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On σ-porous and Φ-angle-small sets in metric spaces

by

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Abstract: It is shown that in metric spaces each (α, ϕ)-meagre set A is uniformly very porous and its index of uniform v-porosity is not smaller than $\frac{k-\alpha}{3k+\alpha}$, provided that ϕ is a strictly k-monotone family of Lipschitz functions and $\alpha < k$. The paper contains also conditions implying that a k-monotone family of Lipschitz functions is strictly k-monotone.

Keywords: prous set, k-monotone family of Lipschitz functions.

Let (X, d) be a metric space and let A be a set contained in X, $A \subset X$. For fixed $x \in X$ and $R > 0$ we denote by $\gamma(x, R, A)$ supremum of those $r > 0$ for which there is $z \in X$, such that

$$B(z, r) \subset B(x, R) \setminus A, \tag{1}$$

where $B(y, \varrho) = \{z \in A : d(z, y) \leq \varrho\}$ is the closed ball with the center at y and with the radius ϱ. We say that the set A is *porous*, if for all $x \in A$

$$\limsup_{R \downarrow 0} \frac{\gamma(x, R, A)}{R} > 0. \tag{2}$$

We say that it is *very porous*, if for all $x \in A$

$$\liminf_{R \downarrow 0} \frac{\gamma(x, R, A)}{R} > 0, \tag{3}$$

(Zajiček, 1976, Argonsky and Brückner, 1985/6). If

$$p\,(A) = \inf_{x \in A} \limsup_{R \downarrow 0} \frac{\gamma(x, R, A)}{R} > 0, \tag{2^u}$$

$$vp\,(A) = \inf_{x \in A} \liminf_{R \downarrow 0} \frac{\gamma(x,R,A)}{R} > 0. \tag{3^u}$$

we say that the set A is *uniformly very porous*. We shall call $p\,(A)$ (resp. $vp\,(A)$) the *index of uniform porosity* (resp. *uniform v-porosity*) of the set A.

A set $A \subset X$ is called *σ-porous*, if it can be represented as a countable union of porous sets.

Porous and σ-porous sets in $\mathbb{R}^n$ were considered earlier by several authors (see the survey paper of Zajiček, 1987/8).

Let $\mathcal{L}$ be the space of all Lipschitzian functions defined on X. We define on $\mathcal{L}$ a quasinorm

$$\|\phi\|_L = \sup_{\substack{x_1,x_2 \in X,\\ x_1 \neq x_2}} \frac{|\phi(x_1) - \phi(x_2)|}{d(x_1,x_2)}. \tag{4}$$

Observe that, if $\|\phi_1 - \phi_2\|_L = 0$, then the difference of ϕ_1 and ϕ_2 is a constant function, i.e., $\phi_1(x) = \phi_2(x) + c$. Thus, we consider the quotient space $\tilde{\mathcal{L}} = {}^{\mathcal{L}}/_{\mathbb{R}}$. The quasinorm $\|\phi\|_L$ induces the norm in the space $\tilde{\mathcal{L}}$. Since this will not lead to any misunderstanding, we shall also denote this norm by $\|\phi\|_L$.

Let Φ be a family of Lipschitz functions defined on X. The quotient space ${}^{\Phi+\mathbb{R}}/_{\mathbb{R}}$ is a subset of the space $\tilde{\mathcal{L}}$. We shall denote it briefly ${}^{\Phi}/_{\mathbb{R}}$. It is a metric space with the distance $d_L(\phi,\psi) = \|\phi - \psi\|_L$.

We say that a Lipschitz function ϕ is *k-monotone*, $0 < k \leq 1$, if for all $x \in X$ and all $t > 0$, there is a $y \in X$ such that $0 < d_X(x,y) < t$ and

$$\phi(y) - \phi(x) \geq k\|\phi\|_L d_X(y,x). \tag{5}$$

If a family Φ consists of k-monotone functions we say that the family Φ is *k-monotone.*

By replacing in (5) the left-hand side of the inequality $\phi(y) - \phi(x)$ by its absolute value we obtain a notion of weak k-monotonicity. Namely, we say that a Lipschitz function ϕ is *weakly k-monotone*, $0 < k \leq 1$, if for all $x \in X$ and all $t > 0$, there is a $y \in X$ such that $0 < d_X(x,y) < t$ and

$$|\phi(y) - \phi(x)| \geq k\|\phi\|_L d_X(y,x). \tag{5^w}$$

If a family Φ consists of weakly k-monotone functions we say that the family Φ is *weakly k-monotone*. Of course, if a function ϕ is k-monotone, then it is also weakly k-monotone. The converse is not true. For example, if X is compact and ϕ is a continuous function, then it is never k-monotone. But it may happen that ϕ is weakly k-monotone.

It is obvious that the linear continuous functionals over a Banach space X are k-monotone for every $0 < k < 1$. If the space X is reflexive they are

Write for any $\phi \in \Phi$, $0 < \alpha < 1$, $x \in X$ (Rolewicz, 1994, 1995, see also Preiss and Zajiček, 1984, for linear continuous functionals ϕ)

$$K(\phi, \alpha, x) = \{y \in X : \phi(y) - \phi(x) \geq \alpha \|\phi\|_L d(y, x)\}. \tag{6}$$

The set $K(\phi, \alpha, x)$ will be called an *α-cone with the vertex at x and the direction ϕ*. Of course, it may happen that $K(\phi, \alpha, x) = \{x\}$. However, if the family Φ is k-monotone and $\alpha < k$, then it is obvious that the set $K(\phi, \alpha, x)$ has the nonempty interior and, even more,

$$x \in \overline{\mathrm{Int} K(\phi, \alpha, x)}. \tag{7}$$

A set $M \subset X$ is said to be *(α, Φ)-meagre* if for every $x \in M$ and arbitrary $\varepsilon > 0$ there are $z \in X$, $d(x, z) < \varepsilon$ and $\phi \in \Phi$ such that

$$M \cap \mathrm{Int}\ K(\phi, \alpha, z) = \emptyset. \tag{8}$$

The arbitrariness of ε and (2) imply that an (α, Φ)-meagre set M is nowhere dense.

A set $M \subset X$ is called *Φ-angle-small* if there is α, $0 < \alpha < 1$, such that the set M is a union of a countable number of (α, Φ)-meagre sets M_n, $M = \bigcup_{n=1}^{\infty} M_n$. Of course, every Φ-angle-small set M is of the first Baire category.

We recall that a real valued function f defined on X is called *Φ-convex* if

$$f(x) = \sup\{\phi(x) + c : \phi \in \Phi, c \in \mathbb{R}, \phi(\cdot) + c \leq f(\cdot)\}, \tag{9}$$

where $\phi(\cdot) + c \leq f(\cdot)$ means that $\phi(y) + c \leq f(y)$ for all $y \in X$. A function $\phi \in \Phi$ will be called a *Φ-subgradient* of the function f at a point x_0 if

$$f(x) - f(x_0) \geq \phi(x) - \phi(x_0) \tag{10}$$

for all $x \in X$.

We shall say that a real-valued function f defined on a metric space (X, d) is *Fréchet Φ-differentiable* at a point x_0 if there are a function $\gamma(t)$ mapping the interval $[0, +\infty)$ into the interval $[0, +\infty]$ such that

$$\lim_{t \downarrow 0} \frac{\gamma(t)}{t} = 0$$

and a function $\phi_{x_0} \in \Phi$ such that

$$|[f(x) - f(x_0)] - [\phi(x) - \phi(x_0)]| \leq \gamma(d(x, x_0)). \tag{11}$$

The function ϕ will be called a *Fréchet Φ-gradient* of the function f at the point x_0.

THEOREM 1 *(Rolewicz, 2002). Let X be a metric space of the second Baire category (in particular, let X be a complete metric space). Let a family Φ be weakly k-monotone and let it be an additive group. Assume that ${}^{\Phi}\!/_{\mathbb{R}}$ is separable in the Lipschitz norm $\|\phi\|_L$.*

If f is a Φ-convex function having at each point a Φ-subgradient, then there is a Φ-angle-small set A such that the function f is Fréchet Φ-differentiable outside the set A. Moreover, the Fréchet Φ-subgradient is unique and it is continuous in the metric d_L on the set $X \setminus A$.

Let (X, d) be a metric space. Let Φ be a family of real-valued functions defined on X. Let $\alpha(t)$ be a function mapping the interval $[0, +\infty)$ into the interval $[0, +\infty]$ such that $\alpha(0) = 0$ and

$$\lim_{t \downarrow 0} \frac{\alpha(t)}{t} = 0. \tag{12}$$

A function $\phi(x) \in \Phi$ is called an $\alpha(\cdot)$-Φ-*subgradient* of the function f at a point x_0 if

$$f(x) - f(x_0) \geq \phi(x) - \phi(x_0) - \alpha(d(x, x_0)). \tag{13}$$

If a real-valued function f has a nonempty $\alpha(\cdot)$-Φ-subdifferential $\partial_\Phi^\alpha f\big|_x$ for all $x \in X$ we say that the function f is $\alpha(\cdot)$-Φ-*subdifferentiable.*

Now we shall extend the definition of α-cone with the vertex at x and the direction ϕ. Namely the set

$$K(\phi, \alpha, x, \varrho) = K(\phi, \alpha, x) \cap \{y : d(x, y) < \varrho\} \tag{14}$$

will be called an (α, ϱ)-*cone with the vertex at x and the direction ϕ.*

A set $M \subset X$ is said to be (α, ϱ, Φ)-meagre if for every $x \in M$ and arbitrary $\varepsilon > 0$ there are $z \in X$, $d(x, z) < \varepsilon$ and $\phi \in \Phi$ such that

$$M \cap \operatorname{Int} K(\phi, \alpha, z, \varrho) = \emptyset. \tag{15}$$

The arbitrariness of ε and (15) imply that an (α, ϱ, Φ)-meagre set M is nowhere dense. We say that $M \subset X$ is *weakly Φ-angle-small* if there are α, $0 < \alpha < 1$, and a sequence $\{\varrho_n\}$ of positive numbers such that M can be represented as a union of a countable number of $(\alpha, \varrho_n, \Phi)$-meagre sets M_n,

$$M = \bigcup_{n=1}^{\infty} M_n. \tag{16}$$

THEOREM 2 *(Rolewicz, 2002). Let X be a metric space of the second Baire category (in particular, let X be a complete metric space). Let Φ be weakly k-monotone and let it be an additive group. Assume that ${}^{\Phi}\!/_{\mathbb{R}}$ is separable in the*

Let f be a continuous $\alpha(\cdot)$-Φ-subdifferentiable function. Then there is a weakly Φ-angle-small set A such that outside of A the function f is Fréchet Φ-differentiable.

Moreover, the Fréchet Φ-subgradient is unique and it is continuous on $X \setminus A$ in the metric d_L.

Thus, we have a natural question of relations between porous sets, Φ-angle-small sets and weakly Φ-angle-small sets. It is obvious that each (α, Φ)-meagre set is simultaneously (α, ϱ, Φ)-meagre for all $\varrho > 0$. As a consequence we obtain that each Φ-angle-small set is also a weakly Φ-angle-small set. Rolewicz (1999) provided an example of an (α, ϱ, Φ)-meagre set, which is not (α_0, Φ)-meagre for any $\alpha_0 > 0$. But then, Rolewicz (2002) shows the following result:

PROPOSITION 1 *Let X be a separable metric space. Let Φ be a fixed family of functions. Then each weakly Φ-angle-small set M is Φ-angle-small.*

It is easy to give an example of a very porous set which is not Φ-angle-small.

EXAMPLE 1

Let $X = [0, 1]$. Then the classical Cantor set $C \subset X$ is obviously very porous since in this case

$$\liminf_{R \downarrow 0} \frac{\gamma(x, R, C)}{R} = \frac{1}{6}\,. \tag{17}$$

On the other hand it is not Φ-angle-small for any k-monotone family Φ . Indeed, suppose that it is Φ-angle-small. It means that $C = \bigcup_{n=1}^{\infty} C_n$, where C_n are (α, Φ)-meagre. Since the set C is uncountable, at least one among the sets C_n, say C_{n_0}, is uncountable, too. Let there be three points $a, b, c \in C_{n_0}, a < b < c$. For any k-monotone family Φ the (α, Φ)-cone with the vertex z is either of the form $[0, z)$ or $(z, 1]$. Thus for $\varepsilon < \min[(c - b), (b - a)]$ there is no cone with a vertex at $z \in C_{n_0}$ such that $|z - b| < \varepsilon$ disjoint with C_{n_0} and we obtain a contradiction.

It is not clear if in general every Φ-angle-small set is porous. We can prove it only under certain assumptions.

We say that a Lipschitz function ϕ is *strictly k-monotone* if for all $x_0 \in X$, there are $\varepsilon_0 > 0$ and $R_{\phi,x_0,\varepsilon_0}$ such that for all t, $0 < t < R_{\phi,x_0,\varepsilon_0}$, and for all $x \in X$ such that $d_X(x, x_0) < \varepsilon_0$ there is a $y \in X$ such that $d_X(x, y) = t$ and

$$\phi(y) - \phi(x) \geq k\|\phi\|_L d(y, x). \tag{5}$$

If a family Φ consists of strictly k-monotone Lipschitz functions we say that the family Φ is *strictly k-monotone.*

It is obvious that the linear continuous functionals over a Banach space X are strictly k-monotone for every $0 < k < 1$. If the space X is reflexive they are

PROPOSITION 2 *Let X be a metric space. Let Φ be a strictly k-monotone family of Lipschitz functions. Let $\alpha < k$. Then each (α, Φ)-meagre set A is uniformly very porous and its index of uniform v-porosity is not smaller than $\frac{k-\alpha}{3k+\alpha}$.*

Proof. Let $\varepsilon_0 \geq \varepsilon > 0$. Since A is (α, Φ)-meagre set for every $x \in A$ there are $z \in X$, $d(x,z) < \varepsilon$ and $\phi \in \Phi$ such that

$$M \cap \text{Int } K(\phi, \alpha, z) = \emptyset. \tag{15}$$

The function ϕ is strictly k-monotone. Thus, there is $x_\varepsilon \in X$ such that $d(x_\varepsilon, z) = \varepsilon$ and

$$\phi(x_\varepsilon) - \phi(z) \geq k\|\phi\|_L \varepsilon. \tag{5'}$$

Let $r = \varepsilon \frac{k-\alpha}{k+\alpha}$ and let $y \in B(x_\varepsilon, r) = \{y \in X : d(y, x_\varepsilon) \leq r\}$. Then

$$\phi(y) \geq \phi(x_\varepsilon) - r\|\phi\|_L.$$

Thus

$$\phi(y) - \phi(z) \geq \phi(x_\varepsilon) - \phi(z) - r\|\phi\|_L \geq \|\phi\|_L(k\varepsilon - r).$$

On the other hand

$$d(y,z) \leq d(x_\varepsilon, z) + r = \varepsilon + r.$$

By the definition of y, y belongs to $K(\phi, \alpha, z)$, provided

$$\|\phi\|_L(k\varepsilon - r) \geq \alpha\|\phi\|_L(\varepsilon + r). \tag{18}$$

Dividing by $\|\phi\|_L$ we get that (18) is equivalent to

$$(k\varepsilon - r) \geq \alpha(\varepsilon + r), \tag{19}$$

which holds because of the definition of r. Thus, $B(x_\varepsilon, r) \subset K(\phi, \alpha, z)$.

Since $K(\phi, \alpha, z)$ is disjoint with A, the ball $B(x_\varepsilon, r)$ is also disjoint with A. Observe that this ball is contained in the ball $B(x, R)$, where $R = 2\varepsilon + r$. Therefore

$$\liminf_{R \downarrow 0} \frac{\gamma(x, R, M)}{R} \geq \frac{r}{R} \geq \frac{\varepsilon \frac{k-\alpha}{k+\alpha}}{2\varepsilon + \varepsilon \frac{k-\alpha}{k+\alpha}} = \frac{k-\alpha}{3(k+\alpha)}. \tag{20}$$

■

It is obvious that if a function ϕ is strictly k-monotone, then it is also k-monotone.

PROBLEM 1 *Suppose that a function ϕ is k-monotone. Is ϕ also strictly k-*

We know the positive answer to this question in a very specific cases.

PROPOSITION 3 *Each k-monotone function ϕ on $(a,b) \subset \mathbb{R}$ is strictly k-monotone.*

Proof. Since ϕ is a Lipschitz function, it is differentiable almost everywhere. The fact that Φ is k-monotone implies that the function ϕ has at most one local minimum. Thus, we have the following three possibilities

(i) $\phi'(x) \geq k$ at each point of differentiability of the function ϕ

(ii) $\phi'(x) \leq -k$ at each point of differentiability of the function ϕ

(iii) there is a point $c, a < c < b$ such that at each point x of differentiability of the $\phi(x)$ function

$$\phi'(x) \begin{cases} \leq -k \; if \; a < x < c \\ \geq k \; if \; c < x < b \end{cases} .$$

It is not difficult to check that in each of those cases $\phi(x)$ is strictly k-monotone. ■

PROPOSITION 4 *Let X be an open set in $\mathbb{R}^n$. Each k-monotone continuously differentiable function ϕ defined on X is strictly $(k-\varepsilon)$-monotone for arbitrary $\varepsilon > 0$.*

Proof. Let K be an arbitrary compact subset of X. Let $S = \{x : \|x\| = 1\}$ be the unit sphere in X and let $r < \inf\{d(x,y) : x \in K, y \notin X\}$. We consider on the set $\mathcal{K}_0 = K \times S \times (0,r]$ the following function $F_\phi(x,h,t) = \frac{\phi(x+th)-\phi(x)}{t}$. Since ϕ is continuously differentiable, the function F_ϕ can be extended in a continuous way on $\mathcal{K}$ being the completion of the set $\mathcal{K}_0$. The set $\mathcal{K}$ is compact. Thus, the function F_ϕ is uniformly continuous on $\mathcal{K}$. Since ϕ is continuously differentiable and k-monotone, for every $x \in K$ there is h_x such that $F(x,h_x,0) \geq k\|\phi\|_L$. The function F_ϕ is uniformly continuous on $\mathcal{K}$, thus there is $r_\phi > 0$ such that for $0 < t < r_\phi$ $F_\phi(x,h_x,t) \geq (k-\varepsilon)\|\phi\|_L$. Then for arbitrary $\varepsilon > 0$ ϕ is strictly $(k-\varepsilon)$-monotone. ■

Proposition 4 can be extended to infinite dimensional Banach spaces, under stronger assumptions about differentiability.

For this purpose we shall introduce a notion of uniform Fréchet differentiable functions. We shall say that a real-valued function f defined on a metric space (X,d) is *uniformly Fréchet Φ-differentiable* if there is a function $\gamma(t)$ mapping the interval $[0,+\infty)$ into the interval $[0,+\infty]$ such that

$$\lim \frac{\gamma(t)}{t} = 0$$

and for arbitrary $x_0 \in X$ there is a function $\phi_{x_0} \in \Phi$ such that

$$|[f(x) - f(x_0)] - [\phi(x) - \phi(x_0)]| \leq \gamma(d(x,x_0)).$$

PROPOSITION 5 *Let X be an open set in a Banach space E. Each k-monotone uniformly Fréchet differentiable function ϕ defined on X is strictly $(k-\varepsilon)$-*

Proof. Let K be an arbitrary closed set in X such that $d = \inf\{d(x,y) : x \in K, y \notin X\} > 0$. Let $S = \{x : \|x\| = 1\}$ be the unit sphere in X and let $r < d$. We consider on the set $\mathcal{K}_0 = K \times S \times (0, r]$ the following function $F_\phi(x,h,t) = \frac{\phi(x+th)-\phi(x)}{t}$. Since ϕ is continuously differentiable and k-monotone, for every $x \in K$ there is h_x such that $F(x, h_x, 0) \geq k\|\phi\|_L$. Since the function ϕ is uniformly Fréchet differentiable, there is $r_\phi > 0$ such that for $0 < t < r_\phi$ $F_\phi(x, h_x, t) \geq (k-\varepsilon)\|\phi\|_L$. Then, for arbitrary $\varepsilon > 0$ ϕ is strictly $(k-\varepsilon)$-monotone. ■

Propositions 4 and 5 can be extended to weakly k-monotone functions. This is the consequence of the following obvious

PROPOSITION 6 *Let X be an open set in a Banach space E. Let ϕ be a weakly k-monotone Gateaux differentiable function. Then for arbitrary $\varepsilon > 0$ ϕ is $(k-\varepsilon)$-monotone.*

Proof. Since ϕ is weakly k-monotone Gateaux differentiable function, then for each $x \in X$ and each $\varepsilon > 0$ there is h_x such that

$$|\partial^G \phi|_x(h_x)| \geq (k - \frac{\varepsilon}{2})\|h_x\|, \tag{21}$$

where $\partial^G \phi|_x$ denote the Gateaux differential of function ϕ at point x.
Thus either

$$\partial^G \phi|_x(h_x) \geq (k - \frac{\varepsilon}{2})\|h_x\|, \tag{21}$$

or

$$\partial^G \phi|_x(h_x) \leq -(k - \frac{\varepsilon}{2})\|h_x\|. \tag{22}$$

In the second case, by replacing h_x by $-h_x$ we obtain

$$\partial^G \phi|_x(-h_x) \geq (k - \frac{\varepsilon}{2})\| - h_x\|. \tag{23}$$

Then, by the definition of the Gateaux differential for each $x \in X$ there is $\delta_x > 0$ and $y \in X$ such that $\|x - y\| < \delta_x$ and

$$\phi(y) - \phi(x) \geq (k - \varepsilon)\|y - x\|. \tag{24}$$

■

As an obvious consequence of Propositions 4, 5, 6 we get

PROPOSITION 4^w *Let X be an open set in $\mathbb{R}^n$. Each weakly k-monotone continuously differentiable function defined on X for every ε, $0 < \varepsilon < \frac{k}{2}$, is strictly $(k-2\varepsilon)$-monotone.*

PROPOSITION 5^w *Let X be an open set in a Banach space E. Each weakly k-monotone uniformly Fréchet differentiable function defined on X for every*

We can generalize strict k-monotonicity to the case of weak k-monotonicity. Namely, we say a Lipschitz function ϕ is *strict weakly k-monotone* if for all $x_0 \in X$, there are $\varepsilon_0 > 0$ and $R_{\phi,x_0,\varepsilon}$ such that for all t, $0 < t < R_{\phi,x_0}$, all $x \in X$ such that $d(x, x_0) < \varepsilon_0$, there is a $y \in X$ such that $d(x, y) = t$ and

$$|\phi(y) - \phi(x)| \geq k\|\phi\|_L d(y, x). \tag{21}$$

Propositions 4^w and 5^w give us a partial positive answer of following problem

PROBLEM 1^w *Suppose that a Lipschitz function ϕ is weakly k-monotone. Is ϕ also strict weakly k-monotone?*

In general the answer is negative as follows from

EXAMPLE 2

Let $X = [0, 1]$. Let

$$\phi(x) = \inf_n 4|x - \frac{1}{2^n}|.$$

It is easy to see that ϕ is a Lipschitz function with constant 4. Take $x = 0$. By simple calculation we obtain that ϕ is weakly $\frac{1}{3}$-monotone, but it is not strict weakly k-monotone for any $k > 0$. Of course, on the set $X' = (0, 1]$ ϕ is strict weakly k-monotone for arbitrary k, $0 < k \leq 1$.

The notion of strict k-monotonicity is similar to the notion of κ-super-metric coupling introduced by Penot (2003). We recall the notion of coupling. Let (X, d_X) and (Y, d_Y) be two metric spaces. By *coupling* we shall understand a function $c(x, y) : X \times Y \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$.

Let $\kappa > 0$. We say that a coupling $c(x, y)$ is *κ-super-metric* at $(x_0, y_0) \in X \times Y$ if $c(x_0, y_0) \in \mathbb{R}$ and for any $r > 0$

$$\sup_{\{y: d_Y(y,y_0) \leq r\}} \left(c(x, y) - c(x_0, y) \right) \geq c(x, y_0) - c(x_0, y_0) + \kappa r d_X(x, x_0) d_Y(y, y_0). \tag{25}$$

We say that $c(x, y)$ is *κ-super-metric coupling* if it is κ-super-metric coupling at (x_0, y_0) for all $(x_0, y_0) \in X \times Y$. By denoting $\phi_1(y) = c(x, y)$ and $\phi_0(y) = c(x_0, y)$ we obtain that $c(x, y)$ is κ-super-metric coupling if and only if the difference $\phi(y) = \phi_1(y) - \phi_0(y)$ for any $r > 0$ satisfies the following inequality

$$\sup_{\{y: d_Y(y,y_0) \leq r\}} \phi(y) \geq \phi(y_0) + \kappa r d_X(x, x_0) d_Y(y, y_0). \tag{26}$$

Now we shall suppose that $\Phi = \{c(x, \cdot) : x \in X\}$ is an additive group consisting of Lipschitz functions and $d_X(x, x_0) = \|\phi_1 - \phi_0\|_L$. Then for every $\varepsilon > 0$ and

$$\phi(y) - \phi(y_0) \geq (\kappa - \varepsilon)\|\phi\|_L d_Y(y, y_0). \tag{5_c}$$

In other words Φ is $(\kappa - \varepsilon)$-monotone.

In the considered case the essential difference between κ-super-metric coupling and $(\kappa - \varepsilon)$-monotonicity is as follous. In the definition of κ-super-metric coupling inequality (25) holds for all $r > 0$ and in the definition of $(\kappa - \varepsilon)$-monotone functions it holds only for sufficiently small t, depending on ϕ and x. Indeed, κ-super-metric coupling implies that the function $\phi_1(y) - \phi_0(y)$, where $\phi_1(y) = c(x, y)$ and $\phi_0(y) = c(x_0, y)$, is unbounded. Thus, in the case when it is Lipschitz the metric space Y is unbounded, too. Observe, that if in the definition of the κ-super-metric coupling we replace the condition that (5) holds, for all $r > 0$ by the condition that there is R, which does not depend on x such that for $0 < r \leq R$ (5) holds then by triangle inequality we obtain also that (5) holds for arbitrary $r > 0$.

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