduce a concept of convergence in the sense of Glowinski):

$$(M1)_h \quad \begin{cases} 1^\circ.\ \text{If } \{v_{h_n}\}_{h_n} \text{ is such that } v_{h_n} \in \mathscr{K}_{h_n}(\Omega) \text{ for any } h_n \text{ and} \\ \quad \{v_{h_n}\}_{n\in N} \text{ is bounded in } V(\Omega), \text{ then the weak cluster} \\ \quad \text{points of } \{v_{h_n}\}_{h_n} \text{ belong to } \mathscr{K}(\Omega). \\ 2^\circ.\ \text{There exists } \Lambda(\Omega) \subset V(\Omega), \overline{\Lambda(\Omega)} = \mathscr{K}(\Omega) \text{and } \mathcal{O}_{h_n} \\ \quad : \Lambda(\Omega) \to \mathscr{K}_{h_n}(\Omega) \\ \quad \text{such that } \lim\limits_{h_n\to 0} \mathcal{O}_{h_n} v = v \text{ strongly in } V(\Omega) \\ \quad \text{for any } v \in \Lambda(\Omega). \end{cases}$$

Now, we may define the following Approximate State Problem: Given any

$$\begin{cases} e_{h_n} = [\mathscr{O}_{h_n}, \mathscr{S}_{h_n}]^T \in U_{ad,h_n}(\Omega), \text{ find } u_{h_n}(e_{h_n}) \in \mathscr{K}_{h_n}(\Omega) \\ \text{such that } \langle A(\mathscr{O}_{h_n})u_{h_n}(e_{h_n}), v_{h_n} - u_{h_n}(e_{h_n})\rangle_{V(\Omega)} \\ \geq \langle L(\mathscr{S}_{h_n}), v_{h_n} - u_{h_n}(e_{h_n})\rangle_{V(\Omega)} \\ \text{holds for all } v_{h_n} \in \mathscr{K}_{h_n}(\Omega). \end{cases} \tag{3.2}$$

Finally, let us define the functionals

$$\begin{cases} \mathscr{L}_{\text{DESIRED DEFLECTION},\langle h\rangle}(e_{h_n}, v_{h_n}) \\ = \mathscr{L}_{\text{DESIRED DEFLECTION}}(e_{h_n}, v_{h_n}), \\ \mathscr{L}_{\text{TOTAL REACTION},\langle h\rangle}(e_{h_n}, v_{h_n}) = \mathscr{L}_{\text{TOTAL REACTION}}(e_{h_n}, v_{h_n}). \end{cases} \tag{3.3}$$

APPROXIMATE OPTIMAL CONTROL PROBLEMS: Given a fixed triangulation $\mathscr{T}_h$, find

$$\left\{ \begin{array}{l} e_{\text{DESIRED DEFLECTION},\langle h\rangle} = \text{Arg Min}_{e_h \in U_{ad,h}(\Omega)} \\ \quad \mathscr{L}_{\text{DESIRED DEFLECTION}}(e_{h_n}, u_h(e_h)), \\ e_{\text{TOTAL REACTION},\langle h\rangle} = \text{Arg Min}_{e_h \in U_{ad,h}(\Omega)} \\ \quad \mathscr{L}_{\text{TOTAL REACTION},\langle R\rangle}(e_h, u_h(e_h)), \end{array} \right. \tag{3.4}$$

where $u_{h_n}(e_{h_n})$ is the solution of the Approximate State Problem (3.2).

THEOREM 4 *The Approximate State Problem (3.2) has a unique solution $u_h(e_h)$ for any $e_h \in U_{ad,h}(\Omega)$ and any h sufficiently small. The Approximate Optimal Control Problem (3.4) has at least one solution for any cost functional (3.3) and for any h sufficiently small.*

Proof. Let us verify the assumptions of Theorem 3, where we set: $U_{ad}(\Omega) := U_{ad,h}(\Omega)$, $e = e_h$, $V(\Omega) := V_h(\Omega)$, $\mathscr{K}_h(\Omega) := \mathscr{K}(\Omega)$ (for any $h > 0$) and define $A(\mathscr{O}_h) : V_h(\Omega) \to V_h^*(\Omega)$, $\mathscr{L}(\mathscr{S}_h)(= B\mathscr{S}_h) : U_h^{\mathscr{S}}(\Omega) \to V_h^*(\Omega)$ by the relations

$$\left\{ \begin{array}{l} \langle \mathscr{A}(\mathscr{O}_h)v_h, z_h\rangle_{V(\Omega)} := a(\mathscr{O}_h, v_h, z_h), \\ \quad \text{and} \\ \langle L(\mathscr{S}_h), v_h\rangle_{V(\Omega)} := \int_\Omega \mathscr{S}_h v_h d\Omega. \end{array} \right. \tag{3.5}$$

The set $\mathscr{K}_h(\Omega)$ $(0 \in \mathscr{K}_h(\Omega))$ is a closed and convex subset of $\mathscr{K}(\Omega)$.

By virtue of Lemma 3 and Theorem 1 the existence $u_h(e_h) \in \mathscr{K}_h(\Omega)$ follows. On the other hand the uniqueness can be proved in the same way as in Lemma 4 and Theorem 2.

Next we note that the cost functional $\mathscr{L}_{\text{DESIRED DEFLECTION},\langle h\rangle}$ for fixed $\langle h\rangle$ satisfies the condition (2.10) (the proof is the same as for Lemma 5). Moreover, we may write

$$\begin{aligned} &\mathscr{L}_{\text{TOTAL REACTION},\langle h\rangle}(e_{h,\langle n\rangle}, v_{h,\langle n\rangle}) = \mathscr{L}_{\text{TOTAL REACTION},\langle h\rangle}(e_h, v_{h,\langle n\rangle}) \\ &+ \mathscr{H}_{h,\langle n\rangle}, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} |\mathscr{H}_{h,\langle n\rangle}| &= \Big| \int_\Omega KG(\mathscr{O}_{h,\langle n\rangle} - \mathscr{O}_h)\,\mathbf{grad}\, v_{h,\langle n\rangle}.\mathbf{grad}\,\theta_h d\Omega \\ &- \int_\Omega (\mathscr{S}_{h,\langle n\rangle} - \mathscr{S}_h)\theta_h d\Omega \Big| \\ &\le \text{const } (\|\mathscr{O}_{h,\langle n\rangle} - \mathscr{O}_h\|_{L_\infty(\Omega)}\|v_{h,\langle n\rangle}\|_{V(\Omega)} + \|\mathscr{S}_{h,\langle n\rangle} - \mathscr{S}_h\|_{L_\infty(\Omega)}) \\ &\to 0. \end{aligned} \tag{3.7}$$

Next, we obtain the following estimate

$$|\mathscr{L}_{\text{TOTAL REACTION},\langle h\rangle}(e_h, v_{h,\langle n\rangle}) - \mathscr{L}_{\text{TOTAL REACTION},\langle h\rangle}(e_h, v_h)|$$

Then, due to (3.6), (3.7) and (3.8), we arrive at

$$\lim_{n\to\infty} \mathscr{L}_{\text{TOTAL REACTION},\langle h\rangle}(e_{h,(n)}, v_{h,(n)}) = \mathscr{L}_{\text{TOTAL REACTION},\langle h\rangle}(e_h, v_h).$$

CONVERGENCE RESULTS

In the following, we will study the convergence of finite elements approximations when the mesh size tends to zero. To this end we establish the crucial

LEMMA 6 *Let* $e_{h_n} \in U_{ad,h}(\Omega)$, $e_{h_n} \to e$ *strongly in* $U(\Omega)$, *as* $h_n \to 0_+$. *Then one has*

$$u_{h_n}(e_{h_n}) \to u(e) \text{ strongly in } V(\Omega), \text{ as } h_n \to 0_+. \tag{3.9}$$

Proof. Since $\mathscr{K}_{h_n}(\Omega) \subset \mathscr{K}(\Omega)$ and $\mathscr{K}(\Omega)$ is weakly closed, the condition $((\mathrm{M1})_h, 1°)$ is trivially satisfied. We shall use the following density result (Glowinski, 1980): $\overline{C^\infty(\bar{\Omega}) \cap \mathscr{K}(\Omega)} = \mathscr{K}(\Omega)$. Then it is natural to take $\Lambda(\Omega) = C^\infty(\bar{\Omega}) \cap \mathscr{K}(\Omega)$. We define $\mathcal{O}_{h_n} : H^1(\Omega) \cap C^0(\bar{\Omega}) \to V_{h_n}(\Omega)$, by the relation (the linear interpolation operator)

$$\begin{cases} \mathcal{O}_{h_n} v \in V_{h_n}(\Omega) \text{ for any } v \in H^1(\Omega) \cap C^0(\bar{\Omega}), \\ \mathcal{O}_{h_n} v(A_j) = v(A_j) \text{ for any } A_j \in \Sigma_h. \end{cases}$$

As the angles of the triangles of $\mathscr{T}_h$ are uniformly bounded below by θ_0 as $h_n \to 0$, then one has (Ciarlet, 1978; Glowinski, 1980)

$$\|\mathcal{O}_{h_n} v - v\|_{V(\Omega)} \leq \text{const.} h_n \|v\|_{H^2(\Omega)} \text{ for any } v \in C^\infty(\bar{\Omega})$$

with constant independent of h_n and v.

This implies that

$$\lim_{n\to\infty} \|\mathcal{O}_{h_n} v - v\|_{V(\Omega)} = 0 \text{ for any } v \in \Lambda(\Omega). \tag{3.10}$$

On the other hand, it is obvious that

$$\mathcal{O}_{h_n} v \in \mathscr{K}_{h_n}(\Omega) \text{ for any } v \in \mathscr{K}(\Omega) \cap C^0(\bar{\Omega}),$$

so that

$$\mathcal{O}_{h_n} v \in \mathscr{K}_{h_n}(\Omega) \text{ for any } v \in \Lambda(\Omega).$$

In conclusion, with the above $\Lambda(\Omega)$ and $\mathcal{O}_{h_n}$, the condition $((\mathrm{M1})_h, 2°)$ is satisfied.

Substituting $v_{h_n} = 0 \in \mathscr{K}_{h_n}(\Omega)$ in the state inequality (3.2), we obtain the estimate

$$KG\text{const}_{\langle 1\rangle,\mathscr{O}} |u_{h_n}(e_{h_n})|^2_{V(\Omega)} \leq \langle A(\mathscr{O}_{h_n}) u_{h_n}(e_{h_n}), u_{h_n}(e_{h_n})\rangle_{V(\Omega)}$$

Then, in view of Lemma 3 we may write

$$
\begin{aligned}
0 &\geq KG\text{const}_{\langle 1\rangle,\mathscr{O}}|u_{h_n}(e_{h_n})|^2_{V(\Omega)} - \langle L(\mathscr{S}_{h_n}), u_{h_n}(e_{h_n})\rangle_{V(\Omega)} \\
&\geq \mathscr{Q}_1\|u_{h_n}(e_{h_n})\|_{V(\Omega)} - \mathscr{Q}_2
\end{aligned}
$$

so that

$$
\|u_{h_n}(e_{h_n})\|_{V(\Omega)} \leq \text{constant } (\equiv (\mathscr{Q}_1/\mathscr{Q}_2)), \tag{3.11}
$$

holds for all h_n sufficiently small.

As a consequence of (3.11), there exist $u_\diamondsuit \in V(\Omega)$ and a subsequence of $\{u_{h_k}(e_{h_k})\}_{k\in N}$ such that

$$
u_{h_k}(e_{h_k}) \to u_\diamondsuit \quad \text{weakly in } V(\Omega). \tag{3.12}
$$

So, (3.12) means that $u_\diamondsuit \in \mathscr{K}(\Omega)$.

We have the following relation (due to $(\mathrm{M1})_h$, (3.10), (3.5) and the weak convergence (3.12))

$$
\begin{aligned}
&|\langle L(\mathscr{S}_{h_k}), \mathcal{O}_{h_k}v - u_{h_k}(e_{h_k})\rangle_{V(\Omega)} - \langle L(\mathscr{S}), v - u_\diamondsuit\rangle_{V(\Omega)}| \\
&= \Big|\int_\Omega [\mathscr{S}_{h_k}(\mathcal{O}_{h_k}v - u_{h_k}(e_{h_k})) - \mathscr{S}(v - u_\diamondsuit)]d\Omega\Big| \\
&\leq \Big|\int_\Omega (\mathscr{S}_{h_k} - \mathscr{S})(\mathcal{O}_{h_k}v - u_{h_k}(e_{h_k}))d\Omega\Big| \\
&+ \Big|\int_\Omega \mathscr{S}[(\mathcal{O}_{h_k}v - u_{h_k}(e_{h_k})) - (u_\diamondsuit - v)]d\Omega\Big| \\
&\leq \text{constant } \|\mathscr{S}_{h_k} - \mathscr{S}\|_{L_\infty(\Omega)}\|\mathcal{O}_{h_k}v - u_{h_k}(e_{h_k})\|_{L_2(\Omega)} \\
&+ \|\mathscr{S}\|_{L_2(\Omega)}\|\mathcal{O}_{h_k}v - v\|_{L_2(\Omega)} + \Big|\int_\Omega \mathscr{S}(u_\diamondsuit - u_{h_k}(e_{h_k}))d\Omega\Big| \to 0
\end{aligned}
\tag{3.13}
$$

Let us substitute $v_{h_k} = 2u_{h_k}(e_{h_k})$ in the state inequality (3.2) and pass to $\liminf_{k\to\infty}$ with $h_k \to 0_+$. The functional $v \to \langle \mathscr{A}(\mathscr{O})v, v\rangle_{V(\Omega)}$ is weakly lower semicontinuous, being convex and differentiable. Thus we see that

$$
\liminf_{k\to\infty}\langle \mathscr{A}(\mathscr{O})u_{h_k}(e_{h_k}), u_{h_k}(e_{h_k})\rangle_{V(\Omega)} \geq \langle \mathscr{A}(\mathscr{O})u_\diamondsuit, u_\diamondsuit\rangle_{V(\Omega)}. \tag{3.14}
$$

Making use of (3.11) and (2.34), we derive that

$$
\begin{aligned}
&|\langle A(\mathscr{O}_{h_k})u_{h_k}(e_{h_k}), u_{h_k}(e_{h_k})\rangle_{V(\Omega)} - \langle \mathscr{A}(\mathscr{O})u_{h_k}(e_{h_k}), u_{h_k}(e_{h_k})\rangle_{V(\Omega)}| \\
&\leq \|A(\mathscr{O}_{h_k})u_{h_k}(e_{h_k}) - \mathscr{A}(\mathscr{O})u_{h_k}(e_{h_k})\|_{V^*(\Omega)}\|u_{h_k}(e_{h_k})\|_{V(\Omega)} \\
&\leq \text{const } \|A(\mathscr{O}_{h_k}) - \mathscr{A}(\mathscr{O})\|_{L(V(\Omega),V^*(\Omega))}\|u_{h_k}(e_{h_k})\|^2_{V(\Omega)} \to 0.
\end{aligned}
\tag{3.15}
$$

Therefore

$$
\begin{aligned}
&\liminf_{k\to\infty}\langle \mathscr{A}(\mathscr{O}_{h_k})u_{h_k}(e_{h_k}), u_{h_k}(e_{h_k})\rangle_{V(\Omega)} \\
&= \liminf_{k\to\infty}(\langle \mathscr{A}(\mathscr{O})u_{h_k}(e_{h_k}), u_{h_k}(e_{h_k})\rangle_{V(\Omega)} \\
&+[\langle \mathscr{A}(\mathscr{O}_{h_k})u_{h_k}(e_{h_k}), u_{h_k}(e_{h_k})\rangle_{V(\Omega)} - \langle \mathscr{A}(\mathscr{O})u_{h_k}(e_{h_k}), u_{h_k}(e_{h_k})\rangle_{V(\Omega)}]) \\
&\geq \liminf \langle \mathscr{A}(\mathscr{O})u_{h_k}(e_{h_k}), u_{h_k}(e_{h_k})\rangle \geq \langle \mathscr{A}(\mathscr{O}) \ldots \rangle
\end{aligned}
\tag{3.16}
$$

follows from (3.14) and (3.15).

Further, by virtue of (3.15), (2.34) and (3.11), (3.12) we may write (for fixed $\mathcal{O} \in V(\Omega)$)

$$\begin{aligned}
&|\langle[\mathscr{A}(\mathscr{O}_{h_k})u_{h_k}(e_{h_k}) - \mathscr{A}(\mathscr{O})u_\diamondsuit], \mathcal{O}\rangle_{V(\Omega)}| \le |\langle[\mathscr{A}(\mathscr{O}_{h_k})u_{h_k}(e_{h_k}) \\
&- \mathscr{A}(\mathscr{O})u_{h_k}(e_{h_k})], \mathcal{O}\rangle_{V(\Omega)}| \\
&+ |\langle[\mathscr{A}(\mathscr{O})u_{h_k}(e_{h_k}) - \mathscr{A}(\mathscr{O})u_\diamondsuit], \mathcal{O}\rangle_{V(\Omega)}| \\
&\le \|A(\mathscr{O}_{h_k})u_{h_k}(e_{h_k}) - \mathscr{A}(\mathscr{O})u_{h_k}(e_{h_k})\|_{V^*(\Omega)}\|\mathcal{O}\|_{V(\Omega)} \\
&+ |\langle[\mathscr{A}(\mathscr{O})u_{h_k}(e_{h_k}) - \mathscr{A}(\mathscr{O})u_\diamondsuit], \mathcal{O}\rangle_{V(\Omega)}| \to 0.
\end{aligned}$$

From this, we conclude that

$$\left\{\begin{array}{l} A(\mathscr{O}_{h_k})u_{h_k}(e_{h_k}) \to \mathscr{A}(\mathscr{O})u_\diamondsuit \text{ weakly in } V^*(\Omega), \\ \quad \text{when} \\ u_{h_k}(e_{h_k}) \to u_\diamondsuit \text{ weakly in } V(\Omega). \end{array}\right. \tag{3.17}$$

Thus, one obtains (coming back to the variational inequality (3.2), inserting $v_{h_k} = \mathcal{O}_{h_k}v$ and passing to limes inferior or limes superior with $h_k \to 0_+$)

$$\begin{aligned}
&-\langle\mathscr{A}(\mathscr{O})u_\diamondsuit, u_\diamondsuit\rangle_{V(\Omega)} \ge \limsup_{k\to\infty}\langle\mathscr{A}(\mathscr{O}_{h_k})u_{h_k}(e_{h_k}), -u_{h_k}(e_{h_k})\rangle_{V(\Omega)} \\
&\ge \limsup_{k\to\infty}(\langle-\mathscr{A}(\mathscr{O}_{h_k})u_{h_k}(e_{h_k}), \mathcal{O}_{h_k}v\rangle_{V(\Omega)} \\
&+ \langle L(\mathscr{S}_{h_k}), \mathcal{O}_{h_k}v - u_{h_k}(e_{h_k})\rangle_{V(\Omega)})
\end{aligned} \tag{3.18}$$

for all $v \in \mathscr{K}(\Omega)$.

Here, by virtue of (3.11) and $((\mathrm{M1})_h, 2°)$ we have

$$\begin{aligned}
&|\langle\mathscr{A}(\mathscr{O}_{h_k})u_{h_k}(e_{h_k}), \mathcal{O}_{h_k}v - v\rangle_{V(\Omega)}| \\
&\le \text{const } \|u_{h_k}(e_{h_k})\|_{V(\Omega)}\|\mathcal{O}_{h_k}v - v\|_{V(\Omega)} \to 0
\end{aligned} \tag{3.19}$$

for $h_k \to 0^+$.

Further, due to (3.19) and (3.17), we deduce that

$$\begin{aligned}
&|\langle\mathscr{A}(\mathscr{O}_{h_k})u_{h_k}(e_{h_k}), \mathcal{O}_{h_k}v\rangle_{V(\Omega)} - \langle\mathscr{A}(\mathscr{O})u_\diamondsuit, v\rangle_{V(\Omega)}| \\
&\le |\langle\mathscr{A}(\mathscr{O}_{h_k})u_{h_k}(e_{h_k}), \mathcal{O}_{h_k}v - v\rangle_{V(\Omega)}| \\
&+ |\langle\mathscr{A}(\mathscr{O}_{h_k})u_{h_k}(e_{h_k}), v\rangle_{V(\Omega)} - \langle\mathscr{A}(\mathscr{O})u_\diamondsuit, v\rangle_{V(\Omega)}| \to 0.
\end{aligned} \tag{3.20}$$

Finally, making use of (3.18), (3.20) and (3.13), we arrive at

$$-\langle\mathscr{A}(\mathscr{O})u_\diamondsuit, u_\diamondsuit\rangle_{V(\Omega)} \ge -\langle\mathscr{A}(\mathscr{O})u_\diamondsuit, v\rangle_{V(\Omega)} + \langle L(\mathscr{S}), v - u_\diamondsuit\rangle_{V(\Omega)}.$$

Thus, $u_\diamondsuit$ is a solution of the inequality (1.4). From the uniqueness of $u(e)$ we conclude that $u_\diamondsuit = u(e)$ and the whole sequence $\{u_{h_n}(e_{h_n})\}_{n\in N}$ tends to

It remains to prove the strong convergence. Since $\mathscr{K}_h(\Omega)$ is a convex cone with a vertex at the origin, we may insert $v_{h_n} := 0$ and $v_{h_n} := 2u_{h_n}(e_{h_n})$ in (3.2) to obtain

$$\langle \mathscr{A}(\mathscr{O}_{h_n})u_{h_n}(e_{h_n}), u_{h_n}(e_{h_n})\rangle_{V(\Omega)} = \langle L(\mathscr{S}_{h_n}), u_{h_n}(e_{h_n})\rangle_{V(\Omega)}. \tag{3.21}$$

Next, according to (3.21) and (3.12), (3.13), we have

$$\lim_{n\to\infty} \langle \mathscr{A}(\mathscr{O}_{h_n})u_{h_n}(e_{h_n}), u_{h_n}(e_{h_n})\rangle_{V(\Omega)} = \langle L(\mathscr{S}), u(e)\rangle_{V(\Omega)}$$
$$= \langle \mathscr{A}(\mathscr{O})u(e), u(e)\rangle_{V(\Omega)}.$$

On the other hand, taking into account (3.15), we get

$$\lim_{n\to\infty} \langle \mathscr{A}(\mathscr{O})u_{h_n}(e_{h_n}), u_{h_n}(e_{h_n})\rangle_{V(\Omega)} = \langle \mathscr{A}(\mathscr{O})u(e), u(e)\rangle_{V(\Omega)}. \tag{3.22}$$

Further, using (3.22) and (1.3), we arrive at

$$\lim_{n\to\infty} a(\mathscr{O}, u_{h_n}(e_{h_n}), u_{h_n}(e_{h_n})) = \lim_{n\to\infty} \langle \mathscr{A}(\mathscr{O})u_{h_n}(e_{h_n}), u_{h_n}(e_{h_n})\rangle_{V(\Omega)}$$
$$= \langle \mathscr{A}(\mathscr{O})u(e), u(e)\rangle_{V(\Omega)} = a(\mathscr{O}, u(e), u(e)). \tag{3.23}$$

On the other hand, we have

$$\lim_{n\to\infty} a(\mathscr{O}, [u_{h_n}(e_{h_n}) - u(e)], [u_{h_n}(e_{h_n}) - u(e)])$$
$$= \lim_{n\to\infty} a(\mathscr{O}, u_{h_n}(e_{h_n}), u_{h_n}(e_{h_n})) - 2a(\mathscr{O}, u_{h_n}(e_{h_n}), u(e))$$
$$+a(\mathscr{O}, u(e), u(e))] = 0,$$

(due to (3.23) and (3.12)). Hence, from (2.33) we conclude that $|u_{h_n}(e_{h_n}) - u(e)|_{V(\Omega)} \to 0$, for $h_n \to 0_+$, which in turn (taking into account (3.12)) implies that $u_{h_n}(e_{h_n}) \to u(e)$ strongly in $V(\Omega)$.

LEMMA 7 *Let $e_{h_n} \in U_{ad,h}(\Omega), e_{h_n} \to e$ strongly in $U(\Omega)$ as $h_n \to 0_+$. Then, we have*

$$\lim_{n\to\infty} \mathscr{L}_{\text{DESIRED DEFLECTION},\langle h\rangle}(e_{h_n}, u_{h_n}(e_{h_n}))$$
$$= \mathscr{L}_{\text{DESIRED DEFLECTION}}(e, u(e))$$

and

$$\lim_{n\to\infty} \mathscr{L}_{\text{TOTAL REACTION},\langle h\rangle}(e_{h_n}, u_{h_n}(e_{h_n})) = \mathscr{L}_{\text{TOTAL REACTION}}(e, u(e)).$$

Proof. It is clear that

$$|\mathscr{L}_{\text{DESIRED DEFLECTION},\langle h\rangle}(e_{h_n}, u_{h_n}(e_{h_n}))$$
$$- \mathscr{L}_{\text{DESIRED DEFLECTION}}(e, u(e))|$$
$$= \left| \int_\Omega ((u_{h_n}(e_{h_n}) - z_{ad})^2 - (u(e) - z_{ad})^2) d\Omega \right|$$

For the second cost functional, we have

$$\begin{aligned}&|\mathscr{L}_{\text{TOTAL REACTION},\langle h\rangle}(e_{h_n}, u_{h_n}(e_{h_n})) - \mathscr{L}_{\text{TOTAL REACTION}}(e, u(e))|\\ &\leq |\mathscr{L}_{\text{TOTAL REACTION},\langle h\rangle}(e_{h_n}, u_{h_n}(e_{h_n})) - \mathscr{L}_{\text{TOTAL REACTION}}(e_{h_n}, u(e))|\\ &+|\mathscr{L}_{\text{TOTAL REACTION},\langle h\rangle}(e_{h_n}, u(e)) - \mathscr{L}_{\text{TOTAL REACTION}}(e, u(e))| \equiv \mathscr{Q}_1 + \mathscr{Q}_2,\end{aligned}$$

where

$$\begin{aligned}&\mathscr{Q}_1 \leq |KG(\mathscr{O}_{h_n}\mathbf{grad}\ (u_{h_n}(e_{h_n}) - u(e)), \mathbf{grad}\ \theta)_{L_2(\Omega)}|\\ &\leq \text{constant}\|u_{h_n}(e_{h_n}) - u(e)\|_{V(\Omega)} \to 0.\end{aligned}$$

Next, we also have

$$\begin{aligned}&\mathscr{Q}_2 \leq |KG((\mathscr{O}_{h_n} - \mathscr{O})\mathbf{grad}\, u(e), \mathbf{grad}\ \theta)_{L_2(\Omega)}| + |(\mathscr{S} - \mathscr{S}_{h_n}, \theta)_{L_2(\Omega)}|\\ &\leq \text{constant} \quad (\|\mathscr{O} - \mathscr{O}_{h_n}\|_{L_\infty(\Omega)}\|u(e)\|_{V(\Omega)}\|\theta\|_{V(\Omega)}\\ &+ \|\mathscr{S} - \mathscr{S}_{h_n}\|_{L_\infty(\Omega)}\|\theta\|_{V(\Omega)}) \to 0,\end{aligned}$$

so that

$$\begin{aligned}&\lim_{n\to\infty} \mathscr{L}_{\text{TOTAL REACTION},\langle h\rangle}(e_{h_n}, u_{h_n}(e_{h_n}))\\ &= \mathscr{L}_{\text{TOTAL REACTION}}(e, u(e)).\end{aligned}$$

■

LEMMA 8 *For any* $e \in [\mathscr{O}, \mathscr{S}]^T \in U_{ad}(\Omega)$ *and any sequence* $\{h_n\}_{n\in N}$, $h_n \to 0_+$ *there exists a sequence* $\{e_{h_n}\}_{n\in N}$ *such that* $e_{h_n} = [\mathscr{O}_{h_n}, \mathscr{S}_{h_n}]^T \in U_{ad,h}(\Omega)$ *and* $e_{h_n} \to e$ *strongly in* $U(\Omega) \equiv C(\bar{\Omega}) \times (\prod_{i=1}^M C(\bar{\Omega}_i))$.

Proof. Let $\Pi_{h_n}\mathscr{O}$ denote the Lagrange linear interpolate of $\mathscr{O}$ over the triangulation $\mathscr{T}_{h_n}$. Since $\mathscr{O} \in W^1_\infty(\Omega)$, the interpolation theory (Ciarlet, 1978; Glowinski, 1980) yields

$$\|\mathscr{O} - \Pi_{h_n}\mathscr{O}\|_{L_\infty(\Omega)} \leq \text{const. } h_n\|\mathscr{O}\|_{W^1_\infty(\Omega)}.$$

Obviously, $\text{const}_{\langle 1\rangle\mathscr{O}} \leq \Pi_{h_n}\mathscr{O} \leq \text{const}_{\langle 2\rangle\mathscr{O}}$ everywhere. For any straight-line segment $\overline{PQ} \in T$ parallel to the X_i-axis and any triangle $T \subset \mathscr{T}_{h_n}$ we have

$$\begin{aligned}&|\partial\Pi_{h_n}\mathscr{O}/\partial x_i| = (1/L)|\mathscr{O}(Q) - \mathscr{O}(P)| \leq (1/L)\int_P^Q |\partial\mathscr{O}/\partial x_i| dx_i\\ &\leq \text{const}_{\langle x_i\rangle,\mathscr{O}},\end{aligned}$$

On the other hand, analogous arguments as in Lemma 13 hold for $\Pi_{h_n}\mathscr{S}$. Thus, setting $e_{h_n} = [\Pi_{h_n}\mathscr{O}, \Pi_{h_n}\mathscr{S}]^T$, we fulfil the conditions of the lemma. ■

THEOREM 5 *Let $\{e_{\langle*\rangle,h_n}\}_{n\in N}$, $n \to \infty$ (or $h_n \to 0_+$) be a sequence of solutions to the Approximate Optimal Control Problem (3.4). Then, a subsequence $\{e_{\langle*\rangle,h_{n_k}}\}_{k\in N} \subset \{e_{\langle*\rangle,h_n}\}_{n\in N}$ exists, such that*

$$\left\{ \begin{array}{l} e_{\langle*\rangle,h_{n_k}} \to e_{\langle*\rangle} \text{ strongly in } U(\Omega)(= C(\bar{\Omega}) \times (\prod_{i=1}^{M} C(\bar{\Omega}_i))), \\ u_{h_{n_k}}(e_{\langle*\rangle,h_{n_k}}) \to u(e_{\langle*\rangle}) \text{ strongly in } V(\Omega), \end{array} \right. \tag{3.24}$$

where $e_{\langle*\rangle}$ is a solution of the Optimal Control Problem (1.10). The limit of each subsequence of $\{e_{\langle*\rangle,h_n}\}_{n\in N}$, converging in $U(\Omega)$ is a solution of the latter problem and an analogue of ((3.24), 2°) holds.

Proof. Since $U_{ad,h}(\Omega) \subset U_{ad}(\Omega)$ is compact in $U(\Omega)$, there exists a subsequence $\{e_{\langle*\rangle,h_{n_k}}\}_{k\in N}$, $k \to \infty$, such that ((3.24), 1°) holds. Let us consider an $e \in U_{ad}(\Omega)$. In view of Lemma 8, there exists a sequence of $\{e_{h_{\mathcal{O}}}\}_{\mathcal{O}\in N}$, $e_{h_{\mathcal{O}}} \in U_{ad,h_{\mathcal{O}}}(\Omega)$, such that $e_{h_{\mathcal{O}}} \to e$ strongly in $U(\Omega)$, as $h_{\mathcal{O}} \to 0_+$. By definition, we have

$$\mathscr{L}_{h_{\mathcal{O}}}(e_{\langle*\rangle,h_{\mathcal{O}}}, u_{h_{\mathcal{O}}}(e_{\langle*\rangle,h_{\mathcal{O}}})) \leq \mathscr{L}_{h_{\mathcal{O}}}(e_{h_{\mathcal{O}}}, u_{h_{\mathcal{O}}}(e_{h_{\mathcal{O}}})).$$

Let us pass to the limit with $h_{\mathcal{O}} \to 0_+$ and apply Lemma 7 to both sides of this inequality. We arrive at

$$\mathscr{L}(e_{\langle*\rangle}, u(e_{\langle*\rangle})) \leq \mathscr{L}(e, u(e)),$$

so that $e_{\langle*\rangle}$ is a solution of the original Optimal Control Problem. Next, by virtue of Lemma 6, we obtain ((3.24), 2°). The previous line of thought may be repeated for any uniformly convergent subsequence of $\{e_{\langle*\rangle,h_n}\}_{n\in N}$. ■

4. Reliable solution of a vibrating pseudoplate

We consider the state problem connected with an unilateral eigenvalue problem. Our problem is to find among all admissible thicknesses of the pseudoplate an extreme one. This can be done by considering a functional/criterion defined on the set of all admissible thicknesses of the pseudoplate and reducing the problem to the minimization of this functional criterion. The volume of the pseudoplate is constant and the thickness of the pseudoplate is bounded.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary $\partial\Omega$, where the boundary $\partial\Omega$ be decomposed as follows:

$$\partial\Omega = \overline{\partial\Omega}_{\text{DISPLACEMENT}} \cup \overline{\partial\Omega}_{\text{CONTACT}},$$

where $\partial\Omega_{\text{DISPLACEMENT}}$ and $\partial\Omega_{\text{CONTACT}}$ are open, non-empty and non-overlapping parts. On $\partial\Omega_{\text{DISPLACEMENT}}$ a homogene kinematic condition is prescribed, whereas on $\partial\Omega_{\text{CONTACT}}$ the pseudoplate is subject to a contact with

Let us assume free vibrations of a thin homogeneous isotropic pseudoplate of the shear model. The displacement function $w = w([x_1, x_2], t)$ is a solution of the hyperbolic equation

$$\rho\mathscr{O}(x_1, x_2)\partial^2 w/\partial t^2 - \operatorname{div}(KG\mathscr{O}(x_1, x_2)\mathbf{grad}\, w) = 0,$$
$$t \in \mathbb{R},\ [x_1, x_2] \in \Omega, \tag{4.1}$$

where ρ is the density, $2\mathscr{O}(x_1, x_2)$ is the variable thickness of the pseudoplate. It is reasonable to suspect that it may be possible to express the displacement $w = w([x_1, x_2], t)$ as the product of two functions, one involving only the space coordinates $[x_1, x_2]$ an the other involving the variable time. This equality may be written

$$w([x_1, x_2], t) = u(x_1, x_2)T(t). \tag{4.2}$$

Then, substituting (4.2) into (4.1), one can readily show that

$$(T(t)/\rho)\mathscr{O}(x_1, x_2)[-\operatorname{div}(KG\mathscr{O}(x_1, x_2)\mathbf{grad}\, u(x_1, x_2))]$$
$$= u(x_1, x_2)\partial^2 T(t)/\partial t^2. \tag{4.3}$$

Next dividing each side of equation (4.3) by the product $u(x_1, x_2)T(t)$, we obtain

$$(1/\rho)\mathscr{O}(x_1, x_2)[-\operatorname{div}(KG\mathscr{O}(x_1, x_2)\mathbf{grad}\, u(x_1, x_2))]/u(x_1, x_2)$$
$$= -(\partial^2 T(t)/\partial t^2)/\partial t^2)T(t)/T(t). \tag{4.4}$$

Thus, from the left-hand side of equation (4.4), we obtain, after some rearranging

$$-\operatorname{div}(KG\mathscr{O}(x_1, x_2)\mathbf{grad}\, u(x_1, x_2)) - \omega^2\rho\mathscr{O}(x_1, x_2)u(x_1, x_2) = 0. \tag{4.5}$$

This is a homogeneous partial differential equation involving the mode shape expression $u(x_1, x_2)$, the pseudoplate properties, and the circular frequency of oscillation ω.

Moreover, by setting $\lambda = \omega^2$, we obtain the following eigenvalue problem for the pseudoplate

$$\begin{cases} -\operatorname{div}(KG\mathscr{O}(x_1, x_2)\mathbf{grad}\, u(x_1, x_2)) = \lambda\rho\mathscr{O}(x_1, x_2)u(x_1, x_2), [x_1, x_2] \in \Omega, \\ \mathscr{M}v = 0 \text{ on } \partial\Omega_{\text{DISPLACEMENT}}, \\ \mathscr{M}_0 v \geq 0 \text{ on } \partial\Omega_{\text{CONTACT}}. \end{cases}$$

We introduce a variational formulation of the eigenvalue problem. To this end we introduce the set (the mode shape)

$$V(\Omega) := \{v \in H^1(\Omega) : \mathscr{M}_0 v = 0 \text{ a.e. on } \partial\Omega_{\text{DISPLACEMENT}}\}$$

and

The set of admissible states contains the functions from the space $V(\Omega)$ non-negative on $\partial\Omega_{\text{CONTACT}}$

$$\mathscr{K}(\Omega) := \{v \in V(\Omega) : \mathscr{M}_0 v \geq 0 \text{ a.e. on } \partial\Omega_{\text{CONTACT}}\} \tag{4.6}$$

The operators $\mathscr{A}(\mathscr{O}) : V(\Omega) \to V^*(\Omega)$, $B(\mathscr{O}) : V(\Omega) \to V^*(\Omega)$ are defined by the relations

$$\begin{cases} \langle \mathscr{A}(\mathscr{O})v, z\rangle_{V(\Omega)} := \int_\Omega KG\mathscr{O}\,\mathbf{grad}\, v.\mathbf{grad}\, z d\Omega, \\ \langle B(\mathscr{O})v, z\rangle_{V(\Omega)} := \int_\Omega \rho\mathscr{O} v z d\Omega, \text{ for any } v, z \in V(\Omega). \end{cases} \tag{4.7}$$

Moreover, the operator $\mathscr{B}(.)$ is compact on the Hilbert space $V(\Omega)$. Indeed, we may prolong the bilinear form ((4.7), 2°) on the space $\hat{\mathscr{H}}(\Omega)$ ($= L_2(\Omega) \times L_2(\Omega)$). Thus we get the prolonged operator $\mathscr{B}_{\hat{\mathscr{H}}} : \hat{\mathscr{H}}(\Omega) \to \hat{\mathscr{H}}^*(\Omega)$. Hence, one has $\mathscr{B} = i^T.\mathscr{B}_{\hat{\mathscr{H}}}.i$, where i is the injection $V(\Omega) \to \hat{\mathscr{H}}(\Omega)$ and $i^T : \hat{\mathscr{H}}^*(\Omega) \to V^*(\Omega)$ and these injections are compact.

The vibration of the pseudoplate is described by the eigenvalue variational inequality

$$\begin{cases} \text{Find a couple} \\ [u(\mathscr{O}), \lambda_*(\mathscr{G})] \in \{\mathscr{K}(\Omega) \setminus \{0\}\} \times \mathbb{R}, u(\mathscr{O}) \neq 0 \\ \text{(the eigensolutions) such that} \\ \langle \mathscr{A}(\mathscr{O})u(\mathscr{O}), v - u(\mathscr{O})\rangle_{V(\Omega)} \geq \lambda_*(\mathscr{O})\langle \mathscr{B}(\mathscr{O})u(\mathscr{O}), v - u(\mathscr{O})\rangle_{\mathscr{K}(\Omega)}, \\ \text{for all } v \in \mathscr{K}(\Omega). \end{cases} \tag{4.8}$$

where $\lambda_*(\mathscr{O})$ is the smallest or first eigenvalue in (4.8).

We will consider the state (eigenvalue) problem (4.8) with some uncertain input data. It may happen that the thickness $\mathscr{O}(x_1, x_2)$ is uncertain, i.e. this is not given uniquely, but the only available information is that it belong to some given set $\mathscr{U}_{ad}(\Omega)$.

Let Ω be decomposed into N disjoint subdomains, i.e.

$$\bar{\Omega} = \bigcup_{k=1}^{N} \overline{\mathscr{Z}}_k, \quad \mathscr{Z}_k \cap \mathscr{Z}_l = \emptyset \quad \text{if } k \neq l,$$

and let

$$\mathscr{U}_{ad}(\Omega) = \{\mathscr{O} \in L_\infty(\Omega) : 0 < \mathscr{O}_{\text{MIN}} \leq \mathscr{O} \leq \mathscr{O}_{\text{MAX}}, \mathscr{O}|_{\mathscr{Z}_k} \in C^{(0),1}(\overline{\mathscr{Z}}_k),$$
$$k = 1, 2, \ldots, N,$$
$$\|\mathscr{O} - \mathscr{O}_0\|_{L_\infty(\Omega)} \leq \text{const}_{1\langle\mathscr{O}\rangle}, \|\partial\mathscr{O}/\partial x_i\|_{L_\infty(\Omega)} \leq \text{const}_{2\langle\mathscr{O}\rangle},\ i = 1, 2\},$$

where $\mathscr{O}_0$ is a given function such that $\mathscr{O}_0|_{\mathscr{Z}_k} \in C^{(0),1}(\overline{\mathscr{Z}}_k)$, $\|\partial\mathscr{O}_0/\partial x_i\|_{L_\infty(\Omega)} \leq \text{const}_{2\langle\mathscr{O}\rangle}$, $i = 1, 2$ and $[\mathscr{O}_{\text{MIN}}, \mathscr{O}_{\text{MAX}}, \text{const}_{1\langle\mathscr{O}\rangle}, \text{const}_{2\langle\mathscr{O}\rangle}]$ are given constants. Note that any $\mathscr{O} \in U_{ad}(\Omega)$ is a piecewise Lipschitz function, which does not

Moreover, we introduce the set

$$\mathscr{U}(\Omega) = (\prod_{k-1}^{N} C(\overline{\mathscr{Z}}_k)).$$

We shall employ a method of reliable solution, which consists of the following main steps

$$\begin{cases} 1^\circ. \text{ choose a functional criterion } [\mathscr{O}, u] \to \Psi(\mathscr{O}, u), \\ 2^\circ. \text{ solve the minimization problem :} \\ \mathscr{O}_* = \underset{\mathscr{O} \in u_{ad}(\Omega)}{\text{Arg Min}}\, \Psi(\mathscr{O}, u), \end{cases} \tag{4.9}$$

where $u(\mathscr{O})$ denotes the (unique) solution of the eigenvalue variational inequality (4.8) for the input data $\mathscr{O}$ (the thickness of the pseudoplate). The choice of the criterion Ψ depends on the technical demands. For instance, in our case Ψ represents the fundamental eigenfrequency of the pseudoplate. Thus, we can define

$$\Psi(\mathscr{O}, u(\mathscr{O})) = \lambda_*(\mathscr{O}), \quad \mathscr{O} \in \mathscr{U}_{ad}(\Omega). \tag{4.10}$$

EXISTENCE OF A RELIABLE SOLUTION OF THE PROBLEM WITH UNCERTAIN DATA NOTATIONS AND PREPARATORY RESULTS

Let $V(\Omega)$ or $\mathscr{H}(\Omega)$ be the real Hilbert spaces with norms $\|.\|_{V(\Omega)}$ (or $\|\cdot\|_{\mathscr{H}(\Omega)}$) and dual $V^*(\Omega)$ (or $\mathscr{H}^*(\Omega)$), also denoted by $\|.,.\|_{V^*(\Omega)}$ (or $\|\cdot\|_{\mathscr{H}^*(\Omega)}$), and $\langle .,. \rangle_{V(\Omega)}$ (or $\langle .,. \rangle_{\mathscr{H}(\Omega)}$) denoting the pairing between $V^*(\Omega)$ and $V(\Omega)$ (or $\mathscr{H}^*(\Omega)$ and $\mathscr{H}(\Omega)$).

The space $V(\Omega)$ is densely and compactly embedded in $\mathscr{H}(\Omega)$ and $\|v\|_{\mathscr{H}(\Omega)} \leq M\|v\|_{V(\Omega)}$ for all $v \in V(\Omega)$.

Let a set $\mathscr{U}_{ad}(\Omega) \subset \mathscr{U}(\Omega)$ of admissible data be given, where $\mathscr{U}(\Omega)$ is a Banach space. Moreover, $\mathscr{U}_{ad}(\Omega)$ is a compact subset of $\mathscr{U}(\Omega)$. Let $\{\mathscr{A}(o)\}_{o \in \mathscr{U}_{ad}(\Omega)}$ and $\{\mathscr{B}(o)\}_{o \in \mathscr{U}_{ad}(\Omega)}$ be the families of linear continuous operators $\mathscr{A}(o) \in L(V(\Omega), V^*(\Omega))$ and $\mathscr{B}(o) \in L(\mathscr{H}(\Omega), \mathscr{H}^*(\Omega))$ satisfying the following properties

$$(\mathrm{H}\mathscr{A}) \quad \begin{cases} 1^\circ. \{\mathscr{A}(o)\}_{o \in \mathscr{U}_{ad}(\Omega)} \subset \Lambda_{V(\Omega)}(\alpha_{\mathscr{A}}, M_{\mathscr{A}}), \\ 2^\circ. o_n \to o \text{ strongly in } \mathscr{U}(\Omega) \Rightarrow \mathscr{A}(o_n)v \to \mathscr{A}(o)v \\ \text{strongly in } V^*(\Omega) \text{ as } n \to \infty, \\ 3^\circ. \langle \mathscr{A}(o)v, z\rangle_{V(\Omega)} = \langle \mathscr{A}(o)z, v\rangle_{V(\Omega)} \text{ for all } o \in \mathscr{U}_{ad}(\Omega), \\ \quad v, z \in V(\Omega), \end{cases}$$

and

$$(\mathrm{H}\mathscr{B}) \quad \begin{cases} 1^\circ. \{\mathscr{B}(o)\}_{o \in U_{ad}(\Omega)} \subset \Lambda_{\mathscr{H}(\Omega)}(\alpha_{\mathscr{B}}, M_{\mathscr{B}}), \\ 2^\circ. o_n \to o \text{ strongly in } U(\Omega) \Rightarrow \mathscr{B}(o_n)v \to \mathscr{B}(o)v \\ \text{strongly in } \mathscr{H}^*(\Omega) \text{ as } n \to \infty, \\ 3^\circ. \langle \mathscr{B}(o)v, z\rangle_{\mathscr{H}(\Omega)} = \langle \mathscr{B}(o)z, v\rangle_{\mathscr{H}(\Omega)} \text{ for all } o \in \mathscr{U}_{ad}(\Omega), \end{cases}$$

where for a Banach space $\mathscr{W}(\Omega)$ and two positive constants $[M_A, M_B]$ such that $0 < M_A < M_B$, we denote by $\Lambda_{\mathscr{W}(\Omega)}(\alpha, M)$ the set of all operators $\mathcal{A} : \mathscr{W}(\Omega) \to \mathscr{W}^*(\Omega)$ for which the inequalities

$$\begin{cases} M_{\mathscr{A}} \|v - w\|^2_{\mathscr{W}(\Omega)} \le \langle \mathcal{A}v - \mathcal{A}w, v - w\rangle_{\mathscr{W}(\Omega)}, \\ \|Av - Aw\|_{\mathscr{W}^*(\Omega)} \le M_B \|v - w\|_{\mathscr{W}(\Omega)} \end{cases} \tag{4.11}$$

are valid.

Let $\mathscr{K}(\Omega) \subset V(\Omega)$ be a closed convex cone with a vertex zero, $\mathscr{K}(\Omega) \neq \{0\}$. We shall deal with the minimization problem

$$\begin{cases} \lambda_{(*)}(\mathcal{O}) = \underset{\substack{v \in \mathscr{K}(\Omega) \\ v \neq 0}}{\mathrm{Min}} \left(\langle \mathscr{A}(\mathcal{O})v, v\rangle_{V(\Omega)} / \langle \mathscr{B}(\mathcal{O})v, v\rangle_{\mathscr{H}(\Omega)}\right) \\ = \left(\langle \mathscr{A}(\mathcal{O})u(\mathcal{O}), u(\mathcal{O})\rangle_{V(\Omega)} / \langle \mathscr{B}(\mathcal{O})u(\mathcal{O}), u(\mathcal{O})\rangle_{\mathscr{H}(\Omega)}\right). \end{cases} \tag{4.12}$$

Further, we have (due to the existence theorem, see Miersemann, 1981; Mysliński, Sokołowski, 1985)

1°. For every $\mathcal{O} \in \mathscr{U}_{ad}(\Omega)$ there exists a solution $[\lambda_*(\mathcal{O}), u(\mathcal{O})]$ (the state) of the problem (4.12).

2°. The set of elements $\{u(\mathcal{O}_n)\}_{n \in N}$ minimizing the functional (4.12) belongs to $\{\mathscr{K}(\Omega)/\{0\}\}$.

3°. $\lambda_{(*)}(\mathcal{O})$ is the least positive number with a nontrivial solution $u(\mathscr{O})$ of the variational inequality.

$$\begin{cases} [\lambda_{(*)}(\mathcal{O}), u(\mathcal{O})] \in \mathbb{R} \times \{\mathscr{K}(\Omega)/\{0\}\},\ u(\mathcal{O}) \neq 0, \\ \langle \mathscr{A}(\mathcal{O})u(\mathcal{O}), v - u(\mathcal{O})\rangle_{V(\Omega)} \\ \ge \lambda_*(\mathcal{O}) \langle \mathscr{B}(\mathcal{O})u(\mathcal{O}), v - u(\mathcal{O})\rangle_{\mathscr{H}(\Omega)} \\ \text{for all } v \in \{\mathscr{K}(\Omega)/\{0\}. \end{cases} \tag{4.13}$$

4°. The variational inequality (4.13) is equivalent to the following inequality:

$$\begin{cases} \langle \mathscr{A}(\mathcal{O})u(\mathcal{O}), v\rangle_{V(\Omega)} \ge \lambda_{(*)}(\mathcal{O}) \langle \mathscr{B}(\mathcal{O})u(\mathcal{O}), v\rangle_{\mathscr{H}(\Omega)} \text{ for all} \\ v \in \{\mathscr{K}(\Omega)/\{0\} \\ \text{and the following equality for } v = u(\mathcal{O}) \\ \langle \mathscr{A}(\mathcal{O})u(\mathcal{O}), u(\mathcal{O})\rangle_{V(\Omega)} = \lambda_{(*)}(\mathcal{O}) \langle \mathscr{B}(\mathcal{O})u(\mathcal{O}), u(\mathcal{O})\rangle_{\mathscr{H}(\Omega)}. \end{cases} \tag{4.14}$$

Define a goal criterion-functional as

$$\Psi(\mathscr{O}) \to \lambda_{(*)}(\mathcal{O}). \tag{4.15}$$

The MINIMIZATION PROBLEM consists in finding a function $\mathcal{O}_*$, such that

$$\mathcal{O}_* \in \mathscr{U}_{ad}(\Omega), \quad \Psi(\mathcal{O}_*) = \underset{\mathcal{O} \in \mathscr{U}_{ad}(\Omega)}{\mathrm{Inf}} \Psi(\mathcal{O}). \tag{4.16}$$

The problem (4.16) means minimization on $\mathscr{U}_{ad}(\Omega)$ of the first eigenvalue

THEOREM 6 *The minimization problem (4.16) has a solution.*

Proof. Let $\{\mathcal{O}_n\}$ be a sequence such that

$$\begin{cases} \mathcal{O}_n \in \mathscr{U}_{ad}(\Omega), \\ \lim_{n\to\infty} \Psi(\mathcal{O}_n) = \operatorname*{Inf}_{\mathcal{O}\in\mathscr{U}_{ad}(\Omega)} \Psi(\mathcal{O}) \text{ or } \Psi(\mathcal{O}_n) \underset{n\to\infty}{\to} \Psi(\mathcal{O}_*). \end{cases} \tag{4.17}$$

On the other hand, we may find a subsequence $\{\mathcal{O}_{n_k}\}_{k\in N}$ such that

$$\mathcal{O}_{n_k} \to \mathcal{O}_* \text{ strongly in } \mathscr{U}(\Omega) \tag{4.18}$$

The set of elements (and eigenfunctions) $\{u(\mathcal{O}_{n_k})\}_{k\in N}$ minimizing the functional (4.12) has the form $\{\mathscr{K}(\Omega)/\{0\}\}$ and $\lambda_{\langle*\rangle}(\mathcal{O}_{n_k})$ is the smallest positive number with a nontrivial solution of a state variational inequality (4.13) or, equivalently, (4.14).

Let us denote

$$\theta(\mathcal{O}) = (u(\mathcal{O})/\langle\mathscr{B}(\mathcal{O})u(\mathcal{O}), u(\mathcal{O})\rangle^{1/2}_{\mathscr{H}(\Omega)}). \tag{4.19}$$

Then, by virtue of (4.14), we may write

$$\lambda_{\langle*\rangle}(\mathcal{O}_n) = \langle\mathscr{A}(\mathcal{O}_n)\theta(\mathcal{O}_n), \theta(\mathcal{O}_n)\rangle_{V(\Omega)}. \tag{4.20}$$

On the other hand, taking the assumptions $((H\mathscr{A}), 1°)$ and $((H\mathscr{B}), 1°)$ and relation (4.12) we get the upper estimate

$$\lambda_{\langle*\rangle}(\mathcal{O}) \le (\langle\mathscr{A}(\mathcal{O})v, v\rangle_{V(\Omega)}/\langle\mathscr{B}(\mathcal{O})v, v\rangle_{V(\Omega)})$$
$$\le (M_{\mathscr{A}}\|v\|^2_{V(\Omega)}/\alpha_{\mathscr{B}}\|v\|^2_{\mathscr{H}(\Omega)}),$$

for all $\mathcal{O} \in \mathscr{U}_{ad}(\Omega)$, $v \in \{\mathscr{K}(\Omega)/\{0\}\}$.

Then taking into account the uniform coercivity of $\{\mathscr{A}(\mathcal{O})\}$, we have boundedness of the sequence $\{\theta(\mathcal{O}_n)\}_{n\in N}$ in the space $V(\Omega)$. This means that there exists a subsequence $\{\lambda_{\langle*\rangle}(\mathcal{O}_{n_k})\}_{k\in N}$ and the elements $\hat{\theta} \in \{\mathscr{K}(\Omega)/\{0\}\}$ and $\hat{\lambda} \in \mathbb{R}$ such that

$$\begin{cases} \lambda_{\langle*\rangle}(\mathcal{O}_{n_k}) \to \hat{\lambda} \text{ in } \mathbb{R}, \\ \theta(\mathcal{O}_{n_k}) \to \hat{\theta} \text{ weakly in } V(\Omega) \\ \text{or} \\ \theta(\mathcal{O}_{n_k}) \to \hat{\theta} \text{ strongly in } \mathscr{H}(\Omega). \end{cases} \tag{4.21}$$

Notice that the function $\hat{\theta} \neq 0$ as a consequence of the relation

$$\langle\mathscr{B}(\mathcal{O}_*)\hat{\theta}, \hat{\theta}\rangle_{\mathscr{H}(\Omega)} = \lim_{k\to\infty} \langle\mathscr{B}(\mathcal{O}_{n_k})\theta(\mathcal{O}_{n_k}), \theta(\mathcal{O}_{n_k})\rangle_{\mathscr{H}(\Omega)} = 1. \tag{4.22}$$

The equality (4.22) follows from the facts

$$\|\mathscr{B}(\mathcal{O}_n)v_n - \mathscr{B}(\mathcal{O})v\|_{\mathscr{H}^*(\Omega)} \le M_{\mathscr{B}}\|v_n - v\|_{\mathscr{H}(\Omega)}$$

for $n \to \infty$ and for $v_n \to v$ strongly in $\mathscr{H}(\Omega)$.

Further, due to the assumptions $((H\mathscr{A}), 2°, 3°)$ and by (4.21), one has

$$\lim_{k\to\infty} \langle \mathscr{A}(o_{n_k})\theta(o_{n_k}), w\rangle_{V(\Omega)}$$
$$= \lim_{k\to\infty} \langle \mathscr{A}(o_{n_k})w, \theta(o_{n_k})\rangle_{V(\Omega)} = \langle \mathscr{A}(o)w, \hat{\theta}\rangle_{V(\Omega)}$$
$$= \langle \mathscr{A}(o)\hat{\theta}, w\rangle_{V(\Omega)}, \tag{4.23}$$

as $o_n \to o$ strongly in $\mathscr{U}(\Omega)$, $w \in V(\Omega)$.

Thus, in view of (4.18) and (4.23), we may write

$$\mathscr{A}(o_{n_k})\theta(o_{n_k}) \to \mathscr{A}(o_*)\hat{\theta} \text{ weakly in } V^*(\Omega). \tag{4.24}$$

Moreover, by virtue of $((H\mathscr{A}), 1°)$, we obtain

$$\langle \mathscr{A}(o_{n_k})\theta(o_{n_k}) - \mathscr{A}(o_{n_k})\hat{\theta}, \theta(o_{n_k}) - \hat{\theta}\rangle_{V(\Omega)} \geq 0.$$

Hence, passing to the limit, the following relation holds (taking into account $((H\mathscr{A}), 2°)$ and (4.21), (4.24))

$$2 \lim_{k\to\infty} \langle \mathscr{A}(o_{n_k})\theta(o_{n_k}), \hat{\theta}\rangle_{V(\Omega)} \leq \liminf_{k\to\infty} \langle \mathscr{A}(o_{n_k})\theta(o_{n_k}), \theta(o_{n_k})\rangle_{V(\Omega)}$$
$$+ \lim_{k\to\infty} \langle \mathscr{A}(o_{n_k})\hat{\theta}, \hat{\theta}\rangle_{V(\Omega)}.$$

Consequently

$$\liminf_{k\to\infty} \langle \mathscr{A}(o_{n_k})\theta(o_{n_k}), \theta(o_{n_k})\rangle_{V(\Omega)} \geq \langle \mathscr{A}(o_*)\hat{\theta}, \hat{\theta}\rangle_{V(\Omega)}. \tag{4.25}$$

Here, from (4.25), (4.19) and the assumptions $((H\mathscr{A}), 2°)$, $((H\mathscr{B}), 2°)$ we conclude

$$\langle \mathscr{A}(o_*)\hat{\theta}, \hat{\theta}\rangle_{V(\Omega)} \leq \liminf_{k\to\infty} \langle \mathscr{A}(o_{n_k})\theta(o_{n_k}), \theta(o_{n_k})\rangle_{V(\Omega)}$$
$$= \liminf_{k\to\infty} (\langle \mathscr{A}(o_{n_k})u(o_{n_k}), u(o_{n_k})\rangle_{V(\Omega)} / \langle \mathscr{B}(o_{n_k})u(o_{n_k}), u(o_{n_k})\rangle_{\mathscr{H}(\Omega)})$$
$$\leq \lim_{k\to\infty} (\langle \mathscr{A}(o_{n_k})v, v\rangle_{V(\Omega)} / \langle \mathscr{B}(o_{n_k})v, v\rangle_{\mathscr{H}(\Omega)})$$
$$= \langle \mathscr{A}(o_*)v, v\rangle_{V(\Omega)} / \langle \mathscr{B}(o_*)v, v\rangle_{\mathscr{H}(\Omega)}, \tag{4.26}$$

for all $v \in \{\mathscr{K}(\Omega) \setminus \{0\}\}$.

Moreover, due to the variational equality $((4.14), 2°)$ the relation (4.22), (4.12) and the estimate (4.26), we have

$$\hat{\lambda} = \langle \mathscr{A}(o_*)\hat{\theta}, \hat{\theta}\rangle_{V(\Omega)} \tag{4.27}$$

where $\hat{\lambda}$ is the smallest eigenvalue of (4.14) for $o_* \in \mathscr{U}_{ad}(\Omega)$, i.e. $\hat{\lambda} = \hat{\lambda}(o_*)$ and $\hat{\theta} = \hat{\theta}(o_*)$ is the corresponding eigenvector. We show that $\hat{\lambda}(o_*)$ is the smallest

This means that there exists $\mathring{\theta} \in \{\mathscr{K}(\Omega)/\{0\}\}$ satisfying $\langle \mathscr{B}(\mathcal{O}_*)\,\mathring{\theta}, \mathring{\theta}\rangle_{\mathscr{H}(\Omega)} = 1$ such that

$$\langle \mathscr{A}(\mathcal{O}_*)\,\mathring{\theta}, \mathring{\theta}\rangle_{V(\Omega)} < \langle \mathscr{A}(\mathcal{O}_*)\hat{\theta}, \hat{\theta}\rangle_{V(\Omega)} = \hat{\lambda}(\mathcal{O}_*). \tag{4.28}$$

By virtue of $((H\mathscr{A}), 2°, (H\mathscr{B}), 2°)$, we may write

$$\begin{aligned} &\lim_{n\to\infty} \langle \mathscr{B}(\mathcal{O}_n)\,\mathring{\theta}, \mathring{\theta}\rangle_{\mathscr{H}(\Omega)} = \langle \mathscr{B}(\mathcal{O}_*)\,\mathring{\theta}, \mathring{\theta}\rangle_{\mathscr{H}(\Omega)} = 1, \\ &\lim_{n\to\infty} \langle \mathscr{A}(\mathcal{O}_n)\,\mathring{\theta}, \mathring{\theta}\rangle_{V(\Omega)} = \langle \mathscr{A}(\mathcal{O}_*)\,\mathring{\theta}, \mathring{\theta}\rangle_{V(\Omega)}. \end{aligned} \tag{4.29}$$

Then taking into account the relation (4.12), we obtain the estimate

$$\begin{aligned} \lambda_{\langle * \rangle}(\mathcal{O}_n) &= \langle \mathscr{A}(\mathcal{O}_n)\theta(\mathcal{O}_n), \theta(\mathcal{O}_n)\rangle_{V(\Omega)} \\ &\le (\langle \mathscr{A}(\mathcal{O}_n)\,\mathring{\theta}, \mathring{\theta}\rangle_{V(\Omega)} / \langle \mathscr{B}(\mathcal{O}_n)\,\mathring{\theta}, \mathring{\theta}\rangle_{V(\Omega)}). \end{aligned} \tag{4.30}$$

Hence (passing to the limit in (4.30), due to (4.21), (4.27) and (4.29)), we get that

$$\hat{\lambda}(\mathcal{O}_*) = \langle \mathscr{A}(\mathcal{O}_*)\hat{\theta}, \hat{\theta}\rangle_{V(\Omega)} \le \langle \mathscr{A}(\mathcal{O}_*)\,\mathring{\theta}, \mathring{\theta}\rangle_{V(\Omega)}.$$

This implies the contradiction with respect to the estimate (4.28). We conclude: $\lambda_{\langle * \rangle}(\mathcal{O}_*) = \hat{\lambda}(\mathcal{O}_*)$ and $\theta(\mathcal{O}_*) = \hat{\theta}(\mathcal{O}_*)$.

In view of $((H\mathscr{A}), 1°)$ and (4.18) we may write

$$\begin{aligned} &\alpha_{\mathscr{A}} \|\theta(\mathcal{O}_n) - \theta(\mathcal{O}_*)\|^2_{V(\Omega)} \\ &\le \langle \mathscr{A}(\mathcal{O}_n)(\theta(\mathcal{O}_n) - \theta(\mathcal{O}_*)), \theta(\mathcal{O}_n) - \theta(\mathcal{O}_*)\rangle_{V(\Omega)}. \end{aligned} \tag{4.31}$$

Moreover, due to (4.20), (4.21), (4.24) and $((H\mathscr{A}), 2°)$, (4.31), we get

$$\begin{aligned} &\lim_{n\to\infty} \alpha_{\mathscr{A}} \|\theta(\mathcal{O}_n) - \theta(\mathcal{O}_*)\|^2_{V(\Omega)} \\ &\le \lim_{n\to\infty} \langle \mathscr{A}(\mathcal{O}_n)(\theta(\mathcal{O}_n) - \theta(\mathcal{O}_*)), \theta(\mathcal{O}_n) - \theta(\mathcal{O}_*)\rangle_{V(\Omega)} \\ &= \lim_{n\to\infty} \{\langle \mathscr{A}(\mathcal{O}_n)\theta(\mathcal{O}_n), \theta(\mathcal{O}_n)\rangle_{V(\Omega)} - 2\langle \mathscr{A}(\mathcal{O}_n)\theta(\mathcal{O}_n), \theta(\mathcal{O}_*)\rangle_{V(\Omega)} \\ &+ \langle \mathscr{A}(\mathcal{O}_n)\theta(\mathcal{O}_*), \theta(\mathcal{O}_*)\rangle_{V(\Omega)}\} = [\lambda_{\langle * \rangle}(\mathcal{O}_*) - 2\lambda_{\langle * \rangle}(\mathcal{O}_*) + \lambda_{\langle * \rangle}(\mathcal{O}_*)] = 0. \end{aligned}$$

Hence, we conclude

$$\theta(\mathcal{O}_n) \to \theta(\mathcal{O}_*) \text{ strongly in } V(\Omega). \tag{4.32}$$

Further, we have the relation (in view of $((4.21), 1°)$)

$$\lambda_{\langle * \rangle}(\mathcal{O}_n) \to \lambda_{\langle * \rangle}(\mathcal{O}_*). \tag{4.33}$$

Thus, taking into consideration (4.17), (4.18), (4.20) and (4.27), (4.32), (4.33), we arrive at

$$\Psi(\mathcal{O}_*) = \operatorname*{Arg\,Min}_{\mathcal{O} \in \mathscr{U}_{ad}(\Omega)} \Psi(\mathcal{O}),$$

Let us apply Theorem 6 to prove the existence of a solution of the eigenvalue problem (4.9) with respect to uncertain data.

LEMMA 9 *The set $\mathscr{K}(\Omega)$ defined in (4.6) is a non-empty, closed and convex subset of $V(\Omega)$.*

Proof. Since $0 \in \mathscr{K}(\Omega)$ (actually, $H_0^1(\Omega) \subset \mathscr{K}(\Omega)$), the set $\mathscr{K}(\Omega)$ is non-empty. The convexity of $\mathscr{K}(\Omega)$ is obvious. Moreover, if $\{v_n\}_{n\in N} \subset \mathscr{K}(\Omega)$ and $v_n \to v$ strongly in $H^1(\Omega)$, then one has $\mathscr{M}_0 v_n \to \mathscr{M}_0 v$, since $\mathscr{M}_0 : V(\Omega) \to L_2(\partial\Omega_{\text{CONTACT}})$ is continuous. On the other hand, $v_n \in \mathscr{K}(\Omega)$, which means that $\mathscr{M}_0 v_n \geq 0$ a.e. on $\partial\Omega_{\text{CONTACT}}$. Therefore, $\mathscr{M}_0 v \geq 0$ a.e. on $\partial\Omega_{\text{CONTACT}}$. Hence, $v \in \mathscr{K}(\Omega)$, which shows that $\mathscr{K}(\Omega)$ is closed.

LEMMA 10 *The family of operators $\{\mathscr{A}(\mathscr{O}_n)\}_{n\in N}$ and $\{\mathscr{B}(\mathscr{O}_n)\}_{n\in N}$, satisfies $\mathscr{O}_n \in \mathscr{U}_{ad}(\Omega)$ the assumptions $(H\mathscr{A})$ and $(H\mathscr{B})$.*

Proof. It is readily seen that (by (4.7))

$$\begin{aligned}\langle \mathscr{A}(\mathscr{O}, v, v\rangle_{V(\Omega)} &\geq KG\mathscr{O}_{\text{MIN}} \int_\Omega |\mathbf{grad}\, v|^2 d\Omega \\ &\geq KG\mathscr{O}_{\text{MIN}} \text{const}_{\text{F}} \|v\|^2_{V(\Omega)},\end{aligned} \tag{4.34}$$

holds for all $v \in V(\Omega)$, since we can employ the Fiedrichs-Poincaré inequality. Then, by virtue of (4.34), we may write

$$\langle \mathscr{A}(\mathscr{O})v, v\rangle_{V(\Omega)} \geq \text{const.} \|v\|^2_{V(\Omega)},$$

for all $\mathscr{O} \in \mathscr{U}_{ad}(\Omega)$, $v \in V(\Omega)$.

Next, we have

$$\begin{aligned}&|\langle \mathscr{A}(\mathscr{O})v, w\rangle_{V(\Omega)} - \langle \mathscr{A}(\mathscr{O})z, w\rangle_{V(\Omega)}| \\ &= \left| \int_\Omega (KG\mathscr{O}\mathbf{grad}\,(v - z).\mathbf{grad}\, w) d\Omega \right| \\ &\leq KG\mathscr{O}_{\text{MAX}} \|v - z\|_{V(\Omega)} \|w\|_{V(\Omega)}.\end{aligned} \tag{4.35}$$

As a consequence, the assumptions $((H\mathscr{A}), 1^\circ, 3^\circ)$ are satisfied. To verify $((H\mathscr{A}), 2^\circ)$, we write

$$\begin{aligned}|\langle \mathscr{A}(\mathscr{O}_n)v - \mathscr{A}(\mathscr{O})v, w\rangle_{V(\Omega)}| &\leq \left| \int_\Omega KG(\mathscr{O}_n - \mathscr{O})\mathbf{grad}\, v.\mathbf{grad}\, w d\Omega \right| \\ &\leq KG\|\mathscr{O}_n - \mathscr{O}\|_{L_\infty(\Omega)} \|v\|_{V(\Omega)} \|w\|_{V(\Omega)}.\end{aligned} \tag{4.36}$$

Hence, one has

$$\|\mathscr{A}(\mathscr{O}_n)v - \mathscr{A}(\mathscr{O})v\|_{V^*(\Omega)} \leq KG\|\mathscr{O}_n - \mathscr{O}\|_{L_\infty(\Omega)} \|v\|_{V(\Omega)} \to 0,$$

as $\mathscr{O}_n \to \mathscr{O}$ strongly in $L_\infty(\Omega)$.

LEMMA 11 *Let $\mathcal{O}_n \in \mathcal{U}_{ad}(\Omega)$, $\mathcal{O}_n \to \mathcal{O}$ strongly in $\mathcal{U}(\Omega)$. Then, we have*

$$\Psi(\mathcal{O}_n, u(\mathcal{O}_n)) \to \Psi(\mathcal{O}, u(\mathcal{O})) \text{ as } n \to \infty. \tag{4.37}$$

Proof. In view of Lemma 10 and Theorem 6, we may write

$$\begin{cases} \lambda_{\langle * \rangle}(\mathcal{O}_n) \to \lambda_{\langle * \rangle}(\mathcal{O}) \text{ in } \mathbb{R}, \\ u(\mathcal{O}_n) \to u(\mathcal{O}) \text{ strongly in } V(\Omega). \end{cases}$$

As a consequence, (4.37) follows. ■

THEOREM 7 *The minimization problem (4.9) has at least one solution.*

Proof. The functional criterion $J(\mathcal{O}) \equiv \Psi(\mathcal{O}, u(\mathcal{O}))$ is continuous on the set $\mathcal{U}_{ad}(\Omega)$ by virtue of Lemma 11. Since the set $\mathcal{U}_{ad}(\Omega)$ is compact in $\mathcal{U}(\Omega)$, there exists a minimizer $\mathcal{O}_*$ in $\mathcal{U}_{ad}(\Omega)$.

5. Finite elements approximation of an eigenvalue problem

The reliable solution (alias worst scenario method) of the eigenvalue problem have to be solved approximately. To this end, we propose to employ the simplest kind of finite elements, namely piecewise linear functions over triangulations. We restrict ourselves to particular domains, namely we suppose that Ω is polygonal. By $\mathcal{T}_h$ we denote a triangulation of Ω which consists of a finite number of closed triangles T.

Here we use again the finite element space $V_h(\Omega) = V(\Omega) \cap \mathcal{H}(\Omega)$ and $\mathcal{U}_{ad,\langle h \rangle}(\Omega) = \mathcal{U}_{ad},(\Omega) \cap \mathcal{H}_h(\Omega)$, respectively. Hence, we have to assume that the triangulations $\mathcal{T}_h$ are consistent with the boundaries $\partial \mathcal{Z}_k$, $k = 1, 2, \ldots, N$, which play role in definition of $\mathcal{U}_{ad}(\Omega)$. As with the partition of the boundary $\partial\Omega = \overline{\partial\Omega}_{\text{DISPLACEMENT}} \cup \overline{\partial\Omega}_{\text{CONTACT}}$, i.e. the number of points $\overline{\partial\Omega}_{\text{DISPLACEMENT}} \cap \overline{\partial\Omega}_{\text{CONTACT}}$ is finite and every point of this kind coincides with a node of $\mathcal{T}_h$.

Thus, we may write

$$\partial\Omega_{\text{CONTACT}} = \bigcup_{j=1}^{N(h)} \overline{A_{j-1}A_j}$$

Then, the set $\mathcal{K}_h(\Omega)$ is defined by

$$\mathcal{K}_h(\Omega) = \Big\{ v_h \in V_h(\Omega) : v_h(A_j) \geq 0 \text{ for all nodes } A \text{ such that}$$

$$\partial\Omega_{\text{CONTACT}} = \bigcup_{j=1}^{N(h)} \overline{A_{j-1}A_j}$$

Moreover, instead of the criterion Ψ we introduce

$$\Psi_h(\mathcal{O}_h, v_h) := \lambda_{*\langle h\rangle}(\mathcal{O}_h).$$

We solve the following APPROXIMATE MINIMIZATION PROBLEM. Find

$$\mathcal{O}_{*\langle h\rangle} = \operatorname*{Arg\,Min}_{O_h \in U_{ad,\langle h\rangle}(\Omega)} \Psi_{\langle h\rangle}(\mathcal{O}_h, u(\mathcal{O}_h)) \tag{5.1}$$

where $[u_h(\mathcal{O}_h), \lambda_{*\langle h\rangle}(\mathcal{O}_h)] \in \{\mathcal{K}_h(\Omega)\setminus\{0\}\} \times \mathbb{R}$ denotes the eigensolution of THE APPROXIMATE STATE EIGENVALUE PROBLEM

$$\begin{aligned} &\langle \mathscr{A}(\mathcal{O}_h)u_h(\mathcal{O}_h), v_h - u_h(\mathcal{O}_h)\rangle_{V(\Omega)} \\ &\geq \lambda_{*\langle h\rangle}(\mathcal{O}_h)\langle \mathscr{B}(\mathcal{O}_h)u_h(\mathcal{O}_h), v_h - u_h(\mathcal{O}_h)\rangle_{\mathscr{H}(\Omega)}, \end{aligned} \tag{5.2}$$

for all $v_h \in \{\mathcal{K}_h(\Omega) \setminus \{0\}\}$.

Thus, a couple of the eigensolutions solving the finite dimensional minimization problem is

$$\begin{aligned} \lambda_{*\langle h\rangle}(\mathcal{O}_h) &= \operatorname*{Min}_{v_h \in \{\mathcal{K}_h(\Omega)\setminus\{0\}\}} (\langle \mathscr{A}(\mathcal{O}_h)v_h, v_h\rangle_{V(\Omega)} / \langle \mathscr{B}(\mathcal{O}_h)v_h, v_h\rangle_{\mathscr{H}(\Omega)}) \\ &= (\langle \mathscr{A}(\mathcal{O}_h)u_h(\mathcal{O}_h), u_h(\mathcal{O}_h)\rangle_{V(\Omega)} / \langle \mathscr{B}(\mathcal{O}_h)u_h(\mathcal{O}_h), u_h(\mathcal{O}_h)\rangle_{\mathscr{H}(\Omega)}). \end{aligned} \tag{5.3}$$

On the other hand, the approximate minimization problem (5.1), taking into account (5.3), is characterized by the relation

$$\begin{aligned} \lambda_{*\langle h\rangle}(\mathcal{O}_{*\langle h\rangle}) &= \operatorname*{Min}_{\mathcal{O}_h \in \mathscr{U}_{ad,\langle h\rangle}(\Omega)} \lambda_{*\langle h\rangle}(\mathcal{O}_h) \\ &= \operatorname*{Min}_{\mathcal{O}_h \in \mathscr{U}_{ad,\langle h\rangle}(\Omega)} \operatorname*{Min}_{v_h \in \{\mathcal{K}_h(\Omega)\setminus\{0\}\}} (\langle \mathscr{A}(\mathcal{O}_h)v_h, v_h\rangle_{V(\Omega)} \\ &\quad / \langle \mathscr{B}(\mathcal{O}_h)v_h, v_h\rangle_{\mathscr{H}(\Omega)}). \end{aligned} \tag{5.4}$$

LEMMA 12 *The eigenvalue problem (5.2) has a unique solution* $[u_h(\mathcal{O}_h), \lambda_{*\langle h\rangle}(\mathcal{O}_h)]$ *for any h sufficiently small. The approximate minimization problem (5.1) has at least one solution for any h sufficiently small.*

Proof. The existence of the approximate minimizer $\mathcal{O}_{*\langle h\rangle}$ in $\mathscr{U}_{ad\langle h\rangle}(\Omega)$ and the corresponding couples $[u_h(\mathcal{O}_{*\langle h\rangle}), \lambda_{*\langle h\rangle}(\mathcal{O}_{*\langle h\rangle})] \in \{\mathcal{K}_h(\Omega) \setminus \{0\}\} \times \mathbb{R}$ is assured due to Theorem 7. ■

LEMMA 13 *For any* $\mathcal{O} \in \mathscr{U}_{ad}(\Omega)$ *and any sequence* $\{h_n\}_{n\in N}$, $h_n \to 0_+$ *there exists a sequence such that* $\{\mathcal{O}_{h_n}\}_{n\in N} \in \mathscr{U}_{ad,\langle h\rangle}(\Omega)$ *and* $\mathcal{O}_{h_n} \to \mathcal{O}$ *strongly in* $u(\Omega)$ $(= (\prod_{k=1}^{N} C(\overline{\mathscr{Z}}_k)))$.

Proof. Let us consider the restriction $\mathcal{O}_k = \mathcal{O}|_{\mathscr{Z}_k}$ of any $\mathcal{O} \in \mathscr{U}_{ad}(\Omega)$ and define: $\mathcal{O}_{h_n} = \Pi_{h_n}\mathcal{O}_{\varepsilon_n}$, where Π_{h_n} is the linear Lagrange interpolant over $\mathscr{T}_h$ and

We have

$$\|\partial\mathcal{O}_{\varepsilon_n}/\partial x_i\|_{L_\infty(\mathscr{T}_k)} \le \varepsilon_n\|\partial\mathcal{O}_0/\partial x_i\|_{L_\infty(\Omega)} + (1-\varepsilon_n)\|\partial\mathcal{O}_k/\partial x_i\|_{L_\infty(\Omega)}$$
$$\le \mathrm{const}_{2\langle\mathcal{O}\rangle}, \quad i = 1,2,, \tag{5.5}$$

by definition of $\mathscr{U}_{ad}(\Omega)$ and $\mathcal{O}_0$. Since

$$\|\mathcal{O}_{\varepsilon_n}\|_{L_\infty(\mathscr{T}_k)} \le \varepsilon_n\|\mathcal{O}_0\|_{L_\infty(\Omega)} + (1-\varepsilon_n)\|\mathcal{O}_k\|_{L_\infty(\Omega)}$$
$$\le \max\{\|\mathcal{O}_0\|_{L_\infty(\Omega)}, \|\mathcal{O}_k\|_{L_\infty(\Omega)}\} \equiv \mathrm{const}_{\langle 3\rangle},$$

holds, we obtain for all ε_n

$$\|\mathcal{O}_{\varepsilon_n}\|_{H^1_\infty(\mathscr{T}_k)} \le \mathrm{const}_{\langle 3\rangle} + 2\mathrm{const}_{1\langle\mathcal{O}\rangle} \equiv \mathrm{const}_{\langle 4\rangle}.$$

Then, taking into account the estimate

$$\|\mathcal{Q} - \Pi_{h_n}\mathcal{Q}\|_{L_\infty(\mathscr{T}_k)} \le Mh_n\|\mathcal{Q}\|_{H^1_\infty(\mathscr{T}_k)},$$

we may write

$$\|\mathcal{O}_{h_n} - \mathcal{O}_0\|_{L_\infty(\mathscr{T}_k)} \le \|\Pi_h\mathcal{O}_{\varepsilon_n} - \mathcal{O}_{\varepsilon_n}\|_{L_\infty(\Omega)} + \|\mathcal{O}_{\varepsilon_n} - \mathcal{O}_0\|_{L_\infty(\Omega)}$$
$$\le M\mathrm{const}_{\langle 4\rangle}h_n + (1-\varepsilon_n)\|\mathcal{O}_k - \mathcal{O}_0\|_{L_\infty(\Omega)}$$
$$\le M\mathrm{const}_{\langle 4\rangle}h_n + (1-\varepsilon_n)\,\mathrm{const}_{1\langle\mathcal{O}\rangle} \le \mathrm{const}_{1\langle\mathcal{O}\rangle}, \tag{5.6}$$

if

$$M\,\mathrm{const}_{\langle 4\rangle}h_n \le \mathrm{const}_{1\langle\mathcal{O}\rangle}\varepsilon_n. \tag{5.7}$$

Further, let $\overline{PQ} \subset T \subset \overline{\mathscr{T}}_k$ be a straight-line segment of the length L, parallel to the x_i-axis. Then one has

$$|\partial\Pi_h\mathcal{O}_{\varepsilon_n}/\partial x_i| = |L^{-1}\int_P^Q(\partial\mathcal{O}_{\varepsilon_n}/\partial x_i)dx_i| \le L^{-1}\int_P^Q|\partial\mathcal{O}_{\varepsilon_n}/\partial x_i|dx_i$$
$$\le \mathrm{const}_{2\langle\mathcal{O}\rangle},$$

following from (5.5), so that

$$\|\partial\mathcal{O}_{h_n}/\partial x_i\|_{L_\infty(\mathscr{T}_k)} \le \mathrm{const}_{2\langle\mathcal{O}\rangle}. \tag{5.8}$$

Thus, we have

$$\|\mathcal{O}_{h_n} - \mathcal{O}_0\|_{L_\infty(\mathscr{T}_k)} \le \|\Pi_{h_n}\mathcal{O}_{\varepsilon_n} - \mathcal{O}_{\varepsilon_n}\|_{L_\infty(\Omega)} + \|\mathcal{O}_{\varepsilon_n} - \mathcal{O}_k\|_{L_\infty(\Omega)}$$
$$\le M\mathrm{const}_{\langle 4\rangle}h_n + \varepsilon_n\|\mathcal{O}_0 - \mathcal{O}_k\|_{L_\infty(\Omega)}$$
$$\le M\mathrm{const}_{\langle 4\rangle}h_n + \varepsilon_n\mathrm{const}_{1\langle\mathcal{O}\rangle} \to 0, \tag{5.9}$$

as $h_n \to 0_+$, $\varepsilon_n \to 0_+$.

Hence, in view of (5.6) to (5.9), we can find a sequence $\{\mathcal{O}_{h_n}\}_{n\in N}$, $h_n \to 0_+$ such that $\mathcal{O}_{h_n} \in \mathscr{U}_{ad.\langle h\rangle}(\Omega)$ and $\mathcal{O}_{h_n} \to \mathcal{O}$ strongly in $(\Pi_{k=1}^N C(\overline{\mathscr{T}}_k))$, concluding

In the following, we introduce the sphere

$$\mathscr{S}(\Omega) = \{v \in \mathscr{H}(\Omega) : \langle \mathscr{B}(\mathscr{O})v, v\rangle_{\mathscr{H}(\Omega)} = 1\}$$

and denote by $o_h(\mathscr{O}) \in \mathscr{K}_h(\Omega) \cap \mathscr{S}(\Omega)$ and $o(\mathscr{O}) \in \mathscr{K}(\Omega) \cap \mathscr{S}(\Omega)$ the normalized functions fulfilling

$$\begin{cases} \lambda_{*\langle h\rangle}(\mathscr{O}_h) = \langle \mathscr{A}(\mathscr{O}_h) o_h(\mathscr{O}_h), o_h(\mathscr{O}_h)\rangle_{V(\Omega)}, \\ \lambda_*(\mathscr{O}) = \langle \mathscr{A}(\mathscr{O}) o(\mathscr{O}), o(\mathscr{O})\rangle_{V(\Omega)}. \end{cases} \tag{5.10}$$

CONVERGENCE RESULTS

Let us study the convergence of finite-element approximations when the mesh size tends to zero. First of all, we have to establish the following,

LEMMA 14 *Let $\mathscr{O}_{h_n} \in \mathscr{U}_{ad,\langle h\rangle}(\Omega)$, $\mathscr{O}_{h_n} \to \mathscr{O}$ strongly in $\mathscr{U}(\Omega)$, as $h_n \to 0_+$. Then one has*

$$\begin{cases} \lambda_{*\langle h_n\rangle}(\mathscr{O}_{h_n}) \to \lambda_*(\mathscr{O}) \text{ in } \mathbb{R}, \\ o_{h_n}(\mathscr{O}_{h_n}) \to o(\mathscr{O}) \text{ strongly in } V(\Omega) \text{ as } h_n \to 0_+. \end{cases} \tag{5.11}$$

Proof. Let a couple (eigenfunction and eigenvalue) $[o(\mathscr{O}), \lambda_*(\mathscr{O})] \in \mathscr{K}(\Omega) \cap \mathscr{S}(\Omega)) \times \mathbb{R}$ be a solution of the following eigenvalue problem

$$\begin{aligned} \lambda_*(\mathscr{O}) &= \operatorname*{Min}_{v \in \{\mathscr{K}(\Omega)\setminus\{0\}\}} (\langle \mathscr{A}(\mathscr{O})v, v\rangle_{V(\Omega)} / \langle B(\mathscr{O})v, v\rangle_{\mathcal{H}(\Omega)}) \\ &= \langle \mathscr{A}(\mathscr{O}) o(\mathscr{O}), o(\mathscr{O})\rangle_{V(\Omega)}. \end{aligned} \tag{5.12}$$

By virtue of (3.10) (we have $\mathscr{K}_h(\Omega) \subset \mathscr{K}(\Omega)$), if element $v \in \{\mathscr{K}(\Omega) \setminus \{0\}\}$ there exists a sequence $\{v_{h_n}\}_{n\in N}$ with $v_{h_n} \in \{\mathscr{K}_{h_n}(\Omega) \setminus \{0\}\}$ such that

$$v_{h_n} \to v \text{ strongly in } V(\Omega) \text{ as } h_n \to 0_+. \tag{5.13}$$

Due to Lemma 13, relation (5.13) and $((H\mathscr{A}), 2°)$, $((H\mathscr{B}), 2°)$ (by passing to the limit) we obtain

$$\begin{aligned} &(\langle \mathscr{A}(\mathscr{O}_{h_n}) v_{h_n}, v_{h_n}\rangle_{V(\Omega)} / \langle \mathscr{B}(\mathscr{O}_{h_n}) v_{h_n}, v_{h_n}\rangle_{\mathscr{H}(\Omega)}) \\ &\to (\langle \mathscr{A}(\mathscr{O})v, v\rangle_{V(\Omega)} / \langle \mathscr{B}(\mathscr{O})v, v\rangle_{\mathscr{H}(\Omega)}) \end{aligned} \tag{5.14}$$

Here, we deduce that the sequence $\{\lambda_{*(h_n)}(\mathcal{O}_{h_n})\}_{n\in N}$ is bounded (in view of (5.3) and $((H\mathscr{A}), 1^\circ)$), $((H\mathscr{B}), 1^\circ)$) and contains the convergent subsequence such that

$$\lambda_{*(h_{n_k})}(\mathcal{O}_{h_{n_k}}) \to \hat{\lambda}_* \text{ for } h_{n_k} \to 0_+. \tag{5.15}$$

Hence the sequence $\{o_{h_{n_k}}(\mathcal{O}_{h_{n_k}})\}_{k\in N}$ is bounded. Then, by virtue of the above assertion for the subsequences $\{\lambda_{*(h_{n_\mathcal{O}})}(\mathcal{O}_{h_{n_\mathcal{O}}})\}_{\mathcal{O}\in N}$ and $\{o_{h_{n_\mathcal{O}}}(\mathcal{O}_{h_{n_\mathcal{O}}})\}_{\mathcal{O}\in N}$ we get

$$o_{h_{n_\mathcal{O}}}(\mathcal{O}_{h_{n_\mathcal{O}}}) \to \hat{o} \text{ weakly in } V(\Omega). \tag{5.16}$$

Next, taking into consideration (5.13) for the sequence $\{a_{h_{n_\mathcal{O}}}\}_{\mathcal{O}\in N}$ with $a_{h_{n_\mathcal{O}}} \in \mathscr{K}_{h_{n_\mathcal{O}}}(\Omega) \cap \mathscr{S}(\Omega)$, we may write

$$a_{h_{n_\mathcal{O}}} \to o(\mathcal{O}) \text{ strongly in } V(\Omega). \tag{5.17}$$

The functional $v \to \langle \mathscr{A}(\mathscr{G})v, v\rangle_{V(\Omega)}$ is weakly lower semicontinuous on $V(\Omega)$ for any $\mathscr{G} \in \mathscr{U}_{ad}(\Omega)$.

Consequently, since $\mathcal{O} \in \mathscr{U}_{ad}(\Omega)$, we may write

$$\liminf_{h_{n_\mathcal{O}}\to 0_+} \langle \mathscr{A}(\mathcal{O})o_{h_{n_\mathcal{O}}}(\mathcal{O}_{h_{n_\mathcal{O}}}), o_{h_{n_\mathcal{O}}}(\mathcal{O}_{h_{n_\mathcal{O}}})\rangle_{V(\Omega)} \geq \langle \mathscr{A}(\mathcal{O})\hat{o}, \hat{o}\rangle_{V(\Omega)}. \tag{5.18}$$

Moreover, we have (in view of (5.16))

$$\begin{aligned}
&|\langle \mathscr{A}(\mathcal{O}_{h_{n_\mathcal{O}}})o_{h_{n_\mathcal{O}}}(\mathcal{O}_{h_{n_\mathcal{O}}}), o_{h_{n_\mathcal{O}}}(\mathcal{O}_{h_{n_\mathcal{O}}})\rangle_{V(\Omega)} \\
&- \langle \mathscr{A}(\mathcal{O})o_{h_{n_\mathcal{O}}}(\mathcal{O}_{h_{n_\mathcal{O}}}), o_{h_{n_\mathcal{O}}}(\mathcal{O}_{h_{n_\mathcal{O}}})\rangle_{V(\Omega)}| \\
&\leq \text{constant } \|\mathcal{O}_{h_{n_\mathcal{O}}} - \mathcal{O}\|_{L_\infty(\Omega)} \|o_{h_{n_\mathcal{O}}}(\mathcal{O}_{h_{n_\mathcal{O}}})\|^2_{V(\Omega)} \to 0.
\end{aligned} \tag{5.19}$$

Further due to (5.10), (5.15) and (5.16), (5.18), (5.19) we arrive at the relations

$$\begin{aligned}
\hat{\lambda}_* &= \lim_{h_{n_\mathcal{O}}\to 0_+} \lambda_{*(h_{n_\mathcal{O}})}(\mathcal{O}_{h_{n_\mathcal{O}}}) \\
&= \lim_{h_{n_\mathcal{O}}\to 0_+} \langle \mathscr{A}(\mathcal{O}_{h_{n_\mathcal{O}}})o_{h_{n_\mathcal{O}}}(\mathcal{O}_{h_{n_\mathcal{O}}}), o_{h_{n_\mathcal{O}}}(\mathcal{O}_{h_{n_\mathcal{O}}})\rangle_{V(\Omega)} \\
&\geq \liminf_{h_{n_\mathcal{O}}\to 0_+} \langle \mathscr{A}(\mathcal{O}_{h_{n_\mathcal{O}}})o_{h_{n_\mathcal{O}}}(\mathcal{O}_{h_{n_\mathcal{O}}}), o_{h_{n_\mathcal{O}}}(\mathcal{O}_{h_{n_\mathcal{O}}})\rangle_{V(\Omega)} \\
&= \liminf_{h_{n_\mathcal{O}}\to 0_+} (\langle \mathscr{A}(\mathcal{O})o_{h_{n_\mathcal{O}}}(\mathcal{O}_{h_{n_\mathcal{O}}}), o_{h_{n_\mathcal{O}}}(\mathcal{O}_{h_{n_\mathcal{O}}})\rangle_{V(\Omega)} \\
&+[\langle \mathscr{A}(\mathcal{O}_{h_{n_\mathcal{O}}})o_{h_{n_\mathcal{O}}}(\mathcal{O}_{h_{n_\mathcal{O}}}), o_{h_{n_\mathcal{O}}}(\mathcal{O}_{h_{n_\mathcal{O}}})\rangle_{V(\Omega)} \\
&-\langle \mathscr{A}(\mathcal{O})o_{h_{n_\mathcal{O}}}(\mathcal{O}_{h_{n_\mathcal{O}}}), o_{h_{n_\mathcal{O}}}(\mathcal{O}_{h_{n_\mathcal{O}}})\rangle_{V(\Omega)}]) \\
&\geq \liminf_{h_{n_\mathcal{O}}\to 0_+} \langle \mathscr{A}(\mathcal{O})o_{h_{n_\mathcal{O}}}(\mathcal{O}_{h_{n_\mathcal{O}}}), o_{h_{n_\mathcal{O}}}(\mathcal{O}_{h_{n_\mathcal{O}}})\rangle_{V(\Omega)} \\
&\geq \langle \mathscr{A}(\mathcal{O})\hat{o}, \hat{o}\rangle_{V(\Omega)} \geq \lambda_*(\mathcal{O}),
\end{aligned} \tag{5.20}$$

On the other hand we deduce

$$\begin{aligned}\hat{\lambda}_* &= \lim_{h_{n_{k_{\mathcal{O}}}} \to 0_+} \langle \mathscr{A}(\mathcal{O}_{h_{n_\mathcal{O}}}) o_{h_{n_\mathcal{O}}}(\mathcal{O}_{h_{n_\mathcal{O}}}), o_{h_{n_\mathcal{O}}}(\mathcal{O}_{h_{n_\mathcal{O}}})\rangle_{V(\Omega)} \\ &\leq \lim_{h_{n_{k_{\mathcal{O}}}} \to 0_+} (\langle \mathscr{A}(\mathcal{O}_{h_{n_\mathcal{O}}}) a_{h_{n_\mathcal{O}}}, a_{h_{n_\mathcal{O}}}\rangle_{V(\Omega)} / \langle \mathscr{B}(\mathcal{O}_{h_{n_\mathcal{O}}}) a_{h_{n_\mathcal{O}}}, a_{h_{n_\mathcal{O}}}\rangle_{\mathscr{H}(\Omega)}) \\ &= \langle \mathscr{A}(\mathcal{O}) o(\mathcal{O}), o(\mathcal{O})\rangle_{V(\Omega)} = \lambda_*(\mathcal{O}),\end{aligned} \tag{5.21}$$

(in view of $((H\mathscr{A}), 2°))$, $((H\mathscr{B}), 2°))$ and (5.10), (5.15), (5.17)).

Thus, from the estimate (5.20) and (5.21) we conclude that

$$\lambda_*(\mathcal{O}) \leq \hat{\lambda}_* \leq \lambda_*(\mathcal{O}). \tag{5.22}$$

Hence, due to (5.10) and (5.22) we have

$$\left\{ \begin{array}{l} \hat{\lambda}_* = \lambda_*(\mathcal{O}) \\ \text{and} \\ \hat{o} = o(\mathcal{O}). \end{array} \right.$$

Further, introduce the variational inequality

$$\begin{aligned}&\langle \mathscr{A}(\mathcal{O}) o(\mathcal{O}), v - o(\mathcal{O})\rangle_{V(\Omega)} \\ &\geq \lambda_*(\mathcal{O}) \langle \mathscr{B}(\mathcal{O}) o(\mathcal{O}), v - o(\mathcal{O})\rangle_{\mathscr{H}(\Omega)} \text{ for all } v \in \mathscr{K}(\Omega),\end{aligned} \tag{5.23}$$

and

$$\begin{aligned}&\langle \mathscr{A}(\mathcal{O}_{h_n}) o_{h_n}(\mathcal{O}_{h_n}), v_{h_n} - o_{h_n}(\mathcal{O}_{h_n})\rangle_{V(\Omega)} \\ &\geq \lambda_{*\langle h_n\rangle}(\mathcal{O}_{h_n}) \langle \mathscr{B}(\mathcal{O}_{h_n}) o_{h_n}(\mathcal{O}_{h_n}), v_{h_n} - o_{h_n}(\mathcal{O}_{h_n})\rangle_{\mathscr{H}(\Omega)},\end{aligned} \tag{5.24}$$

for all $v_{h_n} \in \mathscr{K}_{h_n}(\Omega)$.

In the following, we substitute $v := o_{h_n}(\mathcal{O}_{h_n})$ in (5.23) and $v_{h_n} := a_{h_n}$ in (5.24). Hence after adding the inequalities, we may write

$$\begin{aligned}&\langle [\mathscr{A}(\mathcal{O}_{h_n}) - \mathscr{A}(\mathcal{O})] o_{h_n}(\mathcal{O}_{h_n}), a_{h_n} - o_{h_n}(\mathcal{O}_{h_n})\rangle_{V(\Omega)} \\ &+ \langle \mathscr{A}(\mathcal{O}) o_{h_n}(\mathcal{O}_{h_n}), a_{h_n} - o(\mathcal{O})\rangle_{V(\Omega)} \\ &+ \langle \mathscr{A}(\mathcal{O})[o(\mathcal{O}) - o_{h_n}(\mathcal{O}_{h_n})], o_{h_n}(\mathcal{O}_{h_n}) - o(\mathcal{O})\rangle_{V(\Omega)} \\ &\geq \lambda_*(\mathcal{O}) \langle \mathscr{B}(\mathcal{O}) o(\mathcal{O}), o_{h_n}(\mathcal{O}_{h_n}) - o(\mathcal{O})\rangle_{\mathscr{H}(\Omega)} \\ &+ \lambda_{*\langle h_n\rangle}(\mathcal{O}_{h_n}) \langle \mathscr{B}(\mathcal{O}_{h_n}) o_{h_n}(\mathcal{O}_{h_n}), a_{h_n} - o_{h_n}(\mathcal{O}_{h_n})\rangle_{\mathscr{H}(\Omega)}.\end{aligned}$$

Then, taking into account the coercivity of the operator $\mathscr{A}(\mathcal{O})$, we get the estimate

$$\begin{aligned}&\alpha_{\mathscr{A}} \| o_{h_n}(\mathcal{O}_{h_n}) - o(\mathcal{O})\|^2_{V(\Omega)} \\ &\leq \langle [\mathscr{A}(\mathcal{O}_{h_n}) - \mathscr{A}(\mathcal{O})] o_{h_n}(\mathcal{O}_{h_n}), a_{h_n} - o_{h_n}(\mathcal{O}_{h_n})\rangle_{V(\Omega)} \\ &+ \langle \mathscr{A}(\mathcal{O}) o_{h_n}(\mathcal{O}_{h_n}), a_{h_n} - o(\mathcal{O})\rangle_{V(\Omega)} \\ &+ \lambda_*(\mathcal{O}) \langle \mathscr{B}(\mathcal{O}) o(\mathcal{O}), o(\mathcal{O}) - o_{h_n}(\mathcal{O}_{h_n})\rangle_{\mathscr{H}(\Omega)}\end{aligned}$$

Passing to the lim sup on both sides and using ((5.11), 1°), (5.16), (5.17) and (4.36), $(H\mathscr{A})$, (HB) (and strong convergence of $\{\mathcal{O}_{h_n}(\mathcal{O}_{h_n})\}_{n\in N}$ in $\mathscr{H}(\Omega)$) we deduce the strong convergence of $\{\mathcal{O}_{h_n}(\mathcal{O}_{h_n})\}_{n\in N}$ in $V(\Omega)$, which concludes the proof. ■

THEOREM 8 *Let* $\{\mathcal{O}_{*\langle h_n\rangle}\}_{n\in N}$, $h_n \to 0_+$ *be a sequence of solutions of approximate minimization problems (5.1). Then a subsequence* $\{\mathcal{O}_{*\langle h_{n_k}\rangle}\}_{k\in N} \subset \{\mathcal{O}_{*\langle h_n\rangle}\}_{n\in N}$ *exists such that*

$$\begin{cases} \mathcal{O}_{*\langle h_{n_k}\rangle} \to \mathcal{O}_* \text{ strongly in } \mathscr{U}(\Omega) \\ \text{and} \\ \lambda_{*\langle h_{n_k}\rangle}(\mathcal{O}_{h_{n_k}}) \to \lambda_*(\mathcal{O}_*) \text{ in } \mathbb{R}, \\ \mathcal{O}_{h_{n_k}}(\mathcal{O}_{h_{n_k}}) \to \mathcal{O}(\mathcal{O}_*) \text{ strongly in } V(\Omega), \end{cases} \tag{5.25}$$

where $\mathcal{O}_* \in \mathscr{U}_{ad}(\Omega)$ *is a solution of the* MINIMIZATION PROBLEM *(4.9) and the couple* $[\mathcal{O}(\mathcal{O}*), \lambda_*(\mathcal{O}_*)] \in \mathscr{K}(\Omega) \cap \mathscr{S}(\Omega) \times \mathbb{R}$ *solves the corresponding state eigenvalue problem.*

Moreover, the limit of each subsequence of $\{\mathcal{O}_{*\langle h_n\rangle}\}_{n\in N}$, converging in $\mathscr{U}(\Omega)$, is a solution of the latter problem and an analogue of ((5.25), 2°, 3°) holds.

Proof. Since $\mathscr{U}_{ad}(\Omega)$ is compact in $\mathscr{U}(\Omega)$, there exists a subsequence $\{\mathcal{O}_{*\langle h_{\mathcal{O}}\rangle}\}_{\mathcal{O}\in N}$, $h_{\mathcal{O}} \to 0_+$, such that ((5.25), 1°) holds. Let us consider a $\mathcal{O} \in \mathscr{U}_{ad}(\Omega)$. By Lemma 13, there exists a sequence $\{\mathcal{O}_{h_{\mathcal{O}}}\}_{\mathcal{O}\in N}$ of $\mathcal{O}_{h_{\mathcal{O}}} \in \mathscr{U}_{ad\langle h_{\mathcal{O}}\rangle}(\Omega)$, such that $\mathcal{O}_{h_{\mathcal{O}}} \to \mathcal{O}$ strongly in $\mathscr{U}(\Omega)$, as $h_{\mathcal{O}} \to 0_+$. In view of the definition (4.9), we have

$$\Psi_{\langle h_{\mathcal{O}}\rangle}(\mathcal{O}_{*\langle h_{\mathcal{O}}\rangle}, \mathcal{O}_{h_{\mathcal{O}}}(\mathcal{O}_{*\langle h_{\mathcal{O}}\rangle})) \le \Psi_{\langle h_{\mathcal{O}}\rangle}(\mathcal{O}_{h_{\mathcal{O}}}, \mathcal{O}_{h_{\mathcal{O}}}(\mathcal{O}_{h_{\mathcal{O}}})).$$

Let us pass to the limit with $h_{\mathcal{O}} \to 0_+$ and apply Lemma 14 to both sides of this inequality. We arrive at

$$\Psi(\mathcal{O}_*, \mathcal{O}(\mathcal{O}_*)) \le \Psi(\mathcal{O}, \mathcal{O}(\mathcal{O})),$$

so that $\mathcal{O}_*$ is a solution of the original MINIMIZATION PROBLEM. Making use of Lemma 14, we obtain ((5.25), 3°). The previous line of thought may be repeated to any uniformly convergent subsequence of $\{\mathcal{O}_{*\langle h_n\rangle}\}_{n\in N}$.

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Acknowledgement

This research was supported by the Grant No. 1/8263/01 of the Grant Agency of the Slovak Republic.