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Mechanical and thermal null controllability of thermoelastic plates and singularity of the associated mimimal energy function

by

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Abstract: The null controllability problem is considered for 2-D thermoelastic plates under *hinged* mechanical boundary conditions. The resulting partial differential equation system generates an *analytic* semigroup on the space of finite energy. Consequently, because the thermoelastic system is associated with an infinite speed of propagation, the *null* controllability question is a suitable one for contemplation. It is shown that all finite energy states can be driven to zero by means of $L^2(Q)$–mechanical or thermal controls. In addition, the singularity of the minimal energy function, as $T \downarrow 0$, is also investigated. Ultimately, we establish the *optimal* blowup rate $\mathcal{O}(T^{-\frac{5}{2}})$ for this function, in the case one control (either mechanical or thermal) is acting upon the system and $\mathcal{O}(T^{-\frac{3}{2}})$, in the case of two controls (thermal and mechanical). This rate of singularity is *optimal* and in fact the same as obtained by considering finite dimensional truncations of the thermoelastic PDE.

Keywords: thermal plates, null controllability, singularity of minimal energy.

1. Introduction

Let H be a Hilbert space and let $A : D(A) \subset H \to H$ be a given selfadjoint, positive definite operator defined on a dense domain $D(A)$ in H. In this paper we consider the following abstract model of thermoelasticity:

$$\left\{ \begin{cases} \omega_{tt} + A^2\omega - \alpha A\theta = au_1 & \text{on } (0,T) \\ \theta_t + A\theta + \alpha A\omega_t = bu_2 & \text{on } (0,T) \end{cases} \right. \tag{1}$$

Here, the function u_1 (resp., u_2)$\in L^2(0,T;H)$ represents a mechanical (resp. thermal) control for the system. Moreover, a and b are nonnegative constants such that $a+b>0$. The coupling parameter α is assumed positive.

It is known that the above uncontrolled system (i.e., with $u_1 = u_2 = 0$) generates an analytic semigroup (Lasiecka, Triggiani, 2000, Liu, Renardy, 1995) on the space

$$\mathbf{H} \equiv D\left(A\right) \times H \times H. \tag{2}$$

The Hilbert space $\mathbf{H}$ does indeed constitute the space of wellposedness for the problem, inasmuch as one has continuity of the map

$$\{[\omega_0,\omega_1,\theta_0]\,,u_i\} \in \mathbf{H} \times L^2(0,T;H) \Rightarrow [\omega,\omega_t,\theta] \in C([0,T];\mathbf{H}), \tag{3}$$

for $i=1$ or $i=2$. (see e.g., Theorem 1 of Avalos, Lasiecka, 1996). Our aim is to study the *null controllability* problem associated with the abstract system (1).

A canonical prototype for the abstract model in (1) is the thermal plate equation with hinged (or simply supported) boundary conditions. Other concrete models, such as plate equations, with periodic boundary conditions or which are defined on the whole of space, can also be considered within this framework.

We illustrate the abstraction (1) with a particular example. Let Ω be a bounded, open subset of $\mathbb{R}^2$, with its C^2-boundary denoted as Γ. On $Q \equiv (0,T)\times\Omega$, we will consider the problem of null controllability for the following thermoelastic partial differential equation (PDE):

$$\begin{cases} \begin{cases} \omega_{tt} + \Delta^2\omega + \alpha\Delta\theta = au_1 \text{ on } Q \\ \theta_t - \Delta\theta - \alpha\Delta\omega_t = bu_2 \text{ on } Q \end{cases} \\ \\ \begin{cases} \omega|_\Gamma = 0 \\ \Delta\omega|_\Gamma = 0 \text{ on } (0,T)\times\Gamma \end{cases} \\ \theta|_\Gamma = 0 \text{ on } (0,T)\times\Gamma \\ \omega(t=0) = \omega_0;\ \omega_t(t=0)=\omega_1;\ \theta(t=0)=\theta_0 \text{ on } \Omega. \end{cases} \tag{4}$$

The (control) functions $u_i \in L^2(Q)$, and $[\omega_0,\omega_1,\theta_0]$ are the given initial data. For this example, we will continue the efforts initiated in Avalos, Lasiecka (2002a), in which the null controllability problem is considered when a sole *thermal* control is imposed upon the system. In this paper, we consider a rather more difficult problem where *mechanical* control is in play.

In order to put (4) into the abstract framework of (1), we set $H \equiv L_2(\Omega)$, and define $A_D : D(A_D) \subset L^2(\Omega) \to L^2(\Omega)$ to be the Laplacian operator with Dirichlet boundary conditions. That is,

$$A_D f = -\Delta f, \text{ for } f \in D(A_D) = H^2(\Omega) \cap H_0^1(\Omega) \tag{5}$$

With this quantity, we can proceed to define the *finite energy space* for the system (4) by

$$\mathbf{H} \equiv D(A_D) \times L^2(\Omega) \times L^2(\Omega). \tag{6}$$

With this elliptic operator, one can then exploit the compatibility between the mechanical boundary conditions imposed on the component ω and the domain of definition $D(A_D)$, so as to rewrite the PDE (4) as the abstract model (1), with $A \equiv A_D$.

As mentioned at the top, we wish to discuss the problem of *null controllability* for the abstract system (1), which would then imply the corresponding results for the coupled equations (4). That is to say, we wish to determine if solutions with arbitrary initial data $[\omega_0, \omega_1, \theta_0] \in \mathbf{H}$ can be driven to the zero state, for an appropriate choice of controls $[u_1, u_2]$ in $H \times H$. This notion of controllability for (1) is appropriate for discussion, as it has recently been shown that the operator theoretic model (1), and consequently the concrete 2-D thermoelastic systems, which are abstractly described by (1), can be associated, with an *analytic* C_0 - semigroup $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0}$ on $\mathbf{H}$ (see Lasiecka, Triggiani, 1998a, 2000). Moreover, since it is known that the system (1) is associated with an infinite speed of propagation, then *should* the system (1) have the null controllability property, this property would then obtain in an arbitrarily short time $T > 0$. This consideration motivates then our formal definition of null controllability:

DEFINITION 1 *The PDE system (1) is said to be null controllable, if for any time $T > 0$ and arbitrary initial data $[\omega_0, \omega_1, \theta_0] \in \mathbf{H}$, there exists a control function $u = [u_1, u_2] \in H \times H$ such that the corresponding solution $[\omega, \omega_t, \theta]$ to (1) satisfies $[\omega(T), \omega_t(T), \theta(T)] = [0, 0, 0]$. If, in addition, one can take $u_1 = 0$ (resp., $u_2 = 0$) then we say that the system (1) is thermally (resp., mechanically) null controllable.*

Assuming for the time being that the null controllability property in Definition 1 holds true for the PDE (1), we wish to subsequently determine the rate of "blowup" for the system's associated *minimal energy function*, denoted here as $\mathcal{E}_{\min}(T)$. To explain our meaning of this quantity (again under the assumption that Definition 1 is valid), we consider the optimization problem associated with the null controllability of (1). Namely, let us set $\mathbf{x}_0 = [\omega_0, \omega_1, \theta_0] \in \mathbf{H}$, $u = [u_1, u_2] \in L^2(0, T; H \times H)$, and let

$$\mathbf{x}(t; \mathbf{x}_0; u) \equiv [\omega(t), \omega(t), \theta(t)]. \tag{7}$$

That is, $\mathbf{x}(t; \mathbf{x}_0; u)$ denotes the solution $[\omega(t), \omega(t), \theta(t)]$ to (1), corresponding to initial data $\mathbf{x}_0 \in H$ and $u \in L^2(0, T; H \times H)$. Therewith, we search for $u_T^0(\mathbf{x}_0) \in L^2(0, T; H \times H)$ which solves

$$\left\|u_T^0(\mathbf{x}_0)\right\|_{L^2(0,T;H\times H)} = \min_{u \in L^2(0,T;H\times H)} \left\|u(\cdot, \mathbf{x}_0, u)\right\|_{L^2(Q)}. \tag{8}$$

This minimization problem is well-understood, and admits a unique solution (see e.g., the Appendix of Lasiecka, Triggiani, 1993; and also Lasiecka, Triggiani, 2000). With $u_T^0(\mathbf{x}_0) \in L^2(0,T;H \times H)$ in hand for given $\mathbf{x}_0 \in \mathbf{H}$, we then proceed to define the minimal energy function as

$$\mathcal{E}_{\min}(T) \equiv \sup_{\|\mathbf{x}_0\|_{\mathbf{H}}=1} \left\| u_T^0(\mathbf{x}_0) \right\|_{L^2(0,T;H\times H)} . \tag{9}$$

It is easy to see that $\mathcal{E}_{\min}(\cdot)$ is bounded away from zero. The issue of concern rather is to determine the behaviour of $\mathcal{E}_{\min}(T)$ as $T \downarrow 0$. It is clear that $\mathcal{E}_{\min}(T)$ should blowup as $T \downarrow 0$. Capturing the exact rate of singularity is the intent of this paper.

As noted in Avalos, Lasiecka (2002a, 2002b), the determination of the precise singularity of the minimal energy function has implications and applications for associated problems arising from stochastic differential equations. In particular, null controllability and the optimal estimates for minimal norm controls provide information on regularity properties for the semigroups and functions of certain Markov processes. We refer the interested reader to Da Prato, Zabczyk (1992) and Da Prato (2001)(see also recent papers by Priola, Zabczyk, 2002 and Gozzi, Loreti).

Our main result is as follows:

THEOREM 1 *(i) Let terminal time $T > 0$. Then given arbitrary initial data $[\omega_0, \omega_1, \theta_0]$ in $\mathbf{H}$ and nonnegative parameters a, b such that $a+b > 0$, there exists a control $u \in L^2(0,T;H \times H)$ such that the corresponding solution $[\omega, \omega_t, \theta]$ of (1) satisfies $[\omega(T), \omega_t(T), \theta(T)] = [0,0,0]$. (In other words, the thermoelastic plate (1) is null controllable within the class of (either mechanical or thermal) controls $L^2(0,T;H)$. If $a = 0$ or $b = 0$, then the minimal norm control $u_T^0(\mathbf{x}_0)$ is of order $\mathcal{O}(T^{-5/2})$. That is to say, $\mathcal{E}_{\min}(T) \sim T^{-5/2}$ with either thermal or mechanical controls. On the other hand, if both mechanical and thermal controls are active; that is, $a > 0$ and $b > 0$, then the corresponding minimal norm control is of order $\mathcal{O}(T^{-3/2})$. In other words, $\mathcal{E}_{\min}(T) \sim T^{-3/2}$ in the case that both L^2-thermal and mechanical controls $[u_1, u_2]$ are acting upon the system.*

The singular rates obtained by our Theorem 1 are optimal and can not be improved. This follows from known finite dimensional results. Indeed, the interest in studying the explosion of minimal norm controls has a long-standing tradition which is rooted in the finite dimensional theory. In particular, the paper Seidman (1988) gives a complete and optimal answer to the question in the finite dimensional case. In particular, Seidman (1988) provides a formula by which one can determine the growth of the minimal norm control, as time $T \downarrow 0$, for the dynamics where $x \in \mathbb{R}^n$, $u \in L^2(0,T;\mathbb{R}^m)$, and A (resp. B) is an $n \times n$ (resp. $n \times m$) matrix, with $m \leq n$. This result depends on Kalman's rank condition, which is the sufficient and necessary controllability condition in finite dimensions. The formula in Seidman (1988) yields that $\mathcal{E}_{\min}(T) \sim T^{-k-1/2}$, where k is the Kalman's rank for the system (10); i.e., k is the smallest integer such that $rank([B, AB, ..., A^k B]) = n$.

The optimal estimate for $\mathcal{E}_{\min}(T)$ in Theorem 1 is in agreement with that "predicted" by finite dimensional theory. Indeed, as pointed out originally in Lasiecka, Triggiani (1998b), the system (1), which models the hinged thermoelastic equation (4), can be equated to the abstract differential equation

$$\frac{d}{dt}\mathrm{y}(t) = \mathcal{A}\mathrm{y} + Bu, \tag{11}$$

where $\mathrm{y} = [A\omega, \omega_t, \theta] \in H \times H \times H$, and

$$\mathcal{A} = A \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 0 \\ a \\ b \end{bmatrix}.$$

Thus, if either a or b are equal to zero, formally, the "Kalman's rank" of $\{\mathcal{A}, B\}$ is 2. Consequently, by Seidman's formula from Seidman (1988), which provides a formula for the blowup of minimal energies for the finite dimensional case, we have $\mathcal{E}_{\min}(T) \sim T^{-2-\frac{1}{2}} = T^{-\frac{5}{2}}$. If, instead $a \neq 0, b \neq 0$ then the "Kalman's rank" of $\{A, B\}$ is 1. Consequently, another appeal to Seidman's formula in Seidman (1988) gives $\mathcal{E}_{\min}(T) \sim T^{-1-\frac{1}{2}} = T^{-\frac{3}{2}}$.

REMARK 1 *The null controllability of the abstract system (1)–which models the hinged case–was originally shown in Lasiecka, Triggiani (1998b), in the case of mechanical or thermal control, although no attempt was made therein to analyze the singularity of $\mathcal{E}_{\min}(T)$. In the special case when A is assumed to have compact resolvent, Triggiani (2002) provides an alternate proof of Theorem 1. However, the proof in Triggiani (2002) depends critically on exact knowledge of spectral properties of the underlying operator. In contrast, our treatment does not require compactness of the resolvent and is "eigenfunction expansion-free"; indeed, it is conceptually very different from the one given in Triggiani (2002). In particular, our proof, provided below, invokes a relatively simple multiplier method with appropriate weight functions. The multipliers used are not differential (in contrast with those employed in standard controllability problems for wave equations or parabolic equations) but rather are the operator theoretic psedo-differential quantities invoked in earlier stability and controllability studies of 2-D thermoelastic systems (see e.g., Avalos, 2000, Avalos, Lasiecka, 1996, 1998, 2000).*

REMARK 2 *One can also work to obtain the optimal estimate of the minimal* *that clamped mechanical boundary conditions are present in the model. (In other words, one considers the problem with control term u in the heat component of (1), and with* ω *satisfying* $\omega|_\Gamma = \frac{\partial\omega}{\partial\nu}|_\Gamma = 0$*). This has in fact been done in Avalos, Lasiecka (2002a).*

REMARK 3 *In the context of obtaining estimates for null controllers of analytic thermoelastic plates, we should also mention Benabdallah et al. (1995), which considers thermal null controllability of the specific (spectral) thermoelastic model (4). Given that the underlying spectrum of (4) can be obtained explicitly, the methods employed in Benabdallah et al. (1995) are based on a spectral decomposition and a related Riesz basis property. Also in Benabdallah et al. (1995), in which the principal result is showing the possibility of using locally distributed control, a preliminary estimate for the minimal norm control, which is distributed over the entire domain* $\Omega \times O(T)$ *in* $L^2(Q)$ *is provided. However, the estimate is suboptimal by a factor of two. Instead, the paper of Avalos, Lasiecka (2002a) provides a proof that the optimal rate of singularity for the associated minimal energy is* $O(T^{-\frac{5}{2}})$*, which is in agreement with that predicted by Seidman's finite dimensional theory.*

REMARK 4 *One can also speak of exactly controlling the dynamics (4), but because of the underlying analyticity, the controls must be taken to be in spaces larger than* $L^2(Q)$*. This is a situation entirely analogous to that of the heat equation (see Bensoussan, et al., 1995). For example, it is shown that the system (4) is exactly controllable via thermal control u, provided that u is taken from* $L^2(0,T;H^{-1}(\Omega))$ *(see Avalos, 2000). Thus Theorem 1 of the present work (see also Avalos, Lasiecka, 2002a) says that in the case that one wishes to reach the zero state only, the controller u which does the job may be taken from the narrower space* $L^2(Q)$*. Again, this circumstance is in line with what is seen in the context of controlling the heat equation.*

2. The Controllability inequality

By a straightforward application of the Lumer Phillips Theorem (see e.g., Pazy, 1983), the abstract thermoelastic system (1) may be associated with a linear C_0-semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$, where the linear operator $\mathcal{A}: D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ models the underlying dynamics. In fact,

$$\mathcal{A} \equiv \begin{bmatrix} 0 & I & 0 \\ -A^2 & 0 & \alpha A \\ 0 & -\alpha A & -A \end{bmatrix}; \quad D(\mathcal{A}) = D(A^2) \times D(A) \times D(A). \qquad (12)$$

In consequence, the solution $\mathbf{x}(t;\mathbf{x}_0;u)$ of (1) may be written explicitly as

$$\mathbf{x}(t;\mathbf{x}_0;u) = e^{\mathcal{A}t}\mathbf{x}_0 + \int_0^t e^{\mathcal{A}(t-\tau)} \begin{bmatrix} 0 \\ au_1(\tau) \end{bmatrix} d\tau. \qquad (13)$$

Moreover, from the wellposedness result in (3), we have that the associated controllability mapping $\mathcal{L}_T : L^2(0,T; H \times H) \to \mathbf{H}$ is well-defined:

$$\mathcal{L}_T u = \mathbf{x}(T; \mathbf{x}_0; u) = \begin{bmatrix} \omega(T) \\ \omega_t(T) \\ \theta(T) \end{bmatrix}. \tag{14}$$

In terms of this notation, a classical argument of functional analysis provides that the null controllability property is equivalent to the statement that

$$Range(e^{\mathcal{A}T}) \subset Range(\mathcal{L}_T).$$

In turn, this inclusion is equivalent to establishing the inequality (for some constant $C_T > 0$)

$$\left\| e^{\mathcal{A}^* T} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \vartheta_0 \end{bmatrix} \right\|_{\mathbf{H}} \leq C_T \left\| \mathcal{L}_T^* \begin{bmatrix} \phi_0 \\ \phi_1 \\ \vartheta_0 \end{bmatrix} \right\|_{L^2(0,T;H\times H)} \quad \text{for all } [\phi_0, \phi_1, \vartheta_0] \in \mathbf{H} \tag{15}$$

(see e.g., Zabczyk, 1992, p. 213, Theorem 2.6). Thus, the proof of Theorem 1 hinges on *(i)* proving the inequality (15) for some constant C_T; *(ii)* finding the optimal estimate for the constant C_T as $T \downarrow 0$. Having accomplished these two tasks, one can then proceed in a straightforward fashion to show that this sharp constant C_T measures the singularity of the minimal energy function $\mathcal{E}_{\min}(T)$ (see e.g., Avalos, Lasiecka, 2002a; also Lasiecka, Triggiani, 2000).

In a now standard way - see e.g., Avalos (2000) - one can show explicitly that

$$\mathcal{L}_T^* \begin{bmatrix} \phi_0 \\ \phi_1 \\ \vartheta_0 \end{bmatrix} = [a\phi_t(\cdot), b\psi(\cdot)]; \quad e^{\mathcal{A}^* t} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \vartheta_0 \end{bmatrix} = \begin{bmatrix} \phi(t) \\ -\phi_t(t) \\ \vartheta(t) \end{bmatrix},$$

where $[\phi, -\phi_t, \vartheta]$ satisfy the following homogeneous equation:

$$\begin{cases} \begin{cases} \phi_{tt} + A^2\phi - \alpha A\vartheta = 0 \text{ on } (0,T) \\ \vartheta_t + A\vartheta + \alpha A\phi_t = 0 \text{ on } (0,T) \end{cases} \\ \\ [\phi(0), -\phi_t(0), \vartheta(0)] = [\phi_0, \phi_1, \vartheta_0] \in \mathbf{H}. \end{cases} \tag{16}$$

(For data $[\phi_0, \phi_1, \vartheta_0] \in \mathbf{H}$, we have by (3) that the corresponding solution $[\phi, \phi_t, \vartheta] \in C([0,T]; \mathbf{H})$.)

Thus, in terms of the homogeneous problem (16) (with given initial data $[\phi_0, \phi_1, \vartheta_0] \in \mathbf{H}$), the inequality (15) will take the form

$$\|[\phi(T), \phi_t(T), \vartheta(T)]\|_{\mathbf{H}}^2 \leq C_T^2 \int^T [a\|\phi_t(t)\|_H^2 + b\|\vartheta(t)\|_H^2]\, dt \tag{17}$$

for some constant $C_T \geq 0$.

The behaviour of the constant C_T as $T \to 0$ determines the singular rates of the minimal energy function. To see this, we recall a standard optimization argument. Assuming the validity of (17), we search for the minimal norm control which can readily be shown to take the form (see Lasiecka, Triggiani, 1993, Appendix B, and Lasiecka, Triggiani, 2000)

$$u_T^0 = -\mathcal{L}_T^*(\mathcal{L}_T\mathcal{L}_T^*)^{-1}e^{\mathcal{A}T}\mathbf{x}_0,$$

where $\mathbf{x}_0 \in \mathbf{H}$ is the initial data of the controlled process. We note that the existence of the pseudoinverse $\Gamma_T \equiv \mathcal{L}_T^*(\mathcal{L}_T\mathcal{L}_T^*)^{-1}e^{\mathcal{A}T}$, and its boundedness as a mapping from $\mathbf{H}$ into $L^2(0,T;U)$ ($U \equiv H \times H$), results from the validity of (17).

On the other hand, as easily verified,

$$\mathcal{E}_{min}(T) = ||\Gamma_T||_{\mathcal{L}(\mathbf{H},L^2(0,T;U)} \leq C_T,$$

where C_T here is the same constant which appears in (17).

Thus, the constant C_T provides the estimate for the singularity of the minimal energy function $\mathcal{E}_{min}$.

Finally, we will denote the "energy" of the system by

$$\mathcal{E}(t) \equiv \frac{1}{2}\left(\|A\phi(t)\|_H^2 + \|\phi_t(t)\|_H^2 + \|\vartheta(t)\|_H^2\right). \tag{18}$$

In regards to this quantity, multiplication of the first (Euler beam) equation in (16) by ϕ_t, multiplication of the second by ϑ, subsequent integration in time and space, and integrations by parts, will collectively yield the following dissipative relation for all $0 \leq s \leq t \leq T$:

$$\mathcal{E}(s) = \mathcal{E}(t) + \int_s^t \left\|A^{1/2}\vartheta(\tau)\right\|_{L^2(\Omega)}^2 d\tau \tag{19}$$

(in particular then, $\vartheta \in L^2(0,\infty;D(A^{\frac{1}{2}})))$.

REMARK 5 *As we noted in Remark 1, the proof of Theorem 1 is driven by a multiplier method. In particular, we will work to generate a string of a priori relations for the adjoint homogeneous problem (16). These relations will ultimately be used to derive the reverse inequality (17), which is necessary for null controllability. However, in contrast to the case of exact controllability, in which it would suffice to generate a preliminary observability inequality that is polluted by lower order terms–given that the backwards uniqueness property for the (analytic) thermoelastic system is known to hold true–our present situation demands that the reverse inequality (17) be derived with no lower order terms present. Indeed, our task of finding the order of singularity for the min-*

(17). However, any compactness-uniqueness argument used to eliminate lower order terms would mean a loss of control of C_T, and consequently our multiplier method would provide no information on $\mathcal{E}_{\min}(T)$.

Below, we provide the proof of Theorem 1. The mechanical case ($a \neq 0$) is treated in Section 3 while the thermal case ($a = 0$) is relegated to Section 4.

3. Proof of Theorem 1: The mechanical case

3.1. A preliminary relation

By the remarks made in the previous section, we know that in order to establish the asserted null controllability of the thermoelastic system (1), it is enough to establish the inequality (17) for the homogeneous problem (16). To this end, we need the function $h(t)$, defined by

$$h(t) = t^4(T-t)^4. \tag{20}$$

Moreover, in what follows we shall use the notation

$$||u|| \equiv \|u\|_H\,; \;\; (u,v) \equiv (u,v)_H.$$

LEMMA 1 *With $h(t)$ as defined in (20), the mechanical displacement and thermal component of the solution to (16) satisfy the following relation for $T > 0$:*

$$\begin{aligned}\int_0^T h(t)\left(\|A\phi\|^2 + \|\vartheta\|^2\right) dt \\ = \frac{(1+\alpha^2)}{\alpha}\int_0^T h(t)\,(A\phi,\vartheta)\,dt + 2\int_0^T h(t)\,\|\phi_t\|^2\,dt \\ + \alpha^{-1}\int_0^T h(t)\,(\phi_t,\vartheta)\,dt + \int_0^T h'(t)\,(\phi_t,\phi)\,dt \\ - \alpha^{-1}\int_0^T h'(t)\left(A^{-1}\phi_t,\vartheta\right) dt.\end{aligned} \tag{21}$$

Proof of Lemma 1: Step 1 (treating the thermal term ϑ): By applying $h(t)A^{-1}\vartheta$ to both sides of the Euler beam equation (16), and subsequently integrating in time and space, we obtain

$$\int_0^T \left(\phi_{tt} + A^2\phi - \alpha A\vartheta,\, h(t)A^{-1}\vartheta\right) dt = 0 \tag{22}$$

We now scrutinize each term on the left hand side of (22): (E.i) An integration by parts and use of the heat equation in (16) gives

$$\begin{aligned}\int_0^T \left(\phi_{tt}, h(t)A^{-1}\vartheta\right) dt \\ &= -\int_0^T \left(\phi_t, h(t)A^{-1}\vartheta_t\right) dt - \int_0^T \left(\phi_t, h'(t)A^{-1}\vartheta\right) dt \\ &= \alpha \int_0^T h(t) \left\|\phi_t\right\|^2 dt \\ &\quad + \int_0^T h(t) \left(\phi_t, \vartheta\right) dt - \int_0^T \left(\phi_t, h'(t)A^{-1}\vartheta\right) dt.\end{aligned}$$

(E.ii) The taking of adjoints gives, moreover,

$$\begin{aligned}\int_0^T \left(A^2\phi - \alpha A\vartheta, h(t)A^{-1}\vartheta\right) dt \\ &= \int_0^T h(t) \left(A\phi, \vartheta\right) dt - \alpha \int_0^T h(t) \left\|\vartheta\right\|^2 dt.\end{aligned}$$

Plugging (E.i) and (E.ii) in (22) now results in

$$\begin{aligned}\int_0^T h(t) \left\|\vartheta\right\|^2 dt \\ &= \int_0^T h(t) \left\|\phi_t\right\|^2 dt + \alpha^{-1} \int_0^T h(t) \left(A\phi + \phi_t, \vartheta\right) dt \\ &\quad - \alpha^{-1} \int_0^T h'(t) \left(\phi_t, A^{-1}\vartheta\right) dt. \qquad (23)\end{aligned}$$

Step 2 (Dealing with the displacement ϕ): Multiplying the Euler beam equation, this time by $h(t)\phi$, and integrating in time and space, yields

$$\int_0^T \left(\phi_{tt} + A^2\phi - \alpha A\vartheta, h(t)\phi\right) dt = 0. \qquad (24)$$

The integration by parts and the taking of adjoints give

$$\begin{aligned}\int_0^T h(t) \left\|A\phi\right\|^2 dt \\ &= \int_0^T h'(t)(\phi_t, \phi) dt + \int_0^T h(t) \left\|\phi_t\right\|^2 d\tau + \alpha \int_0^T h(t) \left(A\phi, \vartheta\right) dt. \qquad (25)\end{aligned}$$

3.2. Additional relations

The first term on the right hand side of (21) is problematic, inasmuch as it does not involve the (good) term ϕ_t. To deal with this, we need the following three Propositions.

By multiplying the heat equation in (16) by $h\phi(t)$, integrating in time and space and subsequently integrating by parts we readily have the following:

PROPOSITION 1 *The solution to (16) satisfies the following relation*

$$\int_0^T h(t)\,(\vartheta, A\phi)\,dt = \int_0^T h'(t)\,(\vartheta,\phi)\,dt + \int_0^T h(t)\,(\vartheta-\alpha A\phi,\phi_t)\,dt. \quad (26)$$

Now to handle the first term on the right hand side of (26), we multiply the Euler-Bernoulli beam in (16) by $h'(t)A^{-1}\phi(t)$, integrate in time and space, and subsequently integrate by parts so as to eventually obtain the following result:

PROPOSITION 2 *The solution to (16) satisfies the following relation:*

$$\begin{aligned}\int_0^T h'(t)\,(\vartheta,\phi)\,dt &= \alpha^{-1}\int_0^T h'(t)\left\|A^{\frac{1}{2}}\phi\right\|^2 dt - \alpha^{-1}\int_0^T h''(t)\left(\phi_t, A^{-1}\phi\right) dt\\ &\quad -\alpha^{-1}\int_0^T h'(t)\left\|A^{-\frac{1}{2}}\phi_t\right\|^2 dt. \qquad (27)\end{aligned}$$

Next, to deal with the first term on the right hand side of (27), we multiply the Euler-Bernoulli beam in (16) this time by $h(t)A^{-1}\phi_t(t)$, again integrate in time and space, and integrate by parts to obtain

PROPOSITION 3 *The mechanical displacement of the solution to (16) satisfies the following relation:*

$$\int_0^T h'(t)\left\|A^{\frac{1}{2}}\phi\right\|^2 dt = -\int_0^T h'(t)\left\|A^{-\frac{1}{2}}\phi_t\right\|^2 dt - 2\alpha\int_0^T h(t)\,(\vartheta,\phi_t)\,dt.$$

3.3. Conclusion of the proof of Theorem 1 when $b = 0$

$$\int_0^T h(t)\left(\|A\phi\|^2+\|\vartheta\|^2\right)dt$$
$$=\frac{1+\alpha^2}{\alpha^2}\left(-\int_0^T h''(t)\left(\phi_t,A^{-1}\phi\right)dt+\int_0^T h(t)\left(\vartheta-\alpha A\phi,\phi_t\right)dt\right)$$
$$-\frac{2(1+\alpha^2)}{\alpha^2}\left(\int_0^T h'(t)\left\|A^{-\frac{1}{2}}\phi_t\right\|^2dt+\alpha\int_0^T h(t)\left(\vartheta,\phi_t\right)dt\right)$$
$$+\alpha^{-1}\int_0^T h(t)\left(\phi_t,\vartheta\right)dt+\int_0^T h'(t)\left(\phi_t,\phi\right)dt$$
$$-\alpha^{-1}\int_0^T h'(t)\left(A^{-1}\phi_t,\vartheta\right)dt+2\int_0^T h(t)\|\phi_t\|^2dt. \tag{28}$$

Majorizing the right hand side via $ab\leq\epsilon a^2+C_\epsilon b^2$ gives, for $0<T<1$,

$$\int_0^T h(t)\left(\|A\phi\|^2+\|\vartheta\|^2\right))dt$$
$$\leq-\left(\frac{1+\alpha^2}{\alpha^2}\right)\int_0^T h''(t)\left(\phi_t,A^{-1}\phi\right)dt+\int_0^T h'(t)\left(\phi_t,\phi-\alpha^{-1}A^{-1}\vartheta\right)dt$$
$$+\epsilon\int_0^T h(t)\mathcal{E}(t)dt+C_{\alpha,\epsilon}T^7\int_0^T\|\phi_t\|^2. \tag{29}$$

Now,

$$-\left(\frac{1+\alpha^2}{\alpha^2}\right)\int_0^T h''(t)\left(\phi_t,A^{-1}\phi\right)dt$$
$$\leq\left(\frac{1+\alpha^2}{\alpha^2}\right)\left\|A^{-2}\right\|_{\mathcal{L}(L^2(\Omega))}\int_0^T\sqrt{h(t)}\frac{h''(t)}{\sqrt{h(t)}}\|A\phi\|\,\|\phi_t\|\,dt$$
$$\leq\epsilon\int_0^T h(t)\mathcal{E}(t)dt+C_{\epsilon,\alpha}\int_0^T\frac{[h''(t)]^2}{h(t)}\|\phi_t\|^2dt$$
$$\leq\epsilon\int_0^T h(t)\mathcal{E}(t)dt+C_{\epsilon,\alpha}T^4\int_0^T\|\phi_t\|^2dt. \tag{30}$$

In the same way, we have

$$\int_0^T h'(t)\left(\phi_t,\phi-\alpha^{-1}A^{-1}\vartheta\right)dt$$

Adding now $\int_0^T h(t)\,\|\phi_t\|^2\,dt$ to both sides of (29), and subsequently applying (30) and (31) thereto, we obtain for $0 < T < 1$,

$$2\int_0^T h(t)\mathcal{E}(t)dt \le 3\epsilon \int_0^T h(t)\mathcal{E}(t)dt + C_{\epsilon,\alpha}T^4 \int_0^T \|\phi_t\|^2\,dt,$$

or

$$(2-3\epsilon)\int_0^T h(t)\mathcal{E}(t)dt \le C_{\epsilon,\alpha}T^4 \int_0^T \|\phi_t\|^2\,dt.$$

Finally, taking $\epsilon < \frac{1}{3}$ and making use of the dissipativity inherent in relation (19), we have

$$(2-3\epsilon)\,\mathcal{E}(T)\int_0^T h(t)dt \le C_{\epsilon,\alpha}T^4 \int_0^T \|\phi_t\|^2\,dt,$$

or

$$\mathcal{E}(T) \le C_{\epsilon,\alpha}T^{-5}\int_0^T \|\phi_t\|^2\,dt. \tag{32}$$

This estimate gives finally the observability inequality (17) with $b = 0$, with associated observability constant

$$C_T = \mathcal{O}\left(\frac{1}{T^{\frac{5}{2}}}\right). \tag{33}$$

This completes the derivation of the inequality (15), and hence the proof of the null mechanical controllability statement in Theorem 1 for $a \neq 0$ and $b = 0$.

3.4. Conclusion of the proof of Theorem 1 when $a \neq 0, b \neq 0$

In this case we need to estimate the constant C_T which leads to

$$\mathcal{E}(T) \le C_T^2 \int_0^T \left[||\phi_t(t)||^2 + ||\vartheta(t)||^2\right] dt. \tag{34}$$

To this end, we return to the relation (25), which implies

$$\begin{aligned}\int_0^T h(t)\left[||A\phi||^2 + ||\phi_t||^2 + ||\vartheta||^2\right] dt \\ \le 2(1+\alpha)\int_0^T h(t)[||\vartheta||^2 + ||\phi_t||^2]\,dt \\ + \int_0^T \left(\epsilon h(t)||\phi||^2 + C\,(h'(t))^2 \ldots ||\phi_t||^2\right) dt \end{aligned} \tag{35}$$

From here, after taking suitably small ϵ and noticing that for small $T > 0$

$$\frac{[h'(t)]^2}{h(t)} + h(t) \le CT^6,$$

we obtain

$$\begin{aligned}\int_0^T h(t)\mathcal{E}(t)dt &= 2\int_0^T h(t)[||A\phi||^2 + ||\phi_t||^2 + ||\vartheta||^2]dt \\ &\le CT^6 \int_0^T [||\vartheta||^2 + ||\phi_t||^2]dt. \end{aligned} \tag{36}$$

Combining (36) with the dissipativity relation

$$\mathcal{E}(T) \le \mathcal{E}(t), \text{ for } t > T,$$

yields

$$\mathcal{E}(T) \int_0^T h(t)dt \le CT^6 \int_0^T [||\vartheta||^2 + ||\phi_t||^2]dt.$$

Since $\int_0^T h(t)dt \sim cT^9$, the above inequality yields

$$\mathcal{E}(T) \le CT^{-3} \int_0^T [||\vartheta||^2 + ||\phi_t||^2]dt,$$

whence we obtain for the inequality (34),

$$C_T \sim T^{3/2}. \tag{37}$$

As we noted in Section 2., with the constant C_T as given by (33) (resp. (37)), we can proceed formally to show that for given initial data $\mathbf{x}_0 \in \mathbf{H}$, the corresponding minimal norm control likewise satisfies

$$\left\| u_T^0(\mathbf{x}_0) \right\|_{L^2(0,T';U)} \le C_T \left\| \mathbf{x}_0 \right\|.$$

(See Avalos, Lasiecka, 2002, Lasiecka, Triggiani, 2000). This concludes the proof of Theorem 1 for $a \neq 0$ and $b \neq 0$.

4. Proof of Theorem 1: The thermal case $a = 0$

In the absence of mechanical control u_1 in (1), we must find the constant C_T which satisfies

$$\mathcal{E}(T) \le C^2 \int^T ||\vartheta(t)||^2 dt \tag{38}$$

where again $[\phi, \phi_t, \vartheta]$ solve the homogeneous system (16). To this end, we use the *a priori* relation (23) which was derived in Section 3.1. above, so as to have

$$\begin{aligned}\int_0^T h(t)\,\|\phi_t\|^2\,dt \\ &= \int_0^T h(t)\,\|\vartheta\|^2\,dt - \alpha^{-1}\int_0^T h(t)\,(A\phi + \phi_t, \vartheta)\,dt \\ &\quad + \alpha^{-1}\int_0^T h'(t)\,\left(\phi_t, A^{-1}\vartheta\right)dt. \end{aligned} \tag{39}$$

Combining this with the relation (25), which was derived for the displacement in Section 3.1., we obtain the preliminary relation for the mechanical variable:

$$\begin{aligned}\int_0^T h(t)\left(\|A\phi\|^2 + \|\phi_t\|^2\right)dt \\ &= \int_0^T h'(t)\,(\phi_t, \phi)\,dt + \alpha^{-1}\int_0^T h(t)\left((\alpha^2-2)A\phi - 2\phi_t, \vartheta\right)dt \\ &\quad + 2\alpha^{-1}\int_0^T h'(t)\left(A^{-1}\phi_t, \vartheta\right)dt + 2\int_0^T h(t)\,\|\vartheta\|^2\,dt. \end{aligned} \tag{40}$$

Apparently, we must deal with the first term on the right hand side of (40). In so doing, we multiply the heat equation in (16) by $h'(t)A^{-1}\phi$, and integrate in time and space. This gives

$$\begin{aligned}\int_0^T h'(t)(\phi_t, \phi)dt \\ &= \alpha^{-1}\int_0^T h'(t)(\vartheta, A^{-1}\phi_t - \phi)dt + \alpha^{-1}\int_0^T h''(t)(\vartheta, A^{-1}\phi)dt. \end{aligned} \tag{41}$$

By combining (40) and (41), we obtain

$$\begin{aligned}\int_0^T h(t)\left(\|A\phi\|^2 + \|\phi_t\|^2\right)dt \\ &= 2\int_0^T h(t)\,\|\vartheta\|^2\,dt + \alpha^{-1}\int_0^T h(t)\left((\alpha^2-2)A\phi - 2\phi_t, \vartheta\right)dt \qquad (42) \\ &\quad + \alpha^{-1}\int_0^T h'(t)\left(3A^{-1}\phi_t - \phi, \vartheta\right)dt + \alpha^{-1}\int_0^T h''(t)\left(A^{-1}\phi, \vartheta\right)dt \end{aligned}$$

Upon adding the thermal term $\int_0^T h(t)\,\|\vartheta\|^2\,dt$ to both sides of this relation, and subsequently estimating via the inequality $ab \le \epsilon a^2 + C_\epsilon b^2$, we obtain

$$\begin{aligned}\int_0^T & h(t)\mathcal{E}(t)dt \\ &\le \epsilon \int_0^T h(t)\mathcal{E}(t)dt + C_\epsilon \int_0^T h(t)\,\|\vartheta\|^2\,dt \\ &\quad + C_\epsilon \int_0^T \frac{[h'(t)]^2}{h(t)}\,\|\vartheta\|^2\,dt + C_\epsilon \int_0^T \frac{[h''(t)]^2}{h(t)}\,\|\vartheta\|^2\,dt.\end{aligned}$$

Since

$$\frac{[h'(t)]^2}{h(t)} = \mathcal{O}(T^6) \quad \text{and} \quad \frac{[h''(t)]^2}{h(t)} = \mathcal{O}(T^4),$$

then for $0 < T < 1$, we have

$$\int_0^T h(t)\mathcal{E}(t)dt \le \epsilon \int_0^T h(t)\mathcal{E}(t)dt + C_\epsilon T^4 \int_0^T \|\vartheta\|^2\,dt.$$

Taking $0 < \epsilon < 1$ gives now

$$(1-\epsilon)\int_0^T h(t)\mathcal{E}(t)dt \le C_\epsilon T^4 \int_0^T \|\vartheta\|^2\,dt.$$

Using finally the dissipative relation $\mathcal{E}(s) \le \mathcal{E}(t)$ for $0 \le t \le s \le T$, we have

$$(1-\epsilon)\mathcal{E}(T)\int_0^T h(t)dt \le C_\epsilon T^4 \int_0^T \|\vartheta\|^2\,dt,$$

whence, $C_T \sim T^{-\frac{5}{2}}$. As we noted in Section 2, once the sharp constant C_T in (38) is known, we can proceed in algorithmic fashion to show that likewise the minimal energy function $\mathcal{E}_{\min}(T) = \mathcal{O}(T^{-\frac{5}{2}})$. This completes the proof of Theorem 1 for $a = 0$.

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