Stable and related matrices in economic theory

by

Giorgio Giorgi

Faculty of Economics
University of Pavia
Via S. Felice 5, 27100 Pavia, Italy
E-mail: ggiorgi@eco.unipv.it

Abstract: It is well known that local stability analysis of a Walrasian multiple markets model is performed by approximating, according to Taylor's expansion, a system of first-order differential equations. So, one has to study the stability of a linear system (with constant coefficients). Since the earlier studies of Walrasian economic equilibrium, economists have suggested numerous conditions ensuring local stability of the same.

The aim of this note is to give a survey of various conditions, used in economic analysis, ensuring that a (real) square matrix is stable. We show, in a unified manner, their inter-relations and make some new remarks on quasi-dominant matrices and on D-stable matrices.

Keywords: stability theory, Walrasian system, stable matrices.

1. Introduction

The study of economic equilibrium has been concerned, since its earlier rigorous treatments, with the "stability" of that price structure which clears the commodity markets. The aim of this paper is to take into consideration various local stability conditions for a Walrasian system in a unified matter. Moreover, we present a diagram which shows the various relationships between the conditions of stability considered, emphasizing the role of the so-called " Metzlerian assumption".

To be more precise: we consider an economy in which there are n goods exchanged at positive prices. In such a system, if $D_i(p_1, p_2, \ldots, p_n)$ is the demand function for the i-th good and $S_i(p_1, p_2, \ldots, p_n)$ is the related supply function, we can express the equilibrium situation as $E_i(p_1^*, p_2^*, \ldots, p_n^*) = 0$, $i = 1, 2, \ldots, n$, were E_i denotes the excess demand function for good i, i.e. $E_i(.) = D_i(.) - S_i(.)$. Suppose now that in the said market there is a disturbance, for example a rise in

functions will be affected and the equilibrium price vector of the old situation will no longer clear all markets. The aim of economic stability analysis is to determine the conditions under which the process of adjustment will converge to a new equilibrium price vector.

The modern approach, due to Samuelson (1941) and Lange (1945), formalizes the Walrasian tâtonnement process by means of a system of differential equations of the type

$$\frac{dp_i}{dt} = f_i(E_i(p(t))), \quad i = 1, \ldots, n, \tag{1}$$

where $p(t) = [p_1(t), \ldots, p_n(t)]$, $f_i(0) = 0$, $f_i'(.) > 0$, $i = 1, \ldots, n$, i.e. the time rate of change of any price is an increasing function of excess demand for that comodity, which vanishes when excess demand vanishes. An important special case is the *linear tâtonnement system*, formally described by

$$\frac{dp_i}{dt} = k_i E_i(p(t)), \quad i = 1, \ldots, n, \tag{2}$$

where factor $k_i > 0$ is the "speed of adjustment" on the i-th market, a system which gives rise to the concept of "stability independent of adjustment speeds" (diagonally stable matrices or D-stable matrices).

If we assume that all speeds of adjustment k_i are the same and equal to unity, we have the process

$$\frac{dp_i}{dt} = E_i(p(t)), \quad i = 1, \ldots, n. \tag{3}$$

Consider now the system of differential equations (3) and its equilibrium point (supposed unique) p^*; we may approximate this system in a neighbourhood of p^* by the Taylor's formula

$$\frac{dp}{dt} = E(p^*) + \frac{\partial E}{\partial p}(p^*)(p - p^*) + \ldots,$$

where the Jacobian matrix $\partial E / \partial p$ is evaluated at the equilibrium point. Since $E(p^*) = 0$, by defining π as the vector of discrepancies between prices and equilibrium prices ($\pi = p - p^*$), we obtain

$$\frac{d\pi}{dt} = A\pi(t),$$

where $A = \frac{\partial E}{\partial p}(p^*)$.

So, the study of global stability of a dynamic linear system (with constant coefficients) leads to consideration of local stability of the original market mechanism. A crucial economic assumption for existence, uniqueness of equilibrium prices, stability and comparative statics results (see the bibliographical references) is that the commodities exchanged in the markets are *gross substitutes*

all p. Economically speaking, an increase in the price of any commody, holding all other prices constant, increases excess demand for any other commodity.

Economists call "Metzlerian matrix" (from the economist L.A. Metzler) a square matrix A, with $a_{ij} > 0$ (or $a_{ij} \geq 0$) for all $i \neq j$. This explains the emphasis on the "Metzlerian case" in the diagram at the end of the present paper. We also give some new results concerning quasi-dominant matrices and D-stable matrices.

2. Definitions and results

It is well known that, given the homogeneous, autonomous linear system with constant coefficients

$$x'(t) = Ax(t) \tag{1}$$

where A is an $n \times n$ matrix, its equilibrium solution $\hat{x} = 0$ is *globally stable* if and only if the real part of any eigenvalue of A is negative: $\mathrm{Re}(\lambda_i(A)) < 0$, $\forall i$.

For simplicity, also in view of economic applications, we shall consider only the case where A is a *real* matrix, even if it is easy to adapt the subsequent considerations to the complex case. If we are given a nonlinear autonomous system of first-order differential equations

$$x'(t) = f(x(t)), \tag{2}$$

with $f : \Re^n \to \Re^n$, f continuously differentiable, it is well known that its equilibrium solution $\hat{x} = a$ (i.e. one has $f_i(a) = 0$, $i = 1, \ldots, n$) is *locally* or *asymptotically stable* if $Jf(a)$, the Jacobian matrix of (2) evaluated at a, has only eigenvalues with negative real parts (if at least one eigenvalue has a positive real part, then a is unstable).

It has been shown, quite recently, that even if Jf has all eigenvalues with negative real part, for *any* x in the domain of f, we cannot deduce the global stability of the equilibrium solution of (2) $\hat{x} = a$ (see Cima, van den Essen, Gasull, Hubbers, Manosas, 1997).

A square matrix such that (1) is globally stable, i.e. $\mathrm{Re}(\lambda_i(A)) < 0$, $\forall i$, is also called a *stable matrix*. Due to the importance of stable matrices in economic theory, there is a vast and scattered literature on conditions assuring the stability of A. Here, we put together the most important ones, make some remarks on the so-called D-stability and matrices with quasi-dominant diagonals and show the relations between the various classes of matrices considered.

First of all we recall that we have some necessary and sufficient conditions asssuring that A is stable. One of them is contained in the famous Routh-Hurwitz conditions (Bellman, 1953, Gantmacher, 1959, Lancaster, 1969, Quirk,

THEOREM 1 *Let A be an $n \times n$ real matrix and let $tr_i(A)$ be the trace of order i of A, i.e. the sum of all its $\binom{n}{i}$ principal minors of order i. Let*

$$K_i = \begin{cases} (-1)^i tr_i(A), \ \forall i = 1, \ldots, n \\ 0, \ \forall i > n \ \text{or for } i < 0 \\ 1 \ \text{for } i = 0 \end{cases}$$

and let

$$H_i = \begin{vmatrix} K_1 & K_3 & K_5 & K_7 & \ldots \\ K_0 = 1 & K_2 & K_4 & K_6 & \ldots \\ K_{-1} = 0 & K_1 & K_3 & K_5 & \ldots \\ \hdotsfor{5} \\ \hdotsfor{4} & K_i \end{vmatrix}, \qquad i = 1, 2, \ldots, n.$$

Then A is stable if and only if $H_1 > 0, H_2 > 0, \ldots, H_n > 0$. (This implies $K_i > 0, \ \forall i = 1, \ldots, n$).

This criterion is also attributed to Liénard-Chipart and Fuller (Gandolfo, 1997, Gantmacher, 1959). See also Murata (1977) for a simpler version.

Before introducing the other necessary and sufficient conditions for the stability of A, let us recall the following definitions:

DEFINITION 1 *Let A be symmetric; A is positive definite (p.d.) if $x^T A x > 0$, $\forall x \neq 0$; A is negative definite (n.d.) if $x^T A x < 0$, $\forall x \neq 0$. The square matrix A is positive quasidefinite (p.q.d.) if $x^T A x > 0$, $\forall x \neq 0$, or, equivalently, if $A + A^T$ is p.d. The square matrix A is negative quasidefinite (n.q.d.) if $x^T A x < 0, \forall x \neq 0$.*

THEOREM 2 (Lyapunov theorem; see Lancaster, 1969) *Let W be a (symmetric) n.d. matrix; then A is stable if and only if there exists a p.d. matrix B such that $A^T B + BA = W$.*

Let us now recall the following definitions and notations.

• $M = M(A) = \lfloor m_{ij} \rfloor$, $i, j = 1, \ldots, n$ is the *comparison matrix* of A, i.e.

$$m_{ij} = \begin{cases} |a_{ii}|, & \text{if } i = j \\ -|a_{ij}|, & \text{if } i \neq j. \end{cases}$$

• The square matrix $Z = \lfloor z_{ij} \rfloor$ is a Z-matrix (belongs to the class Z) if it is $z_{ij} \leq 0$, $\forall i \neq j$. Note that $M(A)$ is a Z-matrix.

• D is the class of diagonal matrices with real diagonal elements; D^+ is the class of diagonal matrices with *positive* diagonal elements.

• The square matrix A is a P-matrix (belongs to the class P) if all its principal minors are positive. A satisfies the *Hawkins-Simon conditions* (H.-S.) if all its

- We shall note (with x vector of $\Re^n$)

  $x \geqq 0$, if $x_i \geq 0$, $\forall i \in N = \{1,\ldots,n\}$ (x non negative vector)

  $x > 0$, if $x_i > 0$, $\forall i \in N = \{1,\ldots,n\}$ (x positive vector)

  $x \geq 0$, if $x \geqq 0$, $x \neq 0$ (x semipositive vector)

  $x \geq_H 0$, if $x \geq 0$, $\displaystyle\sum_{i \in H} x_i > 0$, ($H$ proper nonempty subset of N)

  $x = u$, if $x_i = 1$, $\forall i \in N = \{1,\ldots,n\}$.

Similarly for the reversed notations and similarly for the comparison between a matrix A of order $m \times n$ and the $m \times n$ zero matrix.

- S is the class of matrices M (not necessarily square) such that there exists a vector $x \geq 0$ solution of $Mx > 0$.
- S^0 is the class of matrices M for which there exists a vector $x \geq 0$ solution of $Mx \geqq 0$.
- $A = \lfloor a_{ij} \rfloor$, $i,j \in N$, is reducible or *H-reducible* if $N = \{1,\ldots,n\}$ admits a nonempty proper subset H such that $(i \in H, j \notin H) \Rightarrow a_{ij} = 0$. Otherwise A is *irreducible* or *connected*.

In economic literature we encounter also the following defininitions:

If B is a P-matrix, then $-B = A$ is called a *Hicksian matrix*, i.e. A has its principal minors of order i with the sign of $(-1)^i$, $i = 1,\ldots,n$. A is also called a *NP-matrix*. If $B \in Z$, then $-B = A$ is a *Metzlerian matrix*, i.e. $a_{ij} \geq 0$, $\forall i \neq j$.

DEFINITION 2 *Let $A \in Z$; then A is a* K-*matrix (belongs to the class* K*) if A is a* P-*matrix. So* K *is the set of* Z-*matrices:* $K = Z \cap P$.

More than 70 tests are available to check whether a Z-matrix is in K (see, e.g., Berman, Plemmons, 1976, Fiedler, Ptak, 1962, 1966, Magnani, Meriggi, 1981, Samuelson, 1947, Uekawa, 1971).

As Metzlerian matrices are of utmost importance in obtaining stability results and as A is Metzlerian iff $-A \in Z$, it is possible and convenient to adapt the above tests in order to check the stability of a Metzlerian matrix. Here we list the most useful conditions.

THEOREM 3 *Let A be Metzlerian; then the following conditions are mutually equivalent.*

1. *The real part of all the eigenvalues of A is negative, i.e. A is stable.*
2. $-A$ *is a* K-*matrix, i.e. A is Hicksian.*
3. $-A$ *satifies the H.-S. conditions, i.e. all the leading principal minors of A have the sign of $(-1)^i$, $i = 1,\ldots,n$.*
4. *There exists a vector $x \geq 0$ such that $Ax < 0$.*
5. *There exists a vector $x > 0$ such that $Ax < 0$.*
6. *For any $y \leqq 0$ there exists an $x \geqq 0$ such that $Ax = y$.*

8. The matrix A is nonsingular and we have $A^{-1} \leq 0$.
9. If A is written in the form $A = [B - \rho I]$, where $B \geq 0$ and $\rho \in \Re$, then it is $\rho > \lambda^*(B)$, where $\lambda^*(B)$ is the Frobenius root of B.
10. If A is written as in 9 above, then

$$[B - \rho I]^{-1} = -\frac{1}{\rho} \sum_{k=0}^{+\infty} \left(\frac{1}{\rho}B\right)^k.$$

11. There exists a diagonal matrix $D \in \mathsf{D}^+$ such that $(A^T D + DA)$ is n.d.

For the proof of the equivalence between Conditions 1 to 10 see, e.g. Berman, Plemmons (1976), Fiedler, Ptak (1962, 1966), Takayama (1985). Condition 11 is due to Tartar (1971).

Obviously, Condition 4 is equivalent to $-A \in \mathsf{S}$. As the classes S and S^0 are linked by the *Ville theorem of the alternative* ($M \in \mathsf{S} \Leftrightarrow -M^T \notin \mathsf{S}^0$), Condition 4 is equivalent to $A^T \notin \mathsf{S}^0$.

It is rather simple to prove the following result:

THEOREM 4 *Let A be Metzlerian; if A satisfies any of the conditions described in Theorem 3, then the same condition holds for A^T, for $\Pi A \Pi^T$, where Π is any permutation matrix, and for DAE, where $D, E \in \mathsf{D}^+$.*

If the Metzlerian matrix A is, in addition, irreducible, then some conditions from Theorem 3 may be expressed in another form. For example, we have:

4'. There exists a vector $x \geq 0$ such that $Ax \leq 0$.
7'. There exists a $y \leq 0$ such that $Ax = y$ admits a solution $x > 0$.
8'. A^{-1} exists and there is $A^{-1} < 0$.

An important generalization of the Metzlerian matrices has been introduced by Morishima (1952).

DEFINITION 3 *The $n \times n$ matrix A is a Morishima matrix, if there exists a permutation matrix Π such that*

$$\Pi A \Pi^T = A_\Pi = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

with $A_{11} \geq 0$, $A_{22} \geq 0$, $A_{12} \leq 0$ and $A_{21} \leq 0$, A_{11} and A_{22} being square matrices.

Note that A_Π and the nonnegative matrix

$$\begin{bmatrix} A_{11} & -A_{12} \\ -A_{21} & A_{22} \end{bmatrix}$$

are similar, and therefore have the same eigenvalues, namely

$$\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \end{bmatrix} \begin{bmatrix} I & 0 \end{bmatrix} = \begin{bmatrix} A_{11} & -A_{12} \end{bmatrix}$$

So, Theorem 3 may be reformulated under the assumption that A is a Morishima matrix with A_{11} and A_{22} Metzlerian matrices (MM matrices): in particular, we may note that an MM matrix A is stable if and only if is Hicksian (i.e. 1 and 2 of Theorem 3 remain unchanged).

It is also possible to prove that if A is written in the form $A = [M - \rho I]$, where M is a Morishima matrix, then A is stable if and only if $\rho > \lambda^*(M)$. In this case $-A^{-1}$ exists and is a Morishima matrix.

In McKenzie (1960) the following important definition which generalizes a previous one, due to Hadamard, was introduced:

DEFINITION 4 *The $n \times n$ matrix A is said to have a row dominant diagonal (rdd) if there exists a vector $d > 0$ such that*

$$d_i|a_{ii}| > \sum_{j \neq i} d_j|a_{ij}|, \quad i = 1, \ldots, n.$$

A is said to have a column dominant diagonal (cdd) if there exists a vector $d > 0$ such that

$$d_j|a_{jj}| > \sum_{i \neq j} d_i|a_{ij}|, \quad j = 1, \ldots, n.$$

Using the comparison matrix the above definition may be rewritten as follows: $\exists d > 0$: $Md > 0$ (row dominant diagonal) or $dM > 0$ (column dominant diagonal). In other words, the Z-matrix M is a K-matrix or M^T is a K-matrix.

In the original Hadamard definition (Hrdd and Hcdd) we have $d = u$. As both sets Z and K are closed under transposition, it is therefore clear that A has an rdd if and only if it has a cdd (this is not true for the Hadamard definition); so, we have no need to specify whether the dominance refers to the rows or to the columns and simply speak of dominant diagonal (dd) matrices. Therefore, the class of dd matrices is closed with respect to transposition. It is not hard to prove that if A has dd , also DAE, with $D, E \in$ D$^+$, has dd, and conversely.

The most important results of McKenzie are contained in the following theorem.

THEOREM 5 *i) If A has a dd, then $|A| \neq 0$; ii) if A is Metzlerian then A is stable if and only if has a negative dd (ndd) , i.e. we have $a_{ii} < 0$, $\forall i$ and A has a dd.*

McKenzie proved also that if A has an ndd, then A is stable, i.e. the sufficient part of ii) in the previous theorem holds without the Metzlerian assumption. Moreover, it can be shown that ii) of Theorem 5 holds also under the assumption that A is an MM-matrix.

McKenzie (1960) introduces also the following definition:

DEFINITION 5 *The $n \times n$ matrix A is said to have a quasidominant diagonal*

    i) $\exists d > 0 : Md \gneq 0$ *or* $dM \gneq 0$ *in case A is irreducible;*

    ii) $\exists d > 0 : Md \geq_H 0$ *or* $dM \geq_H 0$ *in case A is H-reducible (for any choice of H).*

The above definition is not acceptable, as, for example, the following irreducible matrix

$$A = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

exhibits nqdd but it is $|A| = 0$ and A is not therefore stable.

Similarly, the following definition of Lancaster (1968) is not acceptable:

DEFINITION 6 *The $n \times n$ matrix A has a qdd if $\exists d > 0 : Md \geq 0$ or $dM \geq 0$.*

Consider, e.g., the following reducible matrix

$$A = \begin{bmatrix} -8 & 2 & 2 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -2 & 0 \\ 1 & 1 & 1 & -5 \end{bmatrix}$$

which has nqdd in the sense of Definition 6, but $|A| = 0$ and A is not stable.

It must be noted that *under the assumption that A is irreducible,* Definition 6 coincides with the one given by Taussky (1949), which does not present the above anomalies.

L.W. McKenzie has revised his Definition 5 along the following lines (the indirect reference is in Uekawa, 1971):

DEFINITION 7 *The $n \times n$ matrix A is said to possess a qdd if for each principal submatrix $A(J)$ of A ($J \subseteq N$) we have $d(J)M(J) \geq 0$ or $M(J)d(J) \geq 0$, with $d(J) > 0$ ($d(J)$ being the vector with components $d_j, j \subset J$).*

Definition 7 is equally applicable to the case where A is irreducible as to the one where it is reducible. Moreover, by adopting the revised Definition 7, we can prove, following the same lines of McKenzie (1960), that A has a dd if and only if A has a qdd (in the sense of Definition 7).

It is clear, from Definition 7, that if A has a qdd , then every principal submatrix of A has a qdd and all the diagonal elements of A are nonzero.

Obviously, the same type of amendment must be applied to the Definition 6 of Lancaster.

It is easy to prove that another definition of the qdd matrix, equivalent to Definition 7, is the following one, in which we use the *Gantmacher normal form* (Gantmacher, 1959): the $n \times n$ matrix A is a qdd matrix if there exists

Gantmacher normal form of A, verifying $M(B_{kk})d(k) \geq 0$ or $d(k)M(B_{kk}) \geq 0$, $k = 1, \ldots, s$.

Some other conditions of stability of A are obtained under the assumption that A is *symmetric*. In this case the following result is well known.

THEOREM 6 *Let $A = A^T$. Then the following conditions are mutually equivalent*
i) A is stable.
ii) A is nd.
iii) A is Hicksian.
iv) $-A$ satisfies the H.-S. conditions.

Some other important definitions related to stable matrices are given here below.

DEFINITION 8 (see Arrow, McManus, 1958) *The $n \times n$ matrix A is weakly D-stable or simply D-stable, if DA is stable for each $D \in \mathsf{D}^+$. A is strongly D-stable if $D \in \mathsf{D} \Rightarrow \{DA$ is stable $\Leftrightarrow D \in \mathsf{D}^+\}$.*

DEFINITION 9 (Quirk, Ruppert, 1965) *The $n \times n$ matrix A is totally stable if every principal submatrix $A(J)$ of A $(J \subseteq N)$ is D-stable.*

Obviously, a totally stable matrix is D-stable; the converse is true if A is Metzlerian. Indeed, in this case we have A stable $\Rightarrow A$ ndqd, whereas the following implications are always true: A ndqd $\Rightarrow A$ totally stable $\Rightarrow AD$-stable $\Rightarrow A$ stable.

DEFINITION 10 (Arrow, McManus, 1958) *A is S-stable if, with $S = S^T$, $\{SA$ is stable $\Leftrightarrow S$ p.d.$\}$. A is totally S-stable if every principal submatrix of A is S-stable.*

DEFINITION 11 (Quirk, Ruppert, 1965, Quirk, Saposnik, 1968) *The matrix A is qualitatively stable or sign stable if A is stable, as well as any other $n \times n$ matrix B having the same sign structure as A (each element of B has the same sign as the corresponding element of A; zero elements correspond to zero elements).*

Of course, if A is qualitatively stable, then A is also D-stable. Moreover, Quirk and Ruppert (1965) have shown that if A is qualitatively stable and $a_{ii} < 0 \ \forall i$, then A is totally stable.

DEFINITION 12 (Quirk, Saposnik, 1968) *The $n \times n$ matrix A is potentially stable if there exists a matrix B having the same sign structure of A, which is*

Of course if A is qualitatively stable, then it is potentially stable.

If there is $a_{ii} < 0\ \forall i = 1, \dots, n$, then A is potentially stable: indeed, one can always choose B with diagonal elements large enough in absolute value, in order to obtain that B is an ndd matrix (therefore stable).

Also with reference to the above definitions some words of comment are useful. First we note that it is indifferent to speak of D-stability when AD is stable and similarly to speak of S-stability when AS is stable for any p.d. matrix S (indeed SA and AS are similar).

In Arrow, McManus (1958) the following results are proved:

THEOREM 7 *A is strongly D-stable if and only if there exists $D \in \mathsf{D}$, $|D| \neq 0$, such that $E = D^{-1}AD$ is strongly D-stable. If there exists a matrix $D \in \mathsf{D}$ such that $D^{-1}AD$ is a stable Metzlerian matrix or a nqd matrix, then A is strongly D-stable.*

In the same paper, the authors formulate also the following conjecture:

CONJECTURE. *A is strongly D-stable if and only if there exist a matrix F, with F Metzlerian stable or qnd , and a matrix $E \in \mathsf{D}$ such that $A = EFE^{-1}$.*

This conjecture is shown false by considering the matrix

$$A = \begin{bmatrix} -1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

which is D-stable, both in the weak and in the strong sense, but which does not admit the representation described in the above conjecture.

We note that stable Metzlerian matrices are strongly D-stable; therefore, if A is Metzlerian and satisfies anyone of the conditions of Theorem 3, then A is strongly D-stable. What we are able to prove is the following result:

THEOREM 8 *If $n = 2$ or if $A = EFE^{-1}$ with $E \in \mathsf{D}$ and F a Metzlerian stable matrix, then A is D-stable if and only if it is strongly D-stable.*

Proof. Obviously, strong D-stability implies weak D-stability. Let us now suppose that, given $n = 2$, A is D-stable only in the weak sense, i.e. $D \in \mathsf{D}^+ \Rightarrow DA$ is stable, but there exists a matrix D^* such that $D^* \in \mathsf{D}$, $D^* \notin \mathsf{D}^+$, D^*A is stable.

Then we have

$$a_{11} \leq 0, a_{22} \leq 0, a_{11} + a_{22} < 0, d_1^* < 0, d_2^* < 0$$

and therefore

Legend:

→ implication always true

→ implication generally false but true if A is metzlerian ($a_{ij} \geqq 0$, $\forall i \neq j$)

Boxes:

- $A + A^T$ neg. def.
- A^T neg. q. def.
- $\exists D \in \mathcal{D}^+$: DA is stable
- $D, E \in \mathcal{D}^+ \Rightarrow$ $\Rightarrow -DAE \in \mathcal{P}$
- A hicksian (i.e. N-P matrix)
- A^{-1} neg. q. def.
- A neg. q. def.
- $-A \in \mathcal{P}$
- $-A^T \in \mathcal{P}$
- A totally $\mathcal{S}$-stable
- BAB^{-1} stable $\forall B$ (regular)
- A $\mathcal{S}$-stable
- A potentially stable
- A verifies the Hawkins-Simon conditions
- A strongly $\mathcal{D}$-stable
- A qualit. stable
- $-A \in \mathcal{S}$
- $-A^T \in \mathcal{S}$
- BAB^{-1} stable for some B (regular)
- A stable
- A weakly $\mathcal{D}$-stable
- A totally stable
- $A^T \not\subset \mathcal{S}^o$
- $A \not\subset \mathcal{S}^o$
- A^T stable
- $|A - \lambda I| = 0 \Rightarrow Re(\lambda) < 0$
- $A^{-1} \leq 0$
- A verifies the Routh-Hurwitz conditions
- A has rdd < 0
- A has cdd < 0
- $\exists \lambda, Q$: $A = Q - \lambda I$ $Q \geqq 0$, $\lambda > \lambda^*(Q)$
- $\exists B, B$ pos.def.: BA neg. q. def. (Lyapunov conditions)
- $\exists D: D \in \mathcal{D}^+$, DA neg. q. def.
- A has Hrdd < 0
- A has Hcdd < 0
- $D, E \in \mathcal{D}^+ \Rightarrow$ $\Rightarrow$ DAE has dqd < 0

whereas the Routh-Hurwitz criterion, applied to D^*A, implies the opposite inequality. Therefore with $n = 2$ the weak D-stability implies the strong D-stability.

Let us now suppose $A = EFA^{-1}$ is weakly D-stable, $E \in \mathsf{D}^+$ and F Metzlerian. Therefore, $IF = F$ is stable, as well as $F = E^{-1}AE$. Hence, F is Metzlerian and stable and from Theorem 7 we have that F is strongly D-stable. ■

Another interesting result on D-stability is due to Fisher and Fuller (1958):

if A is Hicksian, there exists a positive diagonal matrix D such that all the roots of DA are real, negative and simple.

Therefore, if A is Hicksian, there exists at least a matrix $D \in \mathsf{D}^+$ such that DA is stable.

Other sufficient conditions for D-stability of A (besides the qualitative stability of A) are the following ones (see Arrow, McManus, 1958, McKenzie, 1960).

i) *If there exist a matrix $D \in \mathsf{D}^+$ such that $(DA + A^T D)$ is n.d. , then A is (weakly) D-stable.*

ii) *If A has a nqdd , then A is D-stable.*

A sufficient condition for S-stability is given in Arrow, McManus (1958): *nqd matrices are S-stable (and therefore strongly D-stable).*

S-stable matrices are characterized by Carlson (1968) and Carlson, Schneider (1963).

We are now ready to present our scheme showing the various relations between the classes introduced. The proofs of the nontrivial implications may be found in the works quoted in the present paper.

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