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Stability properties of weak sharp minima

by

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Abstract: In this paper, stability theorems of Studniarski (1989) are extended to include the stability of weak sharp local minimum points for a nonsmooth mathematical programming problem.

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1. Introduction

This paper is devoted to the study of stability properties of a special type of minimizers for the following optimization problem:

$$P(f,S): \text{ minimize } f(x) \text{ subject to } x \in S,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and S is a nonempty subset of $\mathbb{R}^n$. We begin by quoting a general definition of these minimizers in two versions: global and local; see Studniarski and Ward (1999).

DEFINITION 1 *Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^n$. Suppose that f is finite and constant on the set $W \subset \mathbb{R}^n$, and let $x_0 \in W \cap S$ and $m \geq 1$. For $x \in \mathbb{R}^n$, let*

$$d_W^m(x) := \inf\{\|y-x\|^m \mid y \in W\}.$$

(a) *We say that x_0 is a* weak sharp minimizer of order m *for $P(f,S)$ if there exists $\beta > 0$ such that*

(b) *For* $\varepsilon > 0$, *let* $B(x, \varepsilon) := \{y \in \mathbb{R}^n \mid \|y - x\| \leq \varepsilon\}$. *We say that* x_0 *is a* weak sharp local minimizer of order m *for* $P(f, S)$ *if there exist* $\beta > 0$ *and* $\varepsilon > 0$ *such that*

$$f(x) - f(x_0) \geq \beta d_W^m(x), \text{ for all } x \in S \cap B(x_0, \varepsilon).$$

In particular, if condition (a) *(respectively,* (b)*) holds for* $W = \{x_0\}$, *we say that* x_0 *is a* strict *(respectively,* strict local*)* minimizer of order m *for* $P(f, S)$.

The notion of a weak sharp minimum of order one was studied by Burke and Ferris (1993). Weak sharp minima of order m occur in many optimization problems and have important consequences for the study of optimization algorithms and for stability and sensitivity analysis in nonlinear programming; see, for example, Klatte (1994) and Ward (1998). Various characterizations of weak sharp minimizers of order m (local or global) in nonconvex optimization were obtained by Bonnans and Ioffe (1995), Studniarski (1999), Studniarski and Ward (1999).

In the optimization literature, stability results are usually formulated for mathematical programming problems depending on some parameters. In this paper however, stability is understood in a somewhat different sense, which does not require the introduction of parameter vectors. We now briefly explain this idea.

Hyers (1978, 1985) considered the following notion of stability of minimum points: a relative minimum point x_0 of a function f is called *stable* if functions $\tilde{f}$ (of a suitable class) which are sufficiently close to f have relative minimum points within a prescribed distance from x_0. For functions f defined on a Banach space, he proved a sufficient condition for the stability of a minimum point in this sense. The function $\tilde{f}$ can be regarded as a function obtained from f by adding an arbitrary (but sufficiently small) perturbation. Studniarski (1989) obtained analogous results for constrained nonsmooth optimization problems in finite-dimensional spaces. He proved some theorems on stability in Hyers' sense with respect to arbitrary perturbations of both the objective function and the set of feasible points. A sufficient condition, which ensures the stability property mentioned is that x_0 is a strict local minimizer of order m for the given problem.

The aim of this paper is to generalize the results of Studniarski (1989) by replacing the assumption of strict local minimality of order m at x_0 with a weaker condition, which is intermediate between conditions (a) and (b) of Definition 1. In this way, we avoid the requirement that the solution of problem $P(f, S)$ must be locally unique.

2. The case of a general constraint

In this section, we deal with the problem $P(f, S)$ without considering any special structure of the constraint set S. First, we show that d_W^m is Lipschitzian on some

LEMMA 1 *Let W be a bounded set and let $m \geq 1$ (we do not require of m to be an integer). Then, for any $\varepsilon > 0$, we have*

$$|d_W^m(x) - d_W^m(y)| \leq m\varepsilon^{m-1}\|x - y\| \tag{1}$$

for all $x, y \in W_\varepsilon := \bigcup_{z \in W} B(z, \varepsilon)$.

Proof. Observe that d_W^m can be represented as the composition of two functions: $d_W^m = \psi \circ d_W$, where $\psi(t) := t^m$ for $t \geq 0$. If $d_W(x) = d_W(y)$, then inequality (1) holds trivially. Otherwise, we have

$$t_1 := \min\{d_W(x), d_W(y)\} < t_2 := \max\{d_W(x), d_W(y)\} \leq \varepsilon \tag{2}$$

and, by using the mean value theorem for ψ,

$$\begin{aligned} &|d_W^m(x) - d_W^m(y)| = |\psi(d_W(x)) - \psi(d_W(y))| \\ &\leq \sup_{t \in (t_1, t_2)} |\psi'(t)||d_W(x) - d_W(y)|. \end{aligned} \tag{3}$$

Since d_W is globally Lipschitzian of rank 1 by Clarke (1983, Proposition 2.4.1) and the function $\psi'(t) = mt^{m-1}$ is nondecreasing, we deduce from (3) and the last inequality in (2) that (1) holds in this case also. ■

THEOREM 1 *Suppose that f is constant on the set $W \subset S$. Let $x_0 \in W$, and let W be compact. Suppose that there exist real numbers $\varepsilon > 0$, $\alpha > 0$ and $m \geq 1$ such that*

$$f(x) \geq f(x_0) + \alpha d_W^m(x) \text{ for all } x \in C := S \cap W_\varepsilon. \tag{4}$$

Suppose also that f is Lipschitzian of rank K_0 on W_ε. Let $K := K_0 + \alpha m\varepsilon^{m-1}$, and let β be an arbitrary (but fixed) number in the open interval $(0, \alpha\varepsilon^m)$. Denote $r := (\alpha\varepsilon^m - \beta)/(K_0 + K)$. Suppose that there are given a function $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}$ and a set $\widetilde{S} \subset \mathbb{R}^n$ such that

(a) *$\widetilde{f}$ is Lipschitzian of rank less than K on W_ε;*

(b) *$\widetilde{S} \cap \operatorname{int} W_r \neq \emptyset$, where $W_r := \bigcup_{y \in W} B(y, r)$ and the set $\widetilde{C} := \widetilde{S} \cap W_\varepsilon$ is closed;*

(c) *for each $x \in W_\varepsilon$, we have the inequalities*

$$|\widetilde{f}(x) - f(x)| < \beta/4, \tag{5}$$

$$d_C(x) \leq d_{\widetilde{C}}(x) + \beta/(2K). \tag{6}$$

Then problem $P(\widetilde{f}, \widetilde{S})$ has a local solution which belongs to $\operatorname{int} W_\varepsilon$.

Proof. Let $\Phi = f - h$, where $h(x) := \alpha d_W^m(x)$. We show first that Φ is Lipschitzian of rank K on W_ε. By Lemma 1, we get

$$|h(x) - h(y)| = \alpha|d_W^m(x) - d_W^m(y)| \leq \alpha m\varepsilon^{m-1}\|x - y\|,$$

Hence, h is Lipschitzian of rank $\alpha m\varepsilon^{m-1}$on W_ε. Then

$$\begin{aligned}&|\Phi(x)-\Phi(y)| = |f(x)-h(x)-(f(y)-h(y))|\\ &\leq |f(x)-f(y)|+|h(x)-h(y)|\\ &\leq K_0\|x-y\|+\alpha m\varepsilon^{m-1}\|x-y\|\\ &= K\|x-y\|.\end{aligned}$$

This shows that Φ is Lipschitzian of rank K on W_ε. Condition (4) means that Φ attains its minimum over C at x_0. Applying Clarke (1983, Proposition 2.4.3), we infer that $\Phi + Kd_C$ attains its minimum over W_ε at x_0; then

$$f(x)+Kd_C(x)-f(x_0)\geq \alpha d_W^m(x),\ \text{for all } x\in W_\varepsilon. \tag{7}$$

By assumption (b), there exists a point $z\in\widetilde{S}\cap \operatorname{int} W_r$. We have

$$r=\frac{\alpha\varepsilon^m-\beta}{K_0+K}\leq\frac{\alpha\varepsilon^m}{2K_0+\alpha m\varepsilon^{m-1}}<\frac{\varepsilon}{m}\leq\varepsilon,$$

which implies that $z\in\widetilde{C}$.

Since $z\in\widetilde{C}$, we have $d_{\widetilde{C}}(z)=0$. Moreover, it follows from the relation $z\in\operatorname{int}W_r$ that there exists $w\in W$ for which $\|z-w\|<r$. Therefore, we obtain from (5)

$$\begin{aligned}&\widetilde{f}(z)+Kd_{\widetilde{C}}(z)=\widetilde{f}(z)<f(z)+\beta/4\leq\ f(z)+Kd_C(z)+\beta/4\\ &=f(w)+((f+Kd_C)(z)-(f+Kd_C)(w))+\beta/4\\ &\leq f(w)+(K_0+K)\|z-w\|+\beta/4\\ &<f(w)+\alpha\varepsilon^m-3\beta/4\\ &=f(x_0)+\alpha\varepsilon^m-3\beta/4.\end{aligned} \tag{8}$$

Take any boundary point u of W_ε; then $d_W(u)=\varepsilon$. Using successively (8), (7), (5), (6), we get

$$\widetilde{f}(z)+Kd_{\widetilde{C}}(z)<f(u)+Kd_C(u)-3\beta/4<\widetilde{f}(u)+Kd_{\widetilde{C}}(u). \tag{9}$$

Since the function $\widetilde{f}+Kd_{\widetilde{C}}$ is continuous, it attains the minimum over the compact set W_ε at some point z_0. It follows from inequalities (9) that z_0 belongs to $\operatorname{int}W_\varepsilon$. To end the proof, we only need to verify that $z_0\in\widetilde{C}$; this will obviously imply that z_0 is a local minimum point for problem $P(\widetilde{f},\widetilde{S})$. (Indeed, it suffices to observe that $\widetilde{f}+Kd_{\widetilde{C}}$ reduces to $\widetilde{f}$ on $\widetilde{C}=\widetilde{S}\cap W_\varepsilon$, and W_ε is a neighborhood of z_0.) Since $\widetilde{C}$ is compact, there exists a point $y\in\widetilde{C}$ such that $\|y-z_0\|=d_{\widetilde{C}}(z_0)$. Let $\widetilde{f}$ be Lipschitzian of rank K_1 on W_ε; by assumption (a), we can choose $K_1<K$. It follows from the definitions of y and z_0, respectively, that

which leads to

$$K\|y-z_0\| \le \widetilde{f}(y)-\widetilde{f}(z_0) \le K_1\|y-z_0\|.$$

Thus, $\|y-z_0\|=0$, and so $z_0=y\in\widetilde{C}$. ∎

REMARK 1 (a) *Suppose that there exists a point* $u \in S\cap \mathrm{bd}W_\varepsilon$ *such that* $\|u-y\|=\varepsilon=d_W(u)$ *for some* $y\in W$. *Then* r *is always smaller than* $\varepsilon/(m+2)$. *Indeed, in this case we get by (4)*

$$K_0\varepsilon = K_0\|u-y\| \ge f(u)-f(y) = f(u)-f(x_0) \ge \alpha d_W^m(u) = \alpha\varepsilon^m,$$

and so $K_0 \ge \alpha\varepsilon^{m-1}$. *Hence*

$$r=\frac{(\alpha\varepsilon^m-\beta)}{(K_0+K)}=\frac{(\alpha\varepsilon^m-\beta)}{(2K_0+\alpha m\varepsilon^{m-1})}\le\frac{(\alpha\varepsilon^m-\beta)}{(m+2)\alpha\varepsilon^{m-1}}<\frac{\varepsilon}{(m+2)}.$$

(b) *The number* β *appearing in Theorem 1 may be chosen arbitrarily. When* β *increases, assumption* (b) *becomes stronger, while assumption* (c) *becomes weaker.*

EXAMPLE 1 *The following example shows that Theorem 1 is not valid without the assumption of compactness of the set* W. *Let* $n=m=1$, $S=\mathbb{R}$, $W=\{x\mid x\le 0\}$, *and let* f *be defined by*

$$f(x):=\begin{cases}0, & \text{if } x<0,\\ x, & \text{if } x\ge 0.\end{cases}$$

Then, inequality (4) is fulfilled with $\varepsilon=\alpha=1$ *and* f *is Lipschitzian of rank* $K_0=1$ *on* W_ε. *We compute* $K=2$ *and choose* $\beta=1/2\in(0,1)=(0,\alpha\varepsilon^m)$; *then* $r=1/6$. *Define the new function* $\widetilde{f}$ *by*

$$\widetilde{f}(x):=\begin{cases}e^x/16, & \text{if } x<0,\\ x+1/16, & \text{if } x\ge 0.\end{cases}$$

It is easy to show that conditions (a)–(c) *are satisfied if* $\widetilde{S}=S$. *However, there is no local solution to problem* $P(\widetilde{f},\widetilde{S})$ *since* $\widetilde{f}$ *does not attain its infimum on the real line.*

3. Sets defined by inequalities and equalities

We shall now deal with the case in which the set S appearing in problem $P(f,S)$ has the form

$$S=\{x\in\mathbb{R}^n \mid g_i(x)\le 0,\ i\in I;\ g_j(x)=0,\ j\in J\}, \tag{10}$$

where I, J are given finite sets and g_i, $i\in I\cup J$, are real-valued functions on $\mathbb{R}^n$. In this section, the symbol $\widetilde{S}$ will denote the set defined via formula (10) where the functions g_i are replaced by other functions $\widetilde{g}_i$. Our aim here is to show that inequality (6) in Theorem 1 can be replaced by a certain assumption

LEMMA 2 *Given any subsets of A, B of $\mathbb{R}^n$, we have*

$$d_A(x) \le d_B(x) + \sup_{b \in B} d_A(b) \text{ for all } x \in \mathbb{R}^n. \tag{11}$$

Proof. See Studniarski (1989, p. 30). ■

LEMMA 3 *Let the functions g_i, $i \in I$, be lower semicontinuous, and let the functions g_j, $j \in J$, be continuous. Let $W \subset S$ and let W be a compact set. Take any fixed $\varepsilon > 0$ and denote $C := S \cap W_\varepsilon$, $W_\varepsilon := \bigcup_{y \in W} B(y, \varepsilon)$. Then for any $\eta > 0$, there exists $\delta > 0$, such that, for any real-valued functions $\widetilde{g}_i$, $i \in I \cup J$, satisfying the inequalities*

$$|\widetilde{g}_i(x) - g_i(x)| < \delta \text{ for all } x \in W_\varepsilon,\ i \in I \cup J, \tag{12}$$

we have

$$d_C(x) \le d_{\widetilde{C}}(x) + \eta \text{ for all } x \in W_\varepsilon \tag{13}$$

(where $\widetilde{C} := \widetilde{S} \cap W_\varepsilon$).

Proof. Suppose that the desired conclusion is false. Then there exists $\eta > 0$ such that, for any positive integer k, there exist functions, $\widetilde{g}_{i,k}$, $i \in I \cup J$, which satisfy the inequalities

$$|\widetilde{g}_{i,k}(x) - g_i(x)| < 1/k, \text{ for all } x \in W_\varepsilon,\ i \in I \cup J, \tag{14}$$

whereas the set $\widetilde{C}_k := \widetilde{S}_k \cap W_\varepsilon$ (where $\widetilde{S}_k$ is defined by $\widetilde{g}_{i,k}$ via (10)) satisfies the inequality

$$d_C(x_k) > d_{\widetilde{C}_k}(x_k) + \eta, \text{ for some } x_k \in W_\varepsilon. \tag{15}$$

Combining (15) with (11), (where we should take $A = C$, $B = \widetilde{C}_k$, $x = x_k$), we obtain

$$\sup_{z \in \widetilde{C}_k} d_C(z) > \eta \quad \text{for every } k.$$

Thus, for every k, there exist a point $z_k \in \widetilde{C}_k$ such that

$$d_C(z_k) > \eta. \tag{16}$$

Since $\widetilde{C}_k \subset W_\varepsilon$ for all k and W_ε is compact, we may assume (by taking a subsequence) that $z_k \to z \in W_\varepsilon$. Then, (16) and the continuity of d_C imply

$$d_C(z) = \lim_{k \to \infty} d_C(z_k) \ge \eta. \tag{17}$$

For every k, we have $z_k \in \widetilde{C}_k \subset \widetilde{S}_k$, and so

It follows from the lower semicontinuity of g_i, $i \in I \cup J$, and from (14), (18) that

$$g_i(z) \leq \liminf_{k\to\infty} g_i(z_k) \leq \liminf_{k\to\infty}(\widetilde{g}_{i,k}(z_k) + |g_i(z_k) - \widetilde{g}_{i,k}(z_k)|)$$
$$\leq \liminf_{k\to\infty} \widetilde{g}_{i,k}(z_k) + \lim_{k\to\infty} 1/k \leq 0 \text{ for all } i \in I \cup J.$$

The same argument applied to the function $-g_j$, $j \in J$, gives $g_j(z) \geq 0$. We have shown that $z \in C$, which contradicts (17). ■

THEOREM 2 *Consider problem $P(f, S)$ where the set S is given by (10). Suppose that f is constant on the set $W \subset S$. Let $x_0 \in W$, and let W be compact. Suppose that there exist real numbers $\varepsilon > 0$, $\alpha > 0$ and a positive integer m such that*

$$f(x) \geq f(x_0) + \alpha d_W^m(x), \text{ for all } x \in C = S \cap W_\varepsilon.$$

Suppose also that f is Lipschitzian of rank K_0 on W_ε, the functions g_i, $i \in I$, are lower semicontinuous, while the functions g_j, $j \in J$, are continuous. Let us denote $K := K_0 + \alpha m \varepsilon^{m-1}$, $r := \alpha\varepsilon^m/(K_0 + K)$. Then there exists $\delta > 0$ for which the following implication is true:

If $\widetilde{f}$ and $\widetilde{g}_i$, $i \in I \cup J$, are real-valued functions on $\mathbb{R}^n$ such that

(a) *$\widetilde{f}$ is Lipschitzian of rank less than K on W_ε, the functions $\widetilde{g}_i$, $i \in I$, are lower semicontinuous, while the functions $\widetilde{g}_j$, $j \in J$, are continuous,*

(b) *$\widetilde{S} \cap \operatorname{int} W_r \neq \emptyset$, where $W_r := \bigcup_{y\in W} B(y, r)$,*

(c) *$|\widetilde{f}(x) - f(x)| < \delta$ for all $x \in W_\varepsilon$, and inequalities (12) are satisfied, then problem $P(\widetilde{f}, \widetilde{S})$ has a local solution which belongs to $\operatorname{int} W_\varepsilon$.*

Proof. Assumption (b) assures the existence of a number $r_1 \in (0, r)$ such that $\widetilde{S} \cap \operatorname{int} W_{r_1} \neq \emptyset$, where $W_{r_1} := \bigcup_{y\in W} B(y, r_1)$. Let $\beta \in (0, \alpha\varepsilon^m)$ be the number defined by the equality $r_1 = (\alpha\varepsilon^m - \beta)/(K_0 + K)$. It follows from assumption (a) that the set $\widetilde{C} := \widetilde{S} \cap W_\varepsilon$ is closed. Moreover, the assumptions of Lemma 3 are satisfied. Let us take $\eta := \beta/(2K)$ and choose a number $\delta > 0$ according to Lemma 3. We may assume that $\delta \leq \beta/4$. In this way, assumption (c) implies conditions (5) and (6). To complete the proof, it suffices to apply Theorem 1. ■

REMARK 2 *It can be shown, similarly as in the proof of Theorem 1, that under the assumptions of Theorem 2, we have $r \leq \varepsilon$. Let us note that assumption* (b) *of Theorem 2 may be difficult to verify in practice. Therefore we shall show that, for some particular case, this assumption can be replaced by another condition, which can be verified more easily.*

THEOREM 3 *Consider problem $P(f, S)$ where the set S is given by (10) and $J = \emptyset$ (i.e., there are no equality constraints). Let us denote* *and suppose that the functions g_i for $i \in I \backslash I(x_0)$ are continuous at x_0. Then, Theorem 2 remains true when condition* (b) *is replaced by the following one:*

(b$'$) *there exists $v \in \mathbb{R}^n$ such that, if we denote $g := \max\{g_i \mid i \in I(x_0)\}$, then*

$$g'_+(x_0; v) = \limsup_{t \to 0^+} t^{-1}(g(x_0 + tv) - g(x_0)) < 0. \tag{19}$$

Proof. Suppose that (b$'$) holds. Then it follows from (19) and from the equality $g(x_0) = 0$ that there exist numbers $\sigma > 0$, $t_0 > 0$ such that

$$g(x_0 + tv) \leq -\sigma t, \text{ for all } t \in (0, t_0). \tag{20}$$

Define $\tau := \max\{g_i(x_0) \mid i \in I \setminus I(x_0)\} < 0$. Let us choose $t_1 \in (0, t_0)$ so small that $x_0 + t_1 v \in \operatorname{int} W_r$ and

$$g_i(x_0 + t_1 v) \leq \tau/2, \text{ for all } i \in I \setminus I(x_0). \tag{21}$$

Suppose further that inequalities (12) (with $J = \emptyset$) are satisfied for some $\delta \in (0, \min\{\sigma t_1, -\tau/2\}]$. Since $r \leq \varepsilon$, we have $x_0 + t_1 v \in W_\varepsilon$. Hence, from (12) and (20), we get

$$\begin{aligned} &\widetilde{g}_i(x_0 + t_1 v) < g_i(x_0 + t_1 v) + \delta \leq g(x_0 + t_1 v) + \delta \\ &\leq -\sigma t_1 + \delta \leq 0, \text{ for all } i \in I(x_0). \end{aligned} \tag{22}$$

Observe also that (12) and (21) imply

$$\widetilde{g}_i(x_0 + t_1 v) < g_i(x_0 + t_1 v) + \delta \leq \tau/2 + \delta \leq 0, \text{ for all } i \in I \setminus I(x_0). \tag{23}$$

Inequalities (22) and (23) mean that $x_0 + t_1 v \in \widetilde{S}$, and consequently, $\widetilde{S} \cap \operatorname{int} W_r \neq \emptyset$. We have thus verified that conditions (b$'$) and (12) (for sufficiently small δ) imply condition (b) of Theorem 2. ∎

REMARK 3 *Consider the case when the functions g_i ($i \in I$) in Theorem 3 are locally Lipschitzian. Given a locally Lipschitzian function $g : \mathbb{R}^n \to \mathbb{R}$, we denote by $\partial g(x_0)$ the generalized gradient of g at x_0; see Clarke (1983). By repeating the argument of Remark 3.6 in Studniarski (1989), we can show that the following condition*

$$0 \notin \operatorname{co}\{\partial g_i(x_0) \mid i \in I(x_0)\}, \tag{24}$$

References

BONNANS, J.F., IOFFE, A. (1995) Second-order sufficiency and quadratic growth for nonisolated minima. *Math. Oper. Res.*, **20**, 801–817.

BURKE, J.V. and FERRIS, M.C. (1993) Weak sharp minima in mathematical programming. *SIAM J. Control Optim.*, **31**, 1340–1359.

CLARKE, F.H. (1983) *Optimization and Nonsmooth Analysis.* New York, Wiley-Interscience.

HYERS, D.H. (1978) On the stability of minimum points. *J. Math. Anal. Appl.*, **62**, 530–537.

HYERS, D.H. (1985) Stability of minimum points for problems with constraints. In: *Differential Geometry, Calculus of Variations, and their Applications, Dedic. Mem. L. Euler 200th Anniv. Death*, Lect. Notes Pure Appl. Math., **100**, 283–289.

KLATTE, D. (1994) On quantitative stability for non-isolated minima. *Control Cybernet.*, **23**, 183–200.

STUDNIARSKI, M. (1989) Sufficient conditions for stability of local minimum points in nonsmooth optimization. *Optimization*, **20**, 27–35.

STUDNIARSKI, M. (1999) Characterizations of weak sharp minima of order one in nonlinear programming. In: *System Modelling and Optimization (Detroit, MI, 1997)*, Chapman & Hall/CRC Res. Notes Math., **396**, 207–215.

STUDNIARSKI, M. and WARD, D.E. (1999) Weak sharp minima: characterizations and sufficient conditions. *SIAM J. Control Optim.* **38**, 219–236.

WARD, D. (1998) Sufficient conditions for weak sharp minima of order two and directional derivatives of the value function. In: A. V. Fiacco, ed. *Mathematical Programming with Data Perturbations.* Lect. Notes Pure Appl. Math., **195**, Dekker, New York, 419–436.