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Rotundity, smoothness and duality

by

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Abstract: The duality between smoothness and rotundity of functions is studied in a nonlinear abstract framework. Here smoothness is enlarged to subdifferentiability properties and rotundity is formulated by means of approximation properties.
Keywords: conjugacy, convexity, coupling, duality, estimates, generalized convexity, rotundity, smoothness, subdifferential.

1. Introduction

The main interest of the many transforms known in mathematics (Fourier transform, Laplace transform, Radon transform, Fenchel transform, ...) lies in the fact that they convert a desired property into another one which may be more tractable. It has been known for several decades that Fenchel duality transforms a smoothness property into a well-posedness property and vice-versa. The first instance of such a phenomenon is probably constituted by the Smulian theorems about Fréchet differentiability of norms (Beauzamy, 1982, Deville, Godefroy and Zizler, 1993, Diestel, 1975, etc.). Asplund and Rockafellar (1969) have extended that result to general convex functions on topological vector spaces in duality. Following previous results of Vladimirov, Nesterov and Chekanov (1978) and Zalinescu (1983), Azé and the author (1995) have stressed a quantitative viewpoint in the preceding connection. Azé (1999) and Azé and Rahmouni (1994, 1995, 1996) have extended the preceding results to the case an element of the subdifferential is replaced by the whole subdifferential. Such questions are considered in the books (Azé, 1997; Zalinescu, 2002). They are linked with well-posedness properties.

It is our purpose here to consider the preceding properties in a unified way and to extend them to the case of general dualities. We introduce subdifferentiability properties which are uniform with respect to a certain set and dually, we use a uniform B-differentiability property (in the terminology of Pang, 1990, 1995; Robinson, 1991). After completing a draft of our paper, papers by S. Rolewicz came to our attention (Rolewicz, 1993, 1996). In these references, and in the monograph (Pallaschke, Rolewicz, 1997) an extension of the correspondence between rotundy and smoothness to a nonlinear framework is given, which anticipated our work. However our approach includes the subdifferential case and the uniform properties just mentioned; moreover, being set in the framework of conjugacies using coupling functions, it is more symmetric, so that it allows to interchange the roles of the spaces.

Our study underlines the interest of general dualities which have been brought to the fore in many articles (Balder, 1977; Dolecki, Kurcyusz, 1978; Fan, 1963; Flores-Bazán, Martínez-Legaz, 1998; Ioffe, 2001; Martínez-Legaz, 1988, 1990, 1991, 1995; Moreau, 1970; Penot, 1982, 1997, 2000, 2001; Penot, Volle, 1987, 1988, 1990; Singer, 1986, 1987, etc.) and treated in a systematic way in several recent monographs (Pallaschke, Rolewicz, 1997; Rubinov, 2000; Singer, 1997). However, for the sake of simplicity, we do not tackle the case of the most general dualities but we limit our study to the case of conjugacies, i.e. dualities of Fenchel-Moreau type, which are obtained by means of coupling functions. In doing so, our results remain close enough to the case of the classical Fenchel duality and they keep their intuitive character.

Let us observe that the efforts in putting duality into a general metric framework in the works quoted above are not isolated in mathematics. Parallel developments extending differential calculus and dynamical systems to metric spaces (Aubin, 1999; Pichard, 2001, etc.), differential geometry to metric spaces (Gromov, 1999; Lafontaine, Pansu, 1981), measure theory and probability theory to general topological spaces or metric spaces (Dellacherie, Meyer, 1975; Elworthy, 1975; Schwartz, 1973, 1980, etc.), criteria for estimates to metric spaces (Azé, Corvellec, Lucchetti, 2002; Ioffe, 2001), have reached remarkable achievements which give some hopes for duality questions.

After recalling some basic facts about conjugacies in the next section, we exhibit two conditions relating the coupling to the metric. They play a key role in the sequel. We also present some examples. The main estimates are obtained in Sections 3 (coercivity properties), 4 (passage from growth properties to upper estimates), and 5 (reverse passage).

2. Preliminaries and metric conjugacies

It will be convenient to denote by A the set of functions α from $\mathbb{R}_+$ into $\mathbb{R}_+ \cup \{+\infty\}$ satisfying $\alpha(0) = 0$ (in Azé and Penot, 1995, such functions are called Asplund functions). Any element α of A is identified with its even extension. The *reduced function* (or slope) associated with α is the function $\widehat{\alpha}$ given by $\widehat{\alpha}(0) = 0$, $\widehat{\alpha}(t) = t^{-1}\alpha(t)$ for $t > 0$. When $\widehat{\alpha}$ is nondecreasing, α is said to be *starshaped*. It is said to be *firm* if $(t_n) \to 0$ whenever $(\alpha(t_n)) \to 0$. An Asplund function γ is said to be a *gage* (or *admissible* or *forcing*) if it is firm and nondecreasing; equivalently, $\gamma \in A$ is a gage if it is nondecreasing and if

$\gamma(t) > 0$ for all $t > 0$. An Asplund function $\mu \in A$ is said to be a *modulus* if μ is nondecreasing and if $\lim_{t \searrow 0} \mu(t) = 0$. It is said to be an *hypermodulus* (or a remainder) if its reduced function $\widehat{\mu}$ is a modulus.

The following result of Asplund (Asplund, 1968), Lemma 1, see also Azé and Penot, 1995, Lemma 2.1) connects the preceding properties when taking conjugates. Recall that the (Fenchel) *conjugate* of α is the function α^* given by $\alpha^*(t) := \sup\{st - \alpha(s) : s \in \mathbb{R}\}$.

LEMMA 2.1. *(a) For any starshaped element γ of the set G of gages, γ^* is an hypermodulus.*

(b) For any element ω of the set Ω of hypermodulus, ω^* is a starshaped gage.

We devote the rest of the present section to some basic facts about conjugacies and we present some material we will use. We refer to Balder (1977), Dolecki, Kurcyusz (1978), Moreau (1970), Pallaschke, Rolewicz (1977), Penot (2000), Penot, Volle (1987), Rubinov (2000), Singer (1997) and to several other items of our bibliography for further information.

In the sequel, X and Y are metric spaces (or semi-metric spaces) whose metrics (or semi-metrics) are denoted by d inasmuch there is no risk of confusion. For a subset S of X, d_S stands for the distance function to $S : d_S(x) := \inf\{d(x,w) : w \in S\}$; the closed ball with center x and radius r is denoted by $B(x,r)$ or $B_X(x,r)$. A *coupling* or pairing of X with Y is a function $c : X \times Y \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. Given a function $f : X \to \overline{\mathbb{R}}$, its conjugate is the function $f^c : Y \to \overline{\mathbb{R}}$ defined by

$$f^c(y) := -\inf\{f(x) - c(x,y) : x \in X\}, \tag{1}$$

where $r - s$ mean $r + (-s)$, the addition of $\overline{\mathbb{R}}$ being the extension of the addition of $\mathbb{R}$ given by $+\infty + s = +\infty$ for each $s \in \overline{\mathbb{R}}$, $-\infty + s = -\infty$ for each $s \in \mathbb{R} \cup \{-\infty\}$. When f (or c) takes only finite values, one gets the more familiar expression $f^c(y) = \sup\{c(x,y) - f(x) : x \in X\}$. We refer to Moreau (1970) for the subtilities of the calculus rules in $\overline{\mathbb{R}}$. For the sake of simplicity, in the following sections we assume that the coupling is real-valued.

The *subdifferential* $\partial^c f$ of $f : X \to \overline{\mathbb{R}}$ at $x_0 \in \mathrm{domf} := \mathrm{f}^{-1}(\mathbb{R})$ associated with a coupling c is defined by: $y_0 \in \partial^c f(x_0)$ iff $c(x_0, y_0) \in \mathbb{R}$ and

$$\forall x \in X \qquad f(x) \geq f(x_0) + c(x, y_0) - c(x_0, y_0). \tag{2}$$

Thus one sees that $y_0 \in \partial^c f(x_0)$ iff $c(x_0, y_0) \in \mathbb{R}$ and the Young-Fenchel equality holds:

$$f(x_0) + f^c(y_0) = c(x_0, y_0).$$

The symmetry between X and Y enables us to define in a similar way the subdifferential $\partial^c g$ of a function g on Y.

The following definitions relate the coupling with the metrics of the spaces.

DEFINITION 2.1. *Given $\lambda > 0$, one says that the coupling c is a λ-submetric coupling at $(x_0, y_0) \in X \times Y$ if $c(x_0, y_0) \in \mathbb{R}$ and for any $(x, y) \in X \times Y$ one has*

$$c(x,y) - c(x_0,y) \leq c(x,y_0) - c(x_0,y_0) + \lambda d(x,x_0)d(y,y_0). \tag{3}$$

If S and T are subsets of X and Y respectively, one says that c is a λ-submetric coupling at $S \times T$ if it is a λ-submetric coupling at (x_0, y_0) for any $(x_0, y_0) \in S \times T$. When $S = X$, $T = Y$, one simply says that c is a λ-submetric coupling.

Changing the metric d in X or Y to λd would enable to take $\lambda = 1$; in such a case we say that c is a *submetric coupling* (at $(x_0, y_0) \in X \times Y$, etc...). For a real-valued coupling, the preceding definition is symmetric in X and Y. Note that relation (3) can also be written in the following non symmetric form which allows for a clear comparison with the next notion: for any $r \in \mathbb{R}_+$, $x \in X$, one has

$$\sup_{y \in B(y_0,r)} (c(x,y) - c(x_0,y)) \leq c(x,y_0) - c(x_0,y_0) + \lambda r d(x,x_0).$$

DEFINITION 2.2. *Given $\kappa > 0$, one says that the coupling c is a κ-super-metric coupling at $(x_0, y_0) \in X \times Y$ if $c(x_0, y_0) \in \mathbb{R}$ and for any $r \in \mathbb{R}_+$, $x \in X$, one has*

$$\sup_{y \in B(y_0,r)} (c(x,y) - c(x_0,y)) \geq c(x,y_0) - c(x_0,y_0) + \kappa r d(x,x_0), \tag{4}$$

where $B(y_0, r)$ denotes the closed ball with center y_0 and radius r. One says that c is a κ-super-metric coupling at $(S,T) \subset (X,Y)$ if it is a κ-super-metric coupling at $(x_0, y_0) \in X \times Y$ for any $(x_0, y_0) \in S \times T$.

Observe that in this second definition the roles of X and Y are not symmetric. If the transposed coupling $c^T : Y \times X \to \overline{\mathbb{R}}$ given by $c^T(y,x) := c(x,y)$ is also a κ-super-metric coupling with respect to (y_0, x_0), then we say that c is *κ-super-metric pairing* with respect to (x_0, y_0). When $\kappa = 1$ we omit to mention it; moreover, c is said to be a *metric coupling* if it is both a submetric coupling and a super-metric coupling.

An anonymous referee raised the question of comparing the preceding notion with the concept of monotonicity property with constant $\kappa \in (0, 1)$ introduced in Rolewicz (1993, 1994). In order to answer this question let us say that c has the *κ-monotonicity property at* (x_0, y_0) if for each $x \in X$ there exists a sequence (r_n) of positive real numbers with limit 0 such that for each $n \in \mathbb{N}$ one has

$$\sup_{y \in B(y_0,r_n)} (c(x,y) - c(x_0,y)) \geq c(x,y_0) - c(x_0,y_0) + \kappa r_n d(x,x_0). \tag{5}$$

Then, following Rolewicz (1993, 1994), one says that c has the *κ-monotonicity property* if it has it at any $(x_0, y_0) \in X \times Y$. Clearly, the κ-monotonicity property at (x_0, y_0) is a consequence of the property that c is a κ-super-metric coupling at (x_0, y_0), hence the κ-monotonicity property is a consequence of the property that c is a κ-super-metric coupling at (X, Y). It may happen that the κ-monotonicity property at (x_0, y_0) holds while c is not a κ-super-metric coupling at (x_0, y_0). This fact occurs when $X = Y = \mathbb{R}$, $x_0 = y_0 = 0$, $c(x, y) = |x|\,|y|$ for $(x, y) \in X \times [-1, 1]$, $c(x, y) = 0$ otherwise (take any $\kappa \in (0, 1)$ and any sequence (r_n) with limit 0). On the other hand, our conditions bear on a particular pair (x_0, y_0) while condition (5) has to be satisfied for any pair $(x_0, y_0) \in X \times Y$. When Y contains an isolated point $\overline{y}$ the κ-monotonicity property at $(x_0, \overline{y})$ cannot hold, whatever x_0 is.

Let us give some examples. As just mentioned, they show that in general one cannot expect that the properties defined above are valid for any couple (x_0, y_0).

EXAMPLE 1 (classical duality)

Let X and Y be normed vector spaces in metric duality. This means that there exists a continuous bilinear map $c : X \times Y \to \mathbb{R}$ such that the associated mappings $c_X : X \to Y^*$, $c_Y : Y \to X^*$ given by $c_X(x)(y) := c(x, y)$, $c_Y(y)(x) := c(x, y)$ are isometric embeddings. Then c and c^T are metric couplings. Note that the present example contains the cases $Y = X^*$ and $X = Y^*$ and other cases. This framework is often convenient. For instance, when $X = L_p(T, E)$, $Y = L_q(T, E^*)$, where T is an interval, E is a Banach space and $1 \le p < \infty$, $q = (1 - 1/p)^{-1}$ one does not need to assume that the Radon-Nikodým property holds for E, as it would be the case if one wanted to have $Y = X^*$.

EXAMPLE 2

Let X and Y be normed vector spaces and let $c : X \times Y \to \mathbb{R}$ be an arbitrary continuous bilinear map. Then, for any $\lambda \ge \|c\|$, c is a λ-submetric coupling. If c_X open at 0 with rate κ onto its image, i.e. if for each $t > 0$ one has $B_{Y^*}(0, \kappa t) \cap c_X(X) \subset c_X(B_X(0, t))$, and if c_X is injective, then c is a κ-super-metric coupling at (x_0, y_0) for each $(x_0, y_0) \in X \times Y$. A similar result holds for c^T.

EXAMPLE 3 (Stepanov duality)

Let X and Y be metric spaces and let $c_Y : Y \to \mathbb{R}^X$ be an arbitrary map. Then, setting $c(x, y) := c_Y(y)(x)$, we get a coupling $c : X \times Y \to \mathbb{R}$. Suppose the image of c_Y is contained in the set $S(X, x_0)$ of Stepanov functions at x_0 (or stable at x_0 functions) which is the set of functions $f : X \to \mathbb{R}$ such that there exists some $k > 0$ for which $|f(x) - f(x_0)| \le k d(x, x_0)$. Endow $S(X, x_0)$ with the semi-norm given by

$$\|f\|_S^0 := \sup_{x \in X \setminus \{x_0\}} \frac{|f(x) - f(x_0)|}{d(x, x_0)}, \qquad f \in S(X, x_0)$$

or the norm given by $\|f\|_S := \|f\|_S^0 + |f(x_0)|$. If c_Y is stable at $y_0 \in Y$ in the sense that there exists some $\lambda > 0$ such that, for each $y \in Y$, one has

$\|c_Y(y) - c_Y(y_0)\|_S^0 \leq \lambda d(y, y_0)$, then c is a λ-submetric coupling at (x_0, y_0). If c_Y is a Lipschitz mapping with rate λ, the coupling c is a λ-submetric coupling at (x_0, Y). If c_Y is such that, for each $y \in Y$, one has $\|c_Y(y) - c_Y(y_0)\|_S^0 \geq \kappa d(y, y_0)$ then c is a κ-super-metric coupling at (x_0, y_0).

EXAMPLE 4

In particular, if X is a metric space not reduced to x_0, $Y = S(X, x_0)$ endowed with the norm $\|\cdot\|_S$ and if c is the evaluation mapping, c is a metric coupling at (x_0, Y), as follows from the fact that for any $r > 0$ the function $y_0 + rd(x_0, \cdot)$ belongs to the closed ball of center y_0 and radius r in Y.

EXAMPLE 5 (Lipschitz duality, Martínez-Legaz, 1988; Pallasche, Rolewicz, 1997; Rolewicz, 1994)

Suppose the mapping c_Y of the preceding example takes its values in the subspace $L(X)$ of Lipschitzian functions on X. Given $x_0 \in X$, endow $L(X)$ with the norm

$$\|f\|_L := \sup_{w,x \in X,\ w \neq x} \frac{|f(w) - f(x)|}{d(w, x)} + |f(x_0)| \,.$$

If c_Y is stable at $y_0 \in Y$, then c is a λ-submetric coupling at (X, y_0) for some $\lambda > 0$. If c_Y is a Lipschitz mapping with rate λ, c is a λ-submetric coupling at (X, Y).

In particular, if X is a metric space not reduced to x_0, $Y = L(X)$ and if c is the evaluation mapping, c is a metric coupling.

EXAMPLE 6 (homogeneous duality, Rubinov, 2000, and its references)

Let X be a normed vector space, let Y be a metric space and let $c : X \times Y \to \mathbb{R}$ be such that the image of c_Y is contained in the space $H(X)$ of positively homogeneous functions on X which are bounded on the unit ball B of X. Let us endow $H(X)$ with the norm given by $\|h\| = \sup\{|h(x)| : x \in B\}$. If c_Y is stable at some $y_0 \in Y$ with rate λ, then c is a λ-submetric coupling at $(0, y_0)$.

In particular, if $Y = H(X)$, c is a metric coupling at $(0, Y)$.

A special case of interest is the choice for Y of the set of infima of finite families of continuous linear forms on X (see Rubinov, 2000, Rubinov, Shveidel, 2000).

EXAMPLE 7 (subaffine duality, Martínez-Legaz, 1988; Penot, Volle, 1987, 1988, 1990)

Let X be a normed vector space with dual X^* and let $Y = X^* \times \mathbb{R}$ endowed with the sup norm; let c be given by $c(x, y) = x^*(x) \wedge s$ for $y := (x^*, s)$, where $r \wedge s := \min(r, s) = \frac{1}{2}(r + s - |r - s|)$. As

$$\begin{aligned} |r \wedge s - r' \wedge s'| &\leq \frac{1}{2}\,|r - r' + s - s'| + \frac{1}{2}\,||r - s| - |r' - s'|| \\ &\leq \frac{1}{2}\,|r - r'| + \frac{1}{2}\,|s - s'| + \frac{1}{2}\,|r - s - r' + s'| \\ &\leq |r - r'| + |s - s'| \end{aligned}$$

and as

$$|x^*(x) \wedge s - x^*(x') \wedge s| \leq \|x^*\| \, \|x - x'\| \leq \|y\| \, \|x - x'\| \,,$$

we see that c induces a submetric coupling at (X, y_0), where $y_0 = (0, 0)$. The Hahn-Banach theorem ensures that it is a 1-super-metric coupling at $(0, y_0)$ when X is endowed with the metric given by $d(x, x') := \|x - x'\| \wedge 1$. It is not the case for the distance associated with the norm.

EXAMPLE 8 (discrete measures)

Given a set E, let X be a subfamily of the set $\mathcal{P}_f(E)$ of finite subsets of E and let Y be a subspace of the space of bounded functions on E, endowed with the sup norm. Let us set for $x, x' \in X$, $y \in Y$,

$$d(x, x') := \#(x \Delta x'), \qquad c(x, y) := \sum_{e \in x} y(e)$$

where $\#S$ denotes the number of elements of the finite set S and $x \Delta x' := (x \backslash x') \cup (x' \backslash x)$. One easily checks that d defines a metric on X and that, for $x, x' \in X$, $y, y' \in Y$, one has

$$c(x, y) - c(x', y) - c(x, y') + c(x', y') \leq \sum_{e \in x \Delta x'} |y - y'| \, (e) \leq d(x, x') \, \|y - y'\| \,,$$

while for $x, x' \in X$, $y \in Y$, one has

$$c(x, y) - c(x', y) - c(x, y') + c(x', y') = \sum_{e \in x \Delta x'} |y - y'| \, (e) \geq r d(x, x')$$

for y' given by $y'(e) := y(e)$ for $e \in S \backslash x \Delta x'$, $y'(e) := y(e) + r$ for $e \in x' \backslash x$, $y'(e) := y(e) - r$ for $e \in x \backslash x'$, so that c is a metric coupling. This example is important for discrete optimization.

EXAMPLE 9 (homotone duality)

Let $(X, \leq)$ be an ordered metric space and let Y be a subset of the set of homotone functions on X, a function f being homotone (or isotone or nondecreasing) if for any $x, x' \in X$ with $x \leq x'$ one has $f(x) \leq f(x')$. Given $x_0 \in X$ let us set for $y, y' \in Y$ $d(y, y') := \sup\{r^{-1} |y(x) - y'(x)| : x \in B(x_0, r), r > 0\}$. Then the evaluation mapping is a submetric coupling at (x_0, Y). The special case when $X = \mathbb{R}$, Y being the set of superadditive homotone functions null at 0 is considered in Wolsey (1981). The special case when X is a distributive lattice and Y is formed of modular or submodular functions is also important (see Fujishige (1984), for instance).

EXAMPLE 10 (shady conjugacy)

Let X be a n.v.s. and let Y be its dual. Let $c^{\vee}$ and c^{∇} be the couplings given by $c^{\vee}(x, y) = -\iota_{[y \leq 1]}(x)$ and $c^{\nabla}(x, y) = -\iota_{[y < 1]}(x)$, where $[y \leq 1] := \{x \in$

$X : y(x) \leq 1\}$, $[y < 1] := \{x \in X : y(x) < 1\}$ and where, for a subset S of X, ι_S is the indicator function of S given by $\iota_S(x) = 0$ for $x \in S$, $\iota_S(x) = +\infty$ for $x \in X \backslash S$. Then $c^\vee$ and c^∇ are submetric couplings at $(0, 0)$. See Penot (1997, 2000, 2001), Thach (1991, 1993, 1994, 1995), Volle (1985).

EXAMPLE 11 (partial sublevel duality)

Let X be a n.v.s. and let $Y := X^* \times \mathbb{R}_-$. Let c^Σ be given by $c^\Sigma(x, (x^*, r)) = -\iota_{[x^* \geq r]}(x)$. Then c^∇ is a submetric coupling at $(0, (0, r))$ for each $r \in \mathbb{R}_-$. Similarly, if $Y := X^* \times (-\mathbb{P})$, where $\mathbb{P}$ is the set of positive numbers and if $c^S(x, (x^*, r)) = -\iota_{[x^* > r]}(x)$, then c^S is a submetric coupling at $(0, (0, r))$ for each $r \in \mathbb{P}$ (see Penot, 2000).

EXAMPLE 12 (augmented duality)

Given a coupling $c : X \times Y \to \overline{\mathbb{R}}$ and a function $k : X \to \mathbb{R}$, let $\widehat{c}$ be given by $\widehat{c}(x, y) := c(x, y) + k(x)$. Then one easily checks that $\widehat{c}$ is a metric coupling iff c is a metric coupling. The case in which X is a Hilbert space and $k(x) := \frac{r}{2} \|x\|^2$ for some $r > 0$ is known to be of special importance (Janin, 1973; Rockafellar, 1974; Penot, Volle, 1988; Poliquin, 1992; Eberhard, Nyblom, 1998, etc.).

Other examples are given in Rubinov (2000), Rubinov, Glover (1997, 1998, 1999), Rubinov, Andramonov and Penot (1999, 2000, 2001), Rubinov, Shveidel (2000), Rubinov, Simsek (1995), Schwartz (1973, 1980), Singer (1986, 1987, 1997), for instance.

3. Coercivity

Hereafter X and Y are metric spaces, c is a coupling between X and Y. In order to illustrate the possible uses of nonlinear dualities, we intend to study the interplay between boundedness, conjugacy and coercivity. This can be done in a quantitative (and simple) way (see also Cominetti, 1994, Fougères, 1977, Penot, 1995, for the classical case).

Let us say that a function f on a metric space X is *coercive* if there exists $x_0 \in X$ such that $f(x) \to \infty$ as $d(x, x_0) \to \infty$. This property is obviously independent of the point x_0: it amounts to the requirement that for each $r \in \mathbb{R}$ the sublevel set $[f \leq r]$ is bounded. We say that f is *super-coercive* if there exists $x_0 \in X$ such that $c_\infty(f) := \liminf_{d(x,x_0)\to\infty} f(x)/d(x, x_0) > 0$. We call $c_\infty(f)$ the *coercivity rate* of f. It does not depend on the choice of x_0. Clearly, when $c_\infty(f)$ is finite, it is the supremum of the real numbers b such that $f(x) \geq bd(x, x_0)$ for $d(x, x_0)$ large enough. Note that, if f is bounded below on balls, then $c_\infty(f)$ is also the supremum of the real numbers b such that for some real number a one has $f(x) \geq bd(x, x_0) - a$ for each $x \in X$. Let us say that f is *hyper-coercive* if $f(x)/d(x, x_0) \to \infty$ as $d(x, x_0) \to \infty$, or, equivalently, when $c_\infty(f) = +\infty$.

In the following two statements we suppose X and Y are provided with base points x_0 and y_0 respectively and that $c(\cdot, y_0) = 0$. This last assumption is satisfied in several of the examples displayed above. It can be obtained by replacing c by $\widehat{c}$ given by $\widehat{c}(x,y) := c(x,y) - c(x,y_0)$.

PROPOSITION 3.1. *(a) Suppose the coupling c is a λ-submetric coupling at (x_0,y_0) and $c(\cdot,y_0) = 0$. If for some $a \in \mathbb{R}$, $b > 0$, a function f on X satisfies $f(x) \geq bd(x,x_0) - a$ for each $x \in X$, then $g := f^c$ is bounded above by $a + a'$ on the ball $B(y_0, \lambda^{-1}b)$ provided $c(x_0,\cdot)$ is bounded above by a' on this ball.*
(b) Suppose the coupling c is a κ-super-metric coupling at (x_0,y_0) and $c(\cdot,y_0) = 0$. If g is a function on Y which is bounded above by a on a ball $B(y_0,b)$ for some $a \in \mathbb{R}$, $b > 0$, then $f := g^c$ satisfies $f(x) \geq \kappa bd(x,x_0) - a + a''$ for each $x \in X$ provided $c(x_0,\cdot)$ is bounded below by a'' on this ball. In particular f is super-coercive and its coercivity rate c_∞ is such that $c_\infty \geq \kappa b$.

When $\kappa = \lambda = 1$ as in the classical case, we get a dual interpretation of the coercivity rate of a function bounded below on balls with center x_0 as the supremum of the radius of the balls centered at y_0 on which the conjugate function is bounded above.
Proof. (a) Let f be such that for some $a \in \mathbb{R}$, $b > 0$, one has $f(x) \geq bd(x,x_0) - a$ for each $x \in X$. If the coupling c is a λ-submetric coupling at (x_0,y_0), and if $c(x_0,\cdot)$ is bounded above by a' on $B(y_0,\lambda^{-1}b)$ then for $y \in B(y_0,\lambda^{-1}b)$ one has

$$\begin{aligned} f^c(y) &\leq \sup_{x\in X} (c(x,y) - bd(x,x_0) + a) \\ &\leq \sup_{x\in X} (c(x_0,y) + \lambda d(x,x_0)d(y,y_0) - bd(x,x_0) + a) \leq a + a'. \end{aligned}$$

(b) Let $g : Y \to \overline{\mathbb{R}}$ be bounded above by a on a ball $B(y_0,b)$ for some $a \in \mathbb{R}$, $b > 0$. If the coupling c is a κ-super-metric coupling at (x_0,y_0) and $c(x_0,\cdot)$ is bounded below by a'' on $B(y_0,b)$, then for each $x \in X$, one has

$$\begin{aligned} g^c(x) &\geq \sup_{y\in B(y_0,b)} c(x,y) - a \\ &\geq \sup_{y\in B(y_0,b)} (c(x,y) - c(x_0,y) + a'') - a \geq \kappa bd(x,x_0) + a'' - a. \end{aligned}$$

■

REMARK 3.1. The preceding proof shows that $f^c - c(x_0,\cdot)$ is bounded above by a on the ball $B(y_0,\lambda^{-1}b)$ whenever f satisfies $f \geq bd(\cdot,x_0) - a$ and the coupling c is such that $c(x,y) - c(x_0,y) \leq \lambda d(x,x_0)d(y,y_0)$ for any $x \in X$, $y \in Y$. This condition is satisfied if c is a λ-submetric coupling at (x_0,y_0) such that $c(\cdot,y_0) = 0$. Similarly, $g^c \geq \kappa bd(\cdot,x_0) - a$ whenever $g - c(x_0,\cdot)$ is bounded above by a on the ball $B(y_0,b)$ and the coupling c is such that $\sup_{y\in B(y_0,b)} (c(x,y) - c(x_0,y)) \geq \kappa bd(x,x_0)$ for any $x \in X$. This condition is satisfied if c is a κ-super-metric coupling at (x_0,y_0) such that $c(\cdot,y_0) = 0$.

COROLLARY 3.1. *(a) Suppose the coupling c is a λ-submetric coupling at (x_0, y_0), $c(\cdot, y_0) = 0$ and $c(x_0, \cdot)$ is bounded above on balls. If a function f on X is hyper-coercive and bounded below on bounded sets, then f^c is bounded above on bounded sets of Y.*

(b) Suppose the coupling c is a κ-super-metric coupling at (x_0, y_0), $c(\cdot, y_0) = 0$ and $c(x_0, \cdot)$ is bounded below on balls. If a function g on Y is bounded above on bounded sets of Y, then g^c is hyper-coercive.

4. From rotundity properties to smoothness properties

In this section we suppose a function f on X satisfies a lower estimate and we get an upper estimate on its conjugate. Several of the estimates we deal with involve functions of the form

$$\sigma_{T,x_0}(x) := \sup_{v \in T} (c(x, v) - c(x_0, v))$$

for some $x_0 \in X$ and some $T \subset Y$. When X has a group structure and $c(\cdot, y_0)$ is additive for each $y_0 \in T$, this term is just the support function of T evaluated at $x - x_0 : \sigma_{T,x_0}(x) = \sigma_T(x - x_0)$ with

$$\sigma_T(u) := \sup_{v \in T} c(u, v).$$

Furthermore, if c is the classical duality, X being a normed space, if f is convex and continuous at x_0 and if $T = \partial f(x_0) := \{y_0 \in X^* : \langle y_0, \cdot \rangle \leq f(x_0 + \cdot) - f(x_0)\}$, one has $\sigma_T(x - x_0) = f'(x_0, x - x_0)$, the lower Hadamard derivative (or contingent derivative) of f at x_0 given by

$$f'(x_0, u) := \liminf_{(t,v) \to (0_+, u)} t^{-1} (f(x_0 + tv) - f(x_0))$$

for $u \in X$.

We start with a variant of Theorem 3.2 of Rolewicz (1994), an extension of the famous Asplund (1968) and Brønsted (1964) theorem; see also Azé, Penot (1995), Proposition 3.5.

PROPOSITION 4.1. *Given $x_0 \in \operatorname{dom} f$, $y_0 \in Y$ such that c is a λ-submetric coupling at (x_0, y_0), let $\gamma : \mathbb{R}_+ \to \overline{\mathbb{R}}$ be such that*

$$f(x) \geq f(x_0) + c(x, y_0) - c(x_0, y_0) + \gamma(d(x, x_0)) \quad \forall x \in X. \tag{6}$$

Then one has

$$f^c(y) \leq f^c(y_0) + c(x_0, y) - c(x_0, y_0) + \gamma^*(\lambda d(y, y_0)) \quad \forall y \in Y. \tag{7}$$

When γ is a starshaped gage, relation (6) can be interpreted as a reinforced subdifferentiability property or a rotundity property; then, by Lemma 2.1, γ^* is an hypermodulus, so that relation (7) is an approximation property. When Y is a normed vector space, $x_0 \in \partial^c f^c(y_0)$ and $y \mapsto c(x_0, y) - c(x_0, y_0)$ is linear and continuous, this last relation is a smoothness property which amounts to Fréchet super-differentiability of f^c at y_0. Moreover, when $c(\cdot, y_0) = 0$, and $\gamma(t) > 0$ for $t > 0$, relation (6) is a conditioning property of the minimizer x_0.

The preceding statement is a consequence of the following result which is more technical but more versatile (take $S = \{x_0\}$, $T = \{y_0\}$).

LEMMA 4.1. *Given $S \subset \operatorname{dom} f$, $T \subset Y$ such that c is a λ-submetric coupling at (S, T) and $\gamma : I\!R \to \overline{I\!R}$ such that*

$$f(x) \geq \inf_{x_0 \in S} \left[f(x_0) + \inf_{y_0 \in T} (c(x, y_0) - c(x_0, y_0)) + \gamma(d(x, x_0)) \right] \quad \forall x \in X \quad (8)$$

one has

$$f^c(y) \leq \sup_{y_0 \in T} \left[f^c(y_0) + \sup_{x_0 \in S} (c(x_0, y) - c(x_0, y_0)) + \gamma^*(\lambda d(y, y_0)) \right] \quad \forall y \in Y. \quad (9)$$

Proof. Let $y \in Y$. When $f^c(y) = -\infty$ inequality (9) is obvious. Let $r \in I\!R$, $r < f^c(y)$. We can find $x \in X$ such that $-r > f(x) - c(x, y)$. Then $f(x) < +\infty$, $c(x, y) > -\infty$ and $f(x) < c(x, y) - r$. By (8) we can find $(x_0, y_0) \in S \times T$ such that

$$c(x, y) - r > f(x_0) - c(x_0, y_0) + c(x, y_0) + \gamma(d(x, x_0)).$$

By the relations $f(x_0) - c(x_0, y_0) \geq -f^c(y_0)$, $\gamma(s) - st \geq -\gamma^*(t)$ we get

$$\begin{aligned} -r &> -f^c(y_0) + c(x, y_0) - c(x, y) + \gamma(d(x, x_0)) \\ &\geq -f^c(y_0) + c(x_0, y_0) - c(x_0, y) - \lambda d(x, x_0) d(y, y_0) + \gamma(d(x, x_0)) \\ &\geq -f^c(y_0) + c(x_0, y_0) - c(x_0, y) - \gamma^*(\lambda d(y, y_0)). \end{aligned}$$

Since we may assume that $f^c(y_0) < +\infty$, hence that $f^c(y_0)$ is finite, reversing signs and taking the supremum over $(x_0, y_0) \in S \times T$ and over r, we obtain inequality (9). ∎

We have seen that when S and T are singletons, one obtains the preceding proposition. If only one of the subsets S or T is a singleton, one also obtains interesting consequences. When $S = \{x_0\}$, relation (8) is a kind of calmness condition. Let us give a statement when T is a singleton $\{y_0\}$.

COROLLARY 4.1. *Given $S \subset \operatorname{dom} f$, $y_0 \in Y$ such that c is a λ-submetric coupling at (S, y_0) and $\gamma : I\!R \to \overline{I\!R}$ such that*

$$f(x) \geq \inf_{x_0 \in S} [f(x_0) + c(x, y_0) - c(x_0, y_0) + \gamma(d(x, x_0))] \quad \forall x \in X \quad (10)$$

one has

$$f^c(y) \leq f^c(y_0) + \sup_{x_0 \in S} [c(x_0, y) - c(x_0, y_0)] + \gamma^*(\lambda d(y, y_0)) \quad \forall y \in Y. \quad (11)$$

When X and Y are normed vector spaces, c is the Fenchel coupling and γ is a starshaped gage, relation (10) is satisfied when f is rotund at some $x_0 \in S$ and then relation (11) is a conical approximation property.

Taking for S a set of minimizers, and setting $d_S(x) := \inf\{d(x, w) : w \in S\}$ one gets a consequence of a conditioning property in the sense of Penot (1998).

COROLLARY 4.2. *Suppose $c(\cdot, y_0) = 0$, γ is nondecreasing and let S be a nonempty subset of the set of minimizers of f. Let $m := \inf f(X)$. Suppose that one has*

$$f(x) \geq m + \gamma(d_S(x)) \qquad \forall x \in X.$$

Then, if c is a λ-metric coupling at (S, y_0), setting $\sigma_S(y) := \sup_{x_0 \in S} c(x_0, y)$, one has

$$f^c(y) \leq f^c(y_0) + \sigma_S(y) + \gamma^*(\lambda d(y, y_0)) \qquad \forall y \in Y.$$

Now let us give a variant of Lemma 4.1 which will yield other statements generalizing results in Azé (1999), Azé, Rahmouni (1994, 1995, 1996).

LEMMA 4.2. *Suppose c is a λ-submetric coupling at (S, T) with $S \subset$ domf, $T \subset Y$. Suppose that for some nondecreasing function $\gamma : \mathbb{R}_+ \to \overline{\mathbb{R}}$ one has for each $x \in X$*

$$f(x) \geq \sup_{x_0 \in S} \left[f(x_0) + \sup_{y_0 \in T} (c(x, y_0) - c(x_0, y_0)) \right] + \gamma(d_S(x)). \qquad (12)$$

Then, for each $y \in Y$, one has

$$f^c(y) \leq \sup_{x_0 \in S} (-f(x_0) + c(x_0, y)) + \gamma^*(\lambda d_T(y)). \qquad (13)$$

Moreover, if for some subset Z of Y one has $z \in \partial^c f(x_0)$ for any $x_0 \in S$, $z \in Z$, then, for each $y \in Y$ one has

$$\begin{aligned} f^c(y) &\leq \sup_{x_0 \in S} \left(\inf_{z \in Z} (f^c(z) - c(x_0, z)) + c(x_0, y) \right) + \gamma^*(\lambda d_T(y)) \\ &\leq \inf_{z \in Z} \left(f^c(z) + \sup_{x_0 \in S} (c(x_0, y) - c(x_0, z)) \right) + \gamma^*(\lambda d_T(y)). \end{aligned}$$

Proof. Given $y \in Y$ such that $f^c(y) > -\infty$, let $r \in \mathbb{R}$, $r < f^c(y)$. We can find $x \in X$ such that $r < c(x, y) - f(x)$. Taking (12) into account, for any $x_0 \in S$, $y_0 \in T$, the property of the coupling yields

$$\begin{aligned} r &< -f(x_0) + c(x, y) - c(x, y_0) + c(x_0, y_0) - \gamma(d_S(x)) \\ r &< -f(x_0) + c(x_0, y) + \lambda d(x, x_0) d(y, y_0) - \gamma(d_S(x)). \end{aligned}$$

Taking the infimum over $y_0 \in T$, and, for a given $\varepsilon > 0$, picking some $x_0 \in S$ such that $d_S(x) \geq d(x, x_0) - \varepsilon$ we get

$$r < -f(x_0) + c(x_0, y) + \lambda d(x, x_0) d_T(y) - \gamma(d(x, x_0) - \varepsilon).$$

Setting $s := d(x, x_0) - \varepsilon$, $t := \lambda d_T(y)$ and observing that $st - \gamma(s) \leq \gamma^*(t)$, we get

$$r \leq -f(x_0) + c(x_0, y) + \gamma^* (\lambda d_T(y)) + \lambda \varepsilon d_T(y).$$

Taking the supremum over $x_0 \in S$ and the infimum over $\varepsilon > 0$ we obtain

$$r \leq \sup_{x_0 \in S} (-f(x_0) + c(x_0, y)) + \gamma^* (\lambda d_T(y)).$$

Since r is arbitrarily close to $f^c(y)$, relation (13) follows. The second assertion is a consequence of (13) and of the Fenchel equality $-f(x_0) = f^c(z) - c(x_0, z)$ for each $z \in Z$. ■

Simplified statements can be given when S, T or Z are singletons or when one takes $Z = T$. This last choice can be justified by the fact that, whenever γ takes nonnegative values, in particular when $\gamma(0) = 0$, relation (12) implies that $T \subset \partial^c f(x_0)$ for each $x_0 \in S$. Let us give a statement for this special case.

PROPOSITION 4.2. *Suppose c is a λ-submetric coupling at (S, T) with $S \subset \mathrm{dom} f$, $T \subset Y$. Suppose that for some nondecreasing function $\gamma : \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ relation (12) holds. Then, for each $y_0 \in T$, one has for any $y \in Y$*

$$f^c(y) \leq f^c(y_0) + \sup_{x_0 \in S} (c(x_0, y) - c(x_0, y_0)) + \gamma^* (\lambda d_T(y)).$$

The case S is a singleton is stated in the following corollary.

COROLLARY 4.3. *Suppose c is a λ-submetric coupling at (x_0, T) for some $x_0 \in X$, $T \subset Y$. Let $f : X \to \overline{\mathbb{R}}$ be an arbitrary function with $x_0 \in \mathrm{dom} f$, $T, Z \subset \partial^c f(x_0)$. Suppose that for some nondecreasing function $\gamma : \mathbb{R}_+ \to \overline{\mathbb{R}}$ one has for each $x \in X$*

$$f(x) \geq f(x_0) + \sup_{y_0 \in T} (c(x, y_0) - c(x_0, y_0)) + \gamma(d(x, x_0)). \tag{14}$$

Then, for each $y \in Y$ one has

$$f^c(y) \leq \inf_{z \in Z} (f^c(z) - c(x_0, z) + c(x_0, y)) + \gamma^* (\lambda d_T(y)). \tag{15}$$

Relation (14) can be interpreted as a kind of firm subdifferentiability property as it can be written

$$f(x) \geq f(x_0) + \sigma_{T, x_0}(x) + \gamma(d(x, x_0)).$$

When Y is a normed space and $c(x_0, \cdot)$ is additive, relation (15) yields, for each $y_0 \in T$

$$\begin{aligned} f^c(y) &\le f^c(y_0) + c(x_0, y - y_0) + \gamma^*(\lambda d_T(y)) \\ &\le f^c(y_0) + c(x_0, y - y_0) + \gamma^*(\lambda \|y - y_0\|), \end{aligned}$$

which says that $f^* := f^c$ is Fréchet super-differentiable at y_0, uniformly on T, provided γ is a starshaped gage, as then γ^* is an hyper-modulus by Lemma 2.1.

With the same choice for c, relation (12) can be interpreted as a form of a firm subdifferentiability property, uniform on S; relation (15) yields a conical upper approximation of f around z_0 when $z_0 \in T$. Such an approximation property has been studied by a number of researchers starting from Mignot (see Dontchev, Hager, 1994, Mignot, 1976, Pang, 1990 and 1995, Penot, 1982, Robinson, 1991, at least in its two-sided version, for the one-sided version used here see Agadi, Penot, 1996).

Now let us take for Z a singleton $\{z_0\}$ in Lemma 4.2.

COROLLARY 4.4. *Suppose c, λ, f, γ, $S \subset \operatorname{dom} f$, $T \subset Y$ are as in Lemma 4.2 and relation (12) holds. Suppose that $z_0 \in Y$ is such that $z_0 \in \partial^c f(x_0)$ for each $x_0 \in S$. Then, for each $y \in Y$, one has*

$$f^c(y) \le f^c(z_0) + \sup_{x_0 \in S} (c(x_0, y) - c(x_0, z_0)) + \gamma^*(\lambda d_T(y)).$$

Taking $T = \{y_0\} = \{z_0\}$, we obtain a further simplification.

COROLLARY 4.5. *Suppose c, λ, γ, $S \subset \operatorname{dom} f$ are as above and let $y_0 \in \partial^c f(x_0)$ for each $x_0 \in S$. Suppose c is a λ-submetric coupling at (S, y_0) and that for each $x \in X$ one has*

$$f(x) \ge \sup_{x_0 \in S} (f(x_0) + c(x, y_0) - c(x_0, y_0)) + \gamma(d_S(x)).$$

Then, for each $y \in Y$, one has

$$f^c(y) \le f^c(y_0) + \sup_{x_0 \in S} (c(x_0, y) - c(x_0, y_0)) + \gamma^*(\lambda d(y, y_0)).$$

5. From upper differentiability properties to rotundity properties

We now study the consequences of a reverse estimate: we suppose a function g on Y satisfies an upper estimate and we get a lower estimate on its conjugate. In view of the symmetry of our framework it would be possible to perform such a study on a function f on X. However, our choice is dictated by the hope to get a converse of the results of the preceding section.

PROPOSITION 5.1. *Let g be a given function on Y and let $\omega : \mathbb{R}_+ \to \overline{\mathbb{R}}$ be nondecreasing. Suppose that for some subset T of Y and some $x_0 \in \bigcap\{\partial^c g(y_0) : y_0 \in T\}$ the following estimate holds for any $y_0 \in T$ and any $y \in Y$*

$$g(y) \le g(y_0) + c(x_0, y) - c(x_0, y_0) + \omega\left(d(y, y_0)\right). \tag{16}$$

If for each $y_0 \in T$ the coupling c is a κ-super-metric coupling at (x_0, y_0), then the conjugate g^c of g satisfies

$$g^c(x) \ge g^c(x_0) + \sup_{y_0 \in T} \left(c(x, y_0) - c(x_0, y_0)\right) + \omega^*\left(\kappa d(x, x_0)\right). \tag{17}$$

We observe that when ω is an hyper-modulus, relation (16) can be interpreted as a super-differentiability property at y_0. Then, relation (17) is a rotundity property. We also note that the preceding result can be obtained from the case $T = \{y_0\}$. We deduce it from the following statement by taking $S := \{x_0\}$.

PROPOSITION 5.2. *Let g be a given function on Y and let $\omega : \mathbb{R}_+ \to \overline{\mathbb{R}}$ be nondecreasing. Suppose that for some subset T of Y and some $S \subset \bigcap\{\partial^c g(y_0) : y_0 \in T\}$ the following estimate holds for any $y_0 \in T$*

$$g(y) \le g(y_0) + \sup_{x_0 \in S} \left(c(x_0, y) - c(x_0, y_0)\right) + \omega\left(d(y, y_0)\right) \quad \forall y \in Y. \tag{18}$$

If c is a κ-super-metric coupling at (S, T), then the conjugate g^c of g satisfies

$$g^c(x) \ge \sup_{x_0 \in S} \left[g^c(x_0) + \sup_{y_0 \in T} \left(c(x, y_0) - c(x_0, y_0)\right)\right] + \omega^*\left(\kappa d_S(x)\right) \quad \forall x \in X. \tag{19}$$

Proof. Let $y_0 \in T$, $x \in X$ and let $t > g^c(x)$, $t \in \mathbb{R}$. For each $y \in Y$ we can find $s_y < g(y)$ such that $t > c(x, y) - s_y$ and $x_y \in S$ such that $s_y < g(y_0) + c(x_y, y) - c(x_y, y_0) + \omega\left(d(y, y_0)\right)$, hence

$$c(x, y) - s_y \ge -g(y_0) + c(x, y) - c(x_y, y) + c(x_y, y_0) - \omega\left(d(y, y_0)\right).$$

Since c is a κ-super-metric coupling at (x_y, y_0) and since ω is nondecreasing, taking the supremum over the ball $B(y_0, r)$ for a given $r \ge 0$, we get

$$\sup_{y \in B(y_0, r)} \left(c(x, y) - s_y\right) - c(x, y_0) \ge -g(y_0) + \kappa r d(x, x_y) - \omega(r),$$

$$t - c(x, y_0) \ge -g(y_0) + \kappa r d_S(x) - \omega(r).$$

Given $x_0 \in S$, using the relation $-g(y_0) = -c(x_0, y_0) + g^c(x_0)$, and taking the supremum over $r \in \mathbb{R}_+$, we get

$$t \ge g^c(x_0) - c(x_0, y_0) + c(x, y_0) + \sup_{r \ge 0} \left(\kappa r d_S(x)\right) - \omega(r)).$$

Taking the supremum over $y_0 \in T$ and the infimum over $t > g^c(x)$, we obtain the result from the very definition of ω^*. ■

Note that assumption (16) in Proposition 5.1 corresponds to the conclusion of Proposition 4.1 with $g = f^c$, $\omega = \gamma^*$, $\lambda = 1$ while the conclusion (17) corresponds to assumption (6) in Proposition 4.1 when $\kappa = 1$, $f = g^c$ and $\gamma = \omega^*$. Therefore we have obtained the following characterization.

THEOREM 5.1. *Let $f : X \to \overline{\mathbb{R}}$ and let $\gamma : \mathbb{R}_+ \to \overline{\mathbb{R}}$ be nondecreasing and such that $\gamma^{**} = \gamma$. Given $x_0 \in \operatorname{dom} f$, $y_0 \in \partial^c f(x_0)$ such that c is a metric coupling at (x_0, y_0), the following two relations are equivalent:*

$$f(x) \geq f(x_0) + c(x, y_0) - c(x_0, y_0) + \gamma(d(x, x_0)) \qquad \forall x \in X. \tag{20}$$

$$f^c(y) \leq f^c(y_0) + c(x_0, y) - c(x_0, y_0) + \gamma^*(d(y, y_0)) \qquad \forall y \in Y. \tag{21}$$

Similarly, from Proposition 5.2 one can get a characterization of well-conditioning. We first state a sufficient condition.

COROLLARY 5.1. *Let f be a given function on X such that $f = f^{cc}$ and let $\omega : \mathbb{R}_+ \to \overline{\mathbb{R}}$ be nondecreasing. Suppose there exists some element y_0 of Y such that $c(\cdot, y_0) = 0$. Suppose that for some subset S of the set of minimizers of f the following estimate holds*

$$f^c(y) \leq f^c(y_0) + \sup_{x_0 \in S} c(x_0, y) + \omega\left(d(y, y_0)\right) \qquad \forall y \in Y. \tag{22}$$

If c is a κ-super-metric coupling at (S, y_0), then

$$f(x) \geq \inf f(X) + \omega^*\left(\kappa d_S(x)\right). \tag{23}$$

Proof. Let us observe that, setting $g := f^c$, we have $S \subset \partial^c g(y_0)$ as $g(y_0) = -\inf f(X)$ because $c(\cdot, y_0) = 0$ and as for any $x_0 \in S$, $y \in Y$ we have $g^c(x_0) = f(x_0) = \inf f(X)$. Then the conclusion follows from Proposition 5.2 with $T := \{y_0\}$. ■

Gathering this sufficient condition with the necessary condition of Corollary 4.2 we get the following characterization of well-conditioning.

THEOREM 5.2. *Let f be a given function on X such that $f = f^{cc}$ and let $\gamma : \mathbb{R}_+ \to \overline{\mathbb{R}}$ be a gage such that $\gamma^{**} = \gamma$. Suppose there exists some element y_0 of Y such that $c(\cdot, y_0) = 0$ and c is a metric coupling at (S, y_0) where S is the set of minimizers of f. Then f satisfies the estimate*

$$f(x) \geq \inf f(X) + \gamma\left(d_S(x)\right) \qquad \forall x \in X$$

iff

$$f^c(y) \leq f^c(y_0) + \sup_{x_0 \in S} c(x_0, y) + \gamma^*\left(d(y, y_0)\right) \qquad \forall y \in Y. \tag{24}$$

Let us end our study with the following corollary generalizing part of Azé, Rahmouni (1995), Lemma 2. It deals with the computation of the conjugate of a function of the form $g := \varphi \circ d_T$, where T is some subset of Y.

COROLLARY 5.2. *Suppose there exists a point $x_0 \in X$ such that $c(x_0, \cdot) = 0$ and the coupling c is a metric coupling at (x_0, y_0) for each $y_0 \in T$, where T is some subset of Y. Let $\varphi : \mathbb{R}_+ \to \overline{\mathbb{R}}$ be nondecreasing and such that $\varphi(0) = 0$. Then, for σ_T and g given by $\sigma_T(x) := \sup_{y_0 \in T} c(x, y_0)$, $g(y) = \varphi(d_T(y))$, one has*

$$g^c(x) = \sigma_T(x) + \varphi^*(d(x, x_0)).$$

In particular, one has $(d_T)^c(x) = \sigma_T(x)$ for $x \in B(x_0, 1)$, else $+\infty$.

Proof. Since $c(x_0, \cdot) = 0$ and since for any $y_0 \in T$, $y \in Y$, one has $g(y_0) = \varphi(0) = 0 \leq \varphi(d_T(y)) = g(y)$, so that $x_0 \in \partial^c g(y_0)$ and $g^c(x_0) = 0$. Thus, Proposition 5.1 with $\omega = \varphi$ yields

$$g^c(x) \geq g^c(x_0) + \sup_{y_0 \in T} c(x, y_0) + \varphi^*\left(d(x, x_0)\right) = \sigma_T(x) + \varphi^*(d(x, x_0)).$$

On the other hand, by substituting Y, X, T, g, φ, x_0, to X, Y, S, f, γ, z_0 in Corollary 4.4 and observing that

$$g(y) \geq \sup_{y_0 \in T} \left(g(y_0) + c(x_0, y) - c(x_0, y_0)\right) + \varphi(d_T(y)),$$

we get

$$g^c(x) \leq g^c(x_0) + \sup_{y_0 \in T} c(x, y_0) + \varphi^*\left(d(x, x_0)\right) = \sigma_T(x) + \varphi^*(d(x, x_0)).$$

The last assertion corresponds to the case $\varphi(t) = |t|$ for which $\varphi^* = \iota_{[0,1]}$, the indicator function of $[0, 1]$ whose value is 0 on $[0, 1]$ and $+\infty$ elsewhere. ■

It is shown in Azé, Rahmouni (1994, 1995, 1996), Penot (1995), that well-conditioning of f is linked with a continuity property of the subdifferential of the conjugate function of f and with a growth property of the subdifferential of f. These questions will not be considered here.

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