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# Minimal order deadbeat functional observers for singular 2D linear systems

by

**Tadeusz Kaczorek**

Institute of Control and Industrial Electronics
Faculty of Electrical Engineering, Warsaw University of Technology
Koszykowa 75, 00-662 Warszawa, Poland
E-mail: kaczorek@isep.pw.edu.pl

**Abstract:** Sufficient conditions for the existence of minimal order deadbeat functional observers for singular 2D linear systems described by the general singular 2D model are established. A procedure for computing matrices of the functional observers is given and illustrated by numerical example.

**Keywords:** observers, minimal order observers, functional observers, singular linear systems, 2D systems.

## 1. Introduction

The observer problem for singular (descriptor) linear systems has been considered in many papers and books (Carvalho, Datta, 2002, Fahmy, O'Reilly, 1989, Kaczorek, 1993, 2002, Luenberger, 1966, O'Reilly, 1983, Shafai, Caroll, 1987, Watson, 1998). Darouach and Boudayeb (1995) have established the necessary and sufficient conditions for the existence of a functional observer for standard continuous-time linear systems. In Kaczorek (2000) a new concept of the perfect observers for singular continuous-time linear systems has been proposed. Next, the concept has been extended in Kaczorek (2001a) for standard continuous-time linear systems and in Kaczorek (2001b,c) for singular two-dimensional (2D) linear systems. Tsui (1986, 1998) has proposed a different algorithm for the design of minimal order functional observers for standard linear systems. An other design approach for minimal order functional observers has been proposed in Kaczorek (2002).

The most popular models of 2D linear systems are the models proposed by Roesser (1975), Fornasini and Marchesini (1976, 1978) and Kurek (1985). The models have been extended for singular 2D models in Kaczorek (1988, 1993).

In this paper a design method of minimal order deadbeat functional observers existence of the functional observers will be established and a procedure for computing the matrices of the observers will be proposed.

## 2. Problem formulation

Let $R^{n\times m}$ be the set of $n\times m$ real matrices and $R^n := R^{n\times 1}$. Consider the singular 2D linear system from Kaczorek (1993)

$$Ex_{i+1,j+1} = A_0x_{i,j}+A_1x_{i+1,j}+A_2x_{i,j+1}+B_0u_{ij}+B_1u_{i+1,j}+B_2u_{i,j+1} \quad (1a)$$
$$y_{i,j} = Cx_{ij}, i,j \in Z_+ \quad \text{(the set of nonnegative integers)} \quad (1b)$$

where $x_{ij} \in R^n$, $u_{ij} \in R^m$, $y_{ji} \in R^p$ are the semistate, input and output vectors, respectively and $E, A_l \in R^{n\times n}$, $B_l \in R^{n\times m}$, $l = 0,1,2$, $C \in R^{p\times n}$ with $E$ possibly singular ($\det E = 0$).

It is assumed that

$$\det[Ez_1z_2 - A_0 - A_1z_1 - A_2z_2] \neq 0 \quad \text{for some } (z_1,z_2) \in \mathbb{C}\times\mathbb{C} \quad (2)$$

($\mathbb{C}$ – the field of complex numbers) and rank $C = p$.

We are looking for a minimal order $r$ deadbeat functional 2D observer described by the equations

$$z_{i+1,j+1} = F_0z_{ij} + F_1z_{i+1,j} + F_2z_{i,j+1} + TB_0u_{ij} + TB_1u_{i+1,j} + TB_2u_{i,j+1}$$
$$+G_0y_{ij} + G_1y_{i+1,j} + G_2y_{i,j+1} \quad (3a)$$
$$w_{ij} = Lz_{ij} + My_{ij}, \quad z_{ij} \in R^r, \quad w_{ij} \in R^q, \quad i,j \in Z_+ \quad (3b)$$

that reconstructs exactly for $i > r, j > r$ the given linear function of

$$Kx_{ij} (K \in R^{q\times n} \text{ is given}) \quad (4)$$

i.e.

$$w_{ij} = Kx_{ij} \quad \text{for } i > r, j > r. \quad (5)$$

The problem can be stated as follows: Given $E, A_l, B_l, l = 0,1,2, C$ and $K$, find $F_l \in R^{r\times r}$, $T \in R^{r\times n}$, $G_l \in R^{r\times p}$, $l = 0,1,2; L \in R^{q\times r}$ and $M \in R^{q\times p}$ of (3) such that (5) holds. Solvability conditions for the problem will be established and a procedure for computation of the matrices of (3) will be derived.

## 3. Problem solution

Let us define

$$e_{ij} = z_{ij} - TEx_{ij}, i,j \in Z_+. \quad (6)$$

Using (6), (1) and (3) we may write

$$e_{i+1,j+1} = z_{i+1,j+1} - TEx_{i+1,j+1} = F_0(e_{ij} + TEx_{ij})$$

$$\begin{aligned}
&TB_0u_{ij} + TB_1u_{i+1,j} + TB_2u_{i,j+1} + G_0Cx_{ij} + G_1Cx_{i+1,j} \\
&+G_2Cx_{i,j+1} - T(A_0x_{ij} + A_1x_{i+1,j} + A_2x_{i,j+1} \\
&B_0u_{ij} + B_1u_{i+1,j} + B_2u_{i,j+1}) = F_0e_{ij} + F_1e_{i+1,j} + F_2e_{i,j+1} \\
&+(F_0TE + G_0C - TA_0)x_{ij} \\
&+(F_1TE + G_1C - TA_1)x_{i+1,j} + (F_2TE + G_2C - TA_2)x_{i,j+1}.
\end{aligned} \tag{7}$$

From (7) it follows that

$$e_{i+1,j+1} = F_0e_{ij} + F_1e_{i+1,j} + F_2e_{i,j+1} \tag{8}$$

if and only if

$$TA_l = F_lTE + G_lC \quad \text{for } l = 0, 1, 2. \tag{9}$$

From (6) we have $z_{ij} = TEx_{ij}$ for $i > r, j > r$ if and only if $e_{ij} = 0$ for $i > r, j > 0$ and then from (3b), (1b) and (5) we obtain

$$K = LTE + MC. \tag{10}$$

Note that the equations (9) and (10) are bilinear with respect to the unknown matrices $F_l, l = 0, 1, 2, T$ and $L$.

The equations (9) and (10) can be written in the form

$$TA_l = [F_l, G_l] \begin{bmatrix} TE \\ C \end{bmatrix} \quad \text{for } l = 0, 1, 2 \tag{11}$$

and

$$K = [L, M] \begin{bmatrix} TE \\ C \end{bmatrix}. \tag{12}$$

By the Kronecker-Capelli theorem the equation (12) has a solution $[L, M]$ if and only if

$$\text{rank} \begin{bmatrix} TE \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} TE \\ C \\ K \end{bmatrix}. \tag{13}$$

LEMMA 1 *There exists a matrix $T_k \in R^{k\times n}$ for some $k = 0, 1, \ldots, n-p$ satisfying the condition*

$$rank \begin{bmatrix} T_kE \\ C \end{bmatrix} = rank \begin{bmatrix} T_kE \\ C \\ K \end{bmatrix} \tag{14}$$

*for any $K \in R^{q\times n}$ if and only if*

$$rank \begin{bmatrix} E \\ C \end{bmatrix} = n. \tag{15}$$

*Proof.* To show the necessity note that the condition (14) is satisfied for any K only if

$$\operatorname{rank}\begin{bmatrix} T_kE \\ C \end{bmatrix} = n. \tag{16}$$

From $\begin{bmatrix} T_kE \\ C \end{bmatrix} = \begin{bmatrix} T_k 0 \\ 0 I_p \end{bmatrix}\begin{bmatrix} E \\ C \end{bmatrix}$ it follows that (16) holds only if (15) is satisfied.

Now we shall show that if (15) is satisfied, then there exists a matrix $T_k, k \in [0, 1, \ldots, n-p]$ such that (14) holds for any $K$. Without loss of generality it can be assumed that $C = [C_1 0]$, $\det C_1 \neq 0$. In this case (15) implies rank $E_2 = n - p$, where $E = [E_1, E_2], E_1 \in R^{n\times p}$, $E_2 \in R^{n\times(n-p)}$. From (16) for $k = n - p$ we have rank $T_{n-p}E_2 = n - p$. Then there exists $T_{n-p}$ satisfying (16) for any given $K$ if (15) holds. ■

LEMMA 2 *Let $\Omega_{T_k}$ be the set of matrices $T_k$ satisfying the condition (14). The equation (11) has a solution $T_k \in \Omega_{T_k}$, $[F_l, G_l], l = 0, 1, 2$ for the given matrices $E, A_l, l = 0, 1, 2$ and $C$ if and only if*

$$rank\begin{bmatrix} T_kE \\ C \\ K \end{bmatrix} = rank\begin{bmatrix} T_kE \\ C \\ T_kA_l \end{bmatrix} \quad for\ l = 0, 1, 2. \tag{17}$$

*Proof.* By the Kronecker-Capelli theorem the equations

$$T_kA_l = [F_l, G_l]\begin{bmatrix} T_kE \\ C \end{bmatrix} \quad \text{for } l = 0, 1, 2 \tag{18}$$

have solutions $[F_l, G_l]$, for $l = 0, 1, 2$ and $T_k \in \Omega_{T_k}$ if and only if

$$\operatorname{rank}\begin{bmatrix} T_kE \\ C \end{bmatrix} = \operatorname{rank}\begin{bmatrix} T_kE \\ C \\ T_kA_l \end{bmatrix} \quad \text{for } l = 0, 1, 2. \tag{19}$$

By assumption, (14) holds. Thus, from (14) and (19) we have

$$\operatorname{rank}\begin{bmatrix} T_kE \\ C \\ K \end{bmatrix} = \operatorname{rank}\begin{bmatrix} T_kE \\ C \end{bmatrix} = \operatorname{rank}\begin{bmatrix} T_kE \\ C \\ T_kA_l \end{bmatrix} \quad \text{for } l = 0, 1, 2 \quad ■$$

REMARK 1 *In a particular case, if*

$$rank\, C = rank\begin{bmatrix} C \\ K \end{bmatrix}, \tag{20}$$

*then the linear function $Kx_{ij}$ can be reconstructed exactly by $My$, where $M$ is the solution of the equation $K = MC$. In this case $k = 0$ and $T_0 = 0$. In what*

REMARK 2 *Let* $C = [C_1 0], K = [K_1, K_2]$ *and* $T = [T_1, T_2]$ *where* $C_1 \in R^{p\times p}$, $K_1 \in R^{q\times p}$, $T_1 \in R^{r\times p}$. *Then from (10) we have* $[K_1, K_2] = L[T_1, T_2]E + M[C_1 0]$ *and*

$$K_2 = LT_2E. \tag{21}$$

*From (21) it follows that rank* $K_2 \leq$ *rank* $T_2$ *and the minimal number* $r$ *of rows of* $T_2$ *(and also of* $T$*) is bounded by the rank of* $K_2$, *i.e.* $r \leq$ *rank* $K_2$.

LEMMA 3 *The solution* $e_{ij}$ *of the equation (8) satisfies the condition*

$$e_{ij} = 0 \textit{ for all } i \geq r \textit{ and } j \geq r \tag{22}$$

*if and only if*

$$\det[I_r z_1 z_2 - F_0 - F_1 z_1 - F_2 z_2] = z_1^r z_2^r. \tag{23}$$

*Proof.* The transition matrix $T_{ij}$ of (8) is defined by (Kaczorek, 1993, Klamka, 1991):

$$T_{ij} = \begin{cases} I_r (\text{the identity matrix}) & \text{for } i = j = 0 \\ A_0 T_{i-1,j-1} + A_1 T_{i,j-1} + A_2 T_{i-1,j} & \text{for } i + j > 0 \\ 0 (\text{the zero matrix}) & \text{for } i < 0 \text{ or/and } j < 0 \end{cases}$$

and

$$[I_r z_1 z_2 - F_0 - F_1 z_1 - F_2 z_2]^{-1} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} T_{ij} z_1^{-(i+1)} z_2^{-(j+1)}. \tag{24}$$

Using the technique of division of polynomials and (24) it is easy to show that

$$T_{ij} = 0 \quad \text{for all } i \geq r \text{ or/and } j \geq r \tag{25}$$

if and only if (23) holds.

From the solution

$$e_{ij} = \sum_{k=1}^{i} (T_{i-k-1,j-1} A_0 + T_{i-k,j-1} A_1) e_{k0}$$
$$+ \sum_{l=1}^{j} (T_{i-1,j-l-1} A_0 + T_{i-l,j-1} A_2) e_{0l} + T_{i-1,j-1} A_0 e_{00}$$

of the equation (8) it follows that (21) holds if and only if the condition (25) is satisfied, what is equivalent to (23). ∎

From the above considerations we have the following recursive procedure for

**Procedure**

STEP 1. Find a minimal $k \in \{1, \dots, n-p\}$, $k \geq \text{rank } K$ and the set $\Omega_{T_k}$ of matrices $T_k$ satisfying the condition (14).

STEP 2. Choose the matrices $F_l, l = 0, 1, 2$ and $T_\in \Omega_{T_k}$ satisfying the condition (23),

$$\sigma(F) \cap \sigma(E, A) = \phi \text{ (the empty set) for } l = 0, 1, 2, \tag{26}$$

where $\sigma(F_l)$ ($\sigma(E, A)$) denotes the spectrum of the matrix $F_l$ (of the pair $(E, A)$) and the condition

$$\text{rank } C = \text{rank} \begin{bmatrix} C \\ T_k A_l - F_l T_k E \end{bmatrix} \quad \text{for } l = 0, 1, 2 \tag{27}$$

is satisfied for the given $T_k \in \Omega_{T_k}, A, E, C$.

STEP 3. If there exists $F_l$ satisfying the conditions (23), (26) and (27), then find $G_l$ from the equation

$$G_l C = T_k A_l - F_l T_k E \quad \text{for } l = 0, 1, 2. \tag{28}$$

If not, go to step 1 and replace $k$ by $k+1$.

STEP 4. By solving the equation

$$K = [LM] \begin{bmatrix} T_k E \\ C \end{bmatrix} \tag{29}$$

find the matrices $L$ and $M$.

STEP 5. Write the equations (3) of the desired functional observer.

REMARK 3 *Note that if the matrices $F_l, l = 0, 1, 2$ and $T_k \in \Omega_{T_k}$ are chosen so that the conditions (23), (26) and (27) are satisfied, then the equation (28) has a solution $G_l$ for $l = 0, 1, 2$.*

THEOREM. *Let the singular 2D system (1) satisfy the conditions (2) and (15). There exists a deadbeat functional 2D observer of the order $r$ of the form (3) if for some $k \in [1, \dots, n-p]$ the conditions (17) and (27) hold, and then $r = k$.*
*Proof.* By Lemma 3 the condition (5) is satisfied and the system (3) is a deadbeat functional observer of (1) if the matrices $F_l$ for $l = 0, 1, 2$ are chosen so that (23) holds. If there exists $k \in [1, \dots, n-p]$ such that (17) and (27) are satisfied, then by solving the equations (28) for the given $T_k, A, E, C$ and $F_l$ we can find the matrices $G_l$ for $l = 0, 1, 2$, and next the matrices $L$ and $M$ can be found from the equation (29). ■

## 4. Example

Find the minimal order deadbeat functional observer (3) which reconstructs the linear function (4) with

$$K = \begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix} \tag{30}$$

for the singular 2D system (1) with

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_0 = \begin{bmatrix} -2 & 2 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ -1/2 & 0 & 0 & -1/2 & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & -1 & 1 & -2 \\ -1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1/2 & 0 \end{bmatrix}, \tag{31}$$

$$B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0. \end{bmatrix}$$

It is easy to check that the system (1) with (31) satisfies the conditions (2) and (15) and the condition (20) is not satisfied.

Using the Procedure we obtain:

STEP 1. In this case rank $K = 2$ and we choose $k = 2$ and $T_2 = [t_{ij}] \in R^{2\times 5}$. From (14) we obtain

$$\Omega_{T_2} = \left\{ \begin{bmatrix} t_{11} & 0 & t_{13} & t_{14} & t_{15} \\ t_{21} & 0 & t_{23} & t_{24} & t_{25} \end{bmatrix} : \operatorname{rank} \begin{bmatrix} t_{13} & t_{14} & t_{15} \\ t_{23} & t_{24} & t_{25} \\ 1 & 0 & 1 \end{bmatrix} = 2 \right\} \tag{32}$$

since

$$\operatorname{rank}\begin{bmatrix} T_2E \\ C \end{bmatrix} = \operatorname{rank}\begin{bmatrix} t_{11} & 0 & t_{13} & t_{14} & t_{15} \\ t_{21} & 0 & t_{23} & t_{24} & t_{25} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} T_2E \\ C \\ K \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} t_{11} & 0 & t_{13} & t_{14} & t_{15} \\ t_{21} & 0 & t_{23} & t_{24} & t_{25} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix} \tag{33}$$

STEP 2. In this case we choose $F_0 = -F_1 = F_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and

$$T_2 = \begin{bmatrix} 0 & 0 & -2 & 2 & -2 \\ 1 & 1 & -2 & 1 & -2 \end{bmatrix}$$

and the condition (27) is satisfied for $l = 0, 1, 2$, since

$$\begin{aligned} T_2A_0 - F_0T_2E &= \begin{bmatrix} 0 & -2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ T_2A_1 - F_1T_2E &= \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ T_2A_2 - F_2T_2E &= \begin{bmatrix} -1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{34}$$

STEP 3. By solving the equations (28) for $l = 0, 1, 2$ and using (34) we obtain

$$G_0 = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}, G_1 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, G_2 = \begin{bmatrix} -1 & -2 \\ 0 & 0 \end{bmatrix}$$

STEP 4. In this case equation (29) has the form

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix} = [L, M]\begin{bmatrix} 0 & 0 & -2 & 2 & -2 \\ 1 & 0 & -2 & 1 & -2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

and its solution is given by

$$[L, M] = \left[\begin{array}{cc|cc} 1/2 & -1 & 2 & 2 \\ -1/2 & 0 & 1 & -1 \end{array}\right].$$

STEP 5. The desired deadbeat functional 2D observer has the form

$$z_{i+1,j+1} = \begin{bmatrix} 0 & 1 \\ \cdot & \cdot \end{bmatrix} z_{ij} + \begin{bmatrix} 0 & -1 \\ \cdot & \cdot \end{bmatrix} z_{i+1,j} + \begin{bmatrix} 0 & 1 \\ \cdot & \cdot \end{bmatrix} z_{i,j+1}$$

$$+\begin{bmatrix} 4 & -6 \\ 4 & -5 \end{bmatrix} u_{ij} + \begin{bmatrix} -6 & 2 \\ -5 & 2 \end{bmatrix} u_{i+1,j} + \begin{bmatrix} -4 & 2 \\ -2 & -1 \end{bmatrix} u_{i,j+1}$$
$$+\begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix} y_{ij} + \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} y_{i+1,j} + \begin{bmatrix} -1 & -2 \\ 0 & 0 \end{bmatrix} y_{i,j+1},$$
$$w_{ij} = \begin{bmatrix} \frac{1}{2} & -1 \\ -\frac{1}{2} & 0 \end{bmatrix} z_{ij} + \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} y_{ij}.$$

## 5. Concluding remarks

An approach to the design of minimal order deadbeat 2D functional observers for singular 2D linear systems described by the general 2D model (1) has been proposed. A procedure for computation of the matrices of the functional observers has been proposed and illustrated by a numerical example. The approach proposed with minor modifications (with replacement of the nilpotent matrices $F_l, l = 0, 1, 2$ by Schur matrices) can be also applied to design the minimal order asymptotic functional observers for singular 2D linear systems. An other approach is based on solving the generalized Sylvester equations (9) with respect to the matrix $T$ by the use of the algorithm presented in Carvalho, Datta (2002), for which finding of the matrices $L$, $M$ from (29) may be proposed. An open problem is to establish the necessary and sufficient conditions for solvability of equations (9) and (29) for given matrices $E, A_l, l = 0, 1, 2$ and $C$.

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