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A note on a Fenchel-Young type conjugacy for convexifiable functions

by

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Abstract: We provide definition of such a Fenchel-Young type duality for a convexifiable function f that its second dual is a given function f. Examples and possible applications are given.
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1. Introduction

In many applications of convex functions it is the Fenchel-Young duality, Rockafellar, Wets (1998), Ekeland, Temam (1976) that appears to be a most useful tool since a second conjugate for a convex l.s.c. function is the same function. The appropriate notion for invex functions has recently been obtained in Galewski (2003). The properties of such duality and comparison with some results in the convex case are provided. Our wish has been to extend these results to the case of convexifiable functions defined on a reflexive Banach space X with the duality pairing $\langle , \rangle$.

The results concerning the generalized duality were considered, although in a different setting, in Pallaschke, Rolewicz (1997). Although our main result, Theorem 2, is to some extent equivalent to a classical result, compare Dolecki, Kurcyusz (1978), it is interesting from the applicational point of view, since both conjugates may be calculated explicitly for a number of functions.

2. Fenchel-Young duality for a convexifiable function

The general framework for a generalized duality is to be found in Rockafellar, Wets (1998), but we only partly follow that way, since there is a lack of formal symmetry of the first and second dual, compare (2.1), (2.2).

First we make clear what we mean by convexifiability. Let $\varphi : X \to X$

$f : X \to \overline{R}$ for which there exists a proper convex and l.s.c. function $g : X \to \overline{R}$ such that $g = f \circ \varphi$, compare Craven (1995) for an equivalent formulation.

Let $f : X \to \overline{R}$. We define $f^{\varphi} : X^* \to \overline{R}$ by

$$f^{\varphi}(v) = \sup_{x \in X}\{\langle v, \varphi(x)\rangle - f(x))\} \tag{2.1}$$

and call it a φ-conjugate.

THEOREM 1 *For $f : X \to \overline{R}$ its φ-regularization reads $f^{\varphi} = g^*$, where $g = f \circ \varphi$.*

Proof. Observe that by the properties of φ

$$f^{\varphi}(v) = \sup_{x \in X}\{\langle v, \varphi(x)\rangle - f(x))\} = \sup_{\xi \in X}\{\langle v, \xi\rangle - g(\xi)\}.$$

This is the definition of a Fenchel-Young conjugate of g. ∎

From the above theorem there follows

COROLLARY 1 *If $f \in Conv_{\varphi}$, then its φ-regularization is convex and l.s.c. on X^*.*

We define the second dual $f^{\varphi\varphi} : X \to \overline{R}$ of a function $f : X \to \overline{R}$ as follows

$$f^{\varphi\varphi}(x) = \underset{v \in X^*}{sup}\{\langle v, \varphi(x)\rangle - f^{\varphi}(v)\} \tag{2.2}$$

and call it a $\varphi\varphi$-conjugate.

THEOREM 2 *Let $f \in Conv_{\varphi}$. Then $f^{\varphi\varphi} = f$.*

Proof. By (2.2) we obtain for any x

$$f^{\varphi\varphi}(x) = \sup_{v \in X^*}\{\langle v, \varphi(x)\rangle - f^{\varphi}(v)\} = \sup_{v \in X^*}\{\langle v, \varphi(x)\rangle - g^*(v)\}.$$

By the classical theorem from Ekeland, Temam (1976), we have

$$f^{\varphi\varphi}(x) = g^{**}(\varphi(x)) = g(\varphi(x)) = f(x).$$ ∎

When a function f is not convexifiable, $f^{\varphi\varphi}$ may be understood as an convexifiable minorant or as a kind of $\varphi\varphi$-regularization.

COROLLARY 2 *For any function $f : X \to \overline{R}$ its $\varphi\varphi$-regularization reads $f^{\varphi\varphi} =$*

3. Properties of the generalized duality

Some properties of the introduced duality mimic those of the classical Fenchel-Young conjugacy.

PROPOSITION 1 *Let f, f_1, $f_2 \in Conv_\varphi$.*

1. *(the Fenchel-Young inequality)*
$$f^\varphi(v) + f(x) \geq \langle v, \varphi(x) \rangle;$$
2. *if $f_1 \leq f_2$ then $f_1^\varphi \geq f_2^\varphi$.*

Proof. Let $g = f \circ \varphi$, $g_1 = f_1 \circ \varphi$, $g_2 = f_2 \circ \varphi$ and take any $x \in X$, $v \in X^*$. Since $g(\varphi(x)) = f(x)$ and $f^\varphi(v) = g^*(v)$ we have by Fenchel-Young inequality

$$f^\varphi(v) + f(x) = g^*(v) + g(\varphi(x)) \geq \langle v, \varphi(x) \rangle.$$

Now, let $f_1 \leq f_2$. We obtain that $g_1^*(v) \geq g_2^*(v)$, which by Definition (2.1) means that $f_1^\varphi(v) \geq f_2^\varphi(v)$. ■

PROPOSITION 2 *Let $f \in Conv_\varphi$ with $g = f \circ \varphi$. Then the following conditions are equivalent*

$$f^\varphi(v) + f(x) = \langle v, \varphi(x) \rangle,$$
$$v \in ((\nabla_x \varphi(x))^{-1})^* \partial f(x),$$
$$\varphi(x) \in \partial f^\varphi(v),$$

where $((\nabla_x \varphi(x))^{-1})^*$ denotes the mapping adjoint to $(\nabla_x \varphi(x))^{-1}$.

Proof. We observe that the equality

$$f^\varphi(v) + f(x) = \langle v, \varphi(x) \rangle$$

reads

$$g^*(v) + g(\varphi(x)) = \langle \tau, \varphi(x) \rangle.$$

This, by convexity of g, is equivalent to

$$v \in \partial_\xi g(\varphi(x))$$

and further to

$$\varphi(x) \in \partial g^*(v).$$

By differentiating the formula $f(x) = g(\varphi(x))$ with respect to x, we obtain the second condition in the assertion. ■

4. Examples and Applications

In the example below we will show how the φ-conjugate and $\varphi\varphi$-conjugate are calculated in the concrete examples.

EXAMPLE 1 Let us consider a convexifiable (and nonconvex) function $f : R \to R$ given by

$$f(x) = \frac{1}{2}x^6 + x^4 + \frac{1}{2}x^2 - 6\sqrt{\frac{1}{2}x^6 + x^4 + \frac{1}{2}x^2}.$$

$\varphi : R \to R$ and $g : R \to R$ read $\varphi(x) = x + x^3$, $g(\xi) = \frac{1}{2}\xi^2 - 6|\xi|$. We obtain by (2.1)

$$f^{\varphi}(v) = 18 + 6|v| + \frac{1}{2}v^2.$$

By (2.2) we then have

$$f^{\varphi\varphi}(x) = \sup_{\nu \in X^*} \{\langle v, \varphi(x)\rangle - f^{\varphi}(v)\} = \frac{1}{2}(\varphi(x))^2 - 6|\varphi(x)|.$$

Hence $f(x) = f^{\varphi\varphi}(x)$.

We shall also provide an example of $\varphi\varphi$-regularization.

EXAMPLE 2 Let us consider a non-convexifiable function $f : R \to R$ given by

$$f(x) = -(x \arctan x + 2)^2 x^2 + (x \arctan x + 2)^4 x^4.$$

We define $\varphi(x) = x^2 \arctan x + 2x$ and $g(\xi) = \xi^4 - \xi^2$ which is not convex. We obtain by (2.2)

$$f^{\varphi\varphi}(x) = \begin{cases} f(x) & \text{for} \quad \begin{array}{l} x^2 \arctan x + 2x \le -\frac{\sqrt{6}}{6} \\ \text{or } \frac{\sqrt{6}}{6} \le x^2 \arctan x + 2x \end{array} \\ -5/36 & \text{for} \quad -\frac{\sqrt{6}}{6} \le x^2 \arctan x + 2x \le \frac{\sqrt{6}}{6}. \end{cases}$$

EXAMPLE 3 Let $f : X \to R$ belong to $Conv_{\varphi}$ and let it be defined by $f(x) = \frac{a}{q}\|\varphi(x)\|^q + b$, where $a > 0, b \in R$, $q > 1$. Then $f^{\varphi}(v) = a^{-\frac{p}{q}} \frac{\|v\|_*^p}{p} - b$, $\frac{1}{p} + \frac{1}{q} = 1$, $\|\cdot\|_*$ denotes the norm in the space X^*.

REMARK 1 (Application to variational methods.) *The concept of such generalized duality appears to be of use when a variational method is applied to show the existence of a solution to a certain differential inclusion in case the action functional is such that its second Fenchel-Young dual is equal $-\infty$. For example, one can construct, basing on the dual method from Nowakowski, Rogowski (2001), the duality theory for the following problem*

$$\begin{cases} 0 \in \frac{d}{dt}(L(t, \varphi(x(t)))\nabla_x \varphi(x(t))\dot{x}(t)) + \\ ((\nabla_x \varphi(x(t)))^{-1})^* \partial_x G(t, x(t)) \text{ for a.e. } t \in [0, T] \end{cases}$$

where $T > 0$, X *is separable reflexive Banach space,* $G : [0,T] \times X \to X$ *and* $L : [0,T] \times X \to X$ *are convexifiable in the second variable with respect to* $\varphi : X \to X$*;* G, L *are Caratheodory functions and are subject to suitable growth conditions involving* φ*, namely for all* x *and a.e.* t

$$\frac{1}{2}\|\varphi(x)\|^2 + c \leq L(t,x) \leq \frac{1}{2}\|\varphi(x)\|^2 + c_1$$
$$\frac{a}{q}\|\varphi(x)\|^q + b \leq G(t,x) \leq \frac{a_1}{q_1}\|\varphi(x)\|^{q_1} + b_1$$

where $q_1 \geq q \geq 2$, $a_1, a > 0$, $b_1, b, c_1, c \geq 0$.

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