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Numerically robust synthesis of discrete-time H_∞ estimators based on dual J-lossless factorisations

by

Piotr Suchomski
Department of Automatic Control
Faculty of Electronics, Telecommunication and Computer Science
Gdańsk University of Technology
Narutowicza 11/12, 80-952 Gdańsk, Poland
e-mail: piotrjs@poczta.onet.pl

Abstract: An approach to the numerically reliable synthesis of the H_∞ suboptimal state estimators for discretised continuous-time processes is presented. The approach is based on suitable dual J-lossless factorisations of chain-scattering representations of estimated processes. It is demonstrated that for a sufficiently small sampling period the standard forward shift operator techniques may become ill-conditioned and numerical robustness of the design procedures can be significantly improved by employing the so-called delta operator models of the process. State-space models of all H_∞ suboptimal estimators are obtained by considering the suitable delta-domain algebraic Riccati equation and the corresponding generalised eigenproblem formulation. A relative condition number of this equation is used as a measure of its numerical conditioning. Both regular problems concerning models having no zeros on the boundary of the delta-domain stability region and irregular (non-standard) problems of models with such zeros are examined. For the first case, an approach based on a dual J-lossless factorisation is proposed while in the second case an extended dual J-lossless factorisation based on a zero compensator technique s required. Two numerical examples are given to illustrate some properties of the considered delta-domain approach.

Keywords: discrete-time systems, state estimation, linear filters, Riccati equation, numerical methods.

1. Introduction

The use of the so called delta (δ) operator in formulation and solving of many discrete-time problems (control, estimation, signal processing, modelling) has a number of advantages as opposed to the use of the conventional forward shift operator (q). First, the errors. Second, the δ-operator formulation allows for describing the asymptotic behaviour of discrete-time models of continuous-time systems as the sampling period converges to zero (Middleton and Goodwin, 1986, 1990; Goodwin et al., 1992; Gevers and Li, 1993; Feuer and Middleton, 1995; Li and Fan, 1997; Chen et al., 1999; Suchomski 2001a). The main motivation of the paper is to provide a concise methodology for numerically reliable synthesis of H_∞ suboptimal state estimators for discretised continuous-time processes, especially with fast sampling rates. There is a well known duality between the optimal control and estimation (filtering) problems and the optimal state estimator is the celebrated Kalman filter (Green and Limebeer, 1995; Hassibi et al., 1999). The standard H_∞ estimation problem differs from the Kalman filtering approach in two respects:

(i) unknown deterministic exogenous signals (disturbances) of finite energy replace the white-noise processes that drive the system and corrupt the observations,

(ii) the aim of the estimator (filter) is to ensure that the energy gain from the disturbances to the estimation error is less than a prespecified level (given in terms of the $l_2^{[0,\infty)}$ induced norm).

Hence, the aim is to find a state (weighted) estimate of the form of a linear function of the observations such that the ratio of the estimation error energy to the disturbance energy is suitably bounded. Usually we have conflicting requirements: a small estimation gain is required for 'good attenuation' of measurement noises but not for 'good detection' of the state being estimated. Some additional knowledge (if it exists) about the measurement noise channels can be utilised to improve the frequency shape of the estimator and make the necessary design tradeoffs more rational and better justified. It is worth noting that the 'generic' H_∞ formulation of the design problem concerning the so-called unknown-input observers (see, for example, Hou and Möller, 1992) can also be considered from this viewpoint.

The first δ-domain formulation of the standard H_∞ problems was presented by Middleton and Goodwin (1990). They derived a full-information algorithm, based on a game theory formulation of the original control design problem. A connection between mixed structured singular value robustness theory in the continuous-time and discrete-time areas has been derived by Collins et al. (1997), where, in order to avoid the inherent numerical ill-conditioning resulting from the use of the q operator, the δ-domain modelling of a discrete-time plant has been employed. A δ-domain version of the generic H_∞ discrete-time algorithm of Green and Limebeer (1995) was derived by Collins and Song (1999).

In this paper, concerning the H_∞ suboptimal synthesis of state estimators for δ-operator representations of continuous-time processes, an approach based on dual J-lossless factorisations of the δ-domain dual chain-scattering models of these processes is presented. The key role in the theory of the dual tion of a rational matrix function of a given dynamic system (Kimura, 1992b, 1995, 1997; Tsai et al., 1993). For the so-called standard H_∞ problems, optimal controllers are obtained via performing two coupled J-lossless factorisations (Tsai and Postlethwaite, 1991; Tsai and Tsai, 1992, 1993; Kimura, 1997). A J-lossless coprime factorisation approach to the standard H_∞ control in δ-domain was reported in Suchomski (2001b, 2002b) where necessary and sufficient conditions for the solvability of the problem were given. The method presented in this paper is based on a dual chain-scattering description of the process being estimated and requires the corresponding δ-domain algebraic Riccati equation to be solved. For a regular process having no invariant zeros on the boundary of the δ-domain stability region the resulting generalised eigenvalue problem is solved by using a methodology based on the standard invariant stable subspace approach applied to a suitable extended δ-domain matrix pencil containing parameters of the state-space model of this process. Such an approach can not be utilised for irregular processes having invariant zeros on the boundary of the δ-domain stability region. Note that such 'non-standard' models may appear in many practical problems of the H_∞ design (Kimura, 1984; Safonov, 1987; Sugie and Hara, 1989; O'Young et al., 1989; Hara and Sugie, 1991; Hara et al., 1992; Scherer, 1992a,b). For example, an extended J-lossless outer factorisation for strictly proper transfer functions with $j\omega$-zeros has been examined in Hara and Sugie (1991). In Hara et al. (1992), after having discussed a 1-block H_∞ control problem concerning plants with $j\omega$-axis poles and zeros, the authors derived a necessary and sufficient condition for the H_∞ model matching problem in the transfer function and the state-space setting. In the state estimation issue, such 'unstable' zeros can appear autonomously if some prior knowledge about the process being estimated is present or can artificially be utilised by the designer as a convenient tool for shaping some frequency attributes of the estimator. For example, in diagnostic systems the step (positional) signals appearing in the measurement noise channels can be regarded as adequate symptoms of sensor faults (Chen and Patton, 1999). Some low frequency 'modelling' zeros introduced in a dynamic system description can facilitate design of a decoupled residual generator.

A continuous-time technique called the 'zero cancelling compensation' was derived to cancel the $j\omega$-axis (including infinity) zeros (Copeland and Safonov, 1992, 1995). This approach allows for the use of the common J-lossless factorisation methodologies to solve the extended J-lossless factorisation problem since the zero-compensated system can be treated as a 'standard' regular plant. Consequently, conditions for the solvability of the extended J-lossless factorisation can be derived in terms of the zero-compensated system and the zero compensator. Since the zero compensator is not unique, it follows that, in general, a set of controllers determined by the compensator parameters can be obtained.

In this paper, an extended J-lossless factorisation for irregular generalised δ-domain plants (processes) with invariant zeros on the boundary of the stability

δ-domain is utilised to cancel such zeros and some additional attempt is made to reduce the complexity of the effective estimator. It is thus observed that estimators resulting from a 'directly' employed extended dual J-lossless factorisation methodology with a left zero compensation have non-minimal realisations with uncontrollable modes. Therefore, any reasonable design methodology should give solutions of the suitable low order without constructing any 'evident' zero compensators.

The rest of the paper is organised as follows. In Section 1, some preliminary properties of the δ-domain modelling are presented. In Section 2, fundamental issues related to numerical conditioning of the δ-domain discrete-time Riccati equations are given. Specifically, it is shown why the δ-domain approach to the discrete-time Riccati equations are much superior to the standard q-domain methods if numerical behaviour is assumed as a basis of comparison. In Section 3, two basic problems of the H_∞ optimisation are stated with respect to scattering and dual chain-scattering models of the optimised dynamic system. Sections 4 and 5 contain main results of the paper. In Section 4, after defining the δ-domain dual J-lossless systems and a dual J-lossless factorisation of their dual chain-scattering models we consider the necessary and sufficient conditions for the solvability of the standard H_∞ problem of optimisation of such systems. Next, an extended dual J-lossless factorisation approach for systems with models having invariant zeros on the boundary of the stability region is presented. Conditions for the existence of dual and extended dual J-lossless factorisations are derived in terms of the suitable δ-domain algebraic Riccati equations. In Section 5, state space formulae for all δ-domain H_∞ suboptimal estimators are presented. Two illustrative examples concerning synthesis of such estimators are given in Section 6. The first simple example deals with a process without zeros on the δ-domain stability circle. In the second example, the mechanism of the extended dual J-lossless factorisation is employed to solve a problem of H_∞ suboptimal estimation for a process with such an 'unstable' zero. Some concluding remarks are given in Section 7.

1.1. Basic properties of the delta operator

Let q be the forward shift operator $q : l_2 \to l_2$, established for a sequence $\{x_k\}_{k=0}^{\infty} \in l_2$ as $qx_k = x_{k+1}$. The delta operator $\delta : l_2 \to l_2$ is defined as the following first-order divided difference

$$\delta = \frac{q-1}{\Delta} \tag{1}$$

where $\Delta \in R$ is the sampling period (Middleton and Goodwin, 1990; Ninness and Goodwin, 1991). Thus, the operators q and δ are affinely connected via the relation $q = \Delta \cdot \delta + 1$. Let (q, z) and (δ, ζ) denote the pairs of discrete-time operators q and δ, and the corresponding complex variables z and ζ. Let of the closed disk $\bar{D}_\Delta$ is denoted $\partial\bar{D}_\Delta$. A δ-domain transfer function matrix $G(\zeta) = C(\zeta I - A)^{-1}B + D$, where A, B, C and D are properly dimensioned real matrices of a realisation (A, B, C, D), can be written as

$$G(\zeta) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \tag{2}$$

The set of all eigenvalues $\lambda_i(A)$, $i \in \{1, ..., n\}$ of a matrix $A \in R^{n\times n}$ is denoted $\lambda(A)$. Matrix A is said to be δ-stable if $\lambda(A) \subset D_\Delta$ while $G(\zeta)$ is stable if all its poles belong to D_Δ. The homographic mapping $\zeta \to \zeta^\sim = -\zeta/(1+\Delta\zeta)$ transforms a complex number into its reflection with regard to $\partial\bar{D}_\Delta$. The conjugate system of $G(\zeta)$ is defined as $G^\sim(\zeta) = G^T(-\zeta/(1+\Delta\zeta))$. Assuming that $I_n + \Delta A$ is non-singular we obtain

$$G^\sim(\zeta) = \left[\begin{array}{c|c} -I_A A^T & -I_A C^T \\ \hline B^T I_A & D^T - \Delta B^T I_A C^T \end{array}\right] \tag{3}$$

where $I_A = (I_n + \Delta A^T)^{-1}$ and $I_n \in R^{n\times n}$ denotes the identity matrix. The Hermitian conjugate of $G(\zeta)$ is defined as $G^*(\zeta) = G^T(\bar{\zeta})$. Hence, for $\zeta \in \partial\bar{D}_\Delta$ $G^*(\zeta) = G^\sim(\zeta)$ holds.

1.2. Delta-domain modelling

Consider a linear continuous-time ($\rho = d/dt$) state-space model

$$\left\{\begin{array}{rcl} \rho x(t) & = & A_\rho x(t) + B_\rho u(t) \\ y(t) & = & C_\rho x(t) + D_\rho u(t) \end{array}\right. \tag{4}$$

where $x(t)$ is the state vector, $u(t)$ is the input and $y(t)$ denotes the output. If $u(t)$ is piece-wise constant and right-continuous the following δ-operator state-space model can be derived (Middleton and Goodwin, 1990; Ninness and Goodwin, 1991; Premaratne et al., 1994; Neuman, 1993)

$$\left\{\begin{array}{rcl} \delta x_k & = & A_\delta x_k + B_\delta u_k \\ y_k & = & C_\delta x_k + D_\delta u_k \end{array}\right. \tag{5}$$

where $x_k = x(k\Delta)$, $u_k = u(k\Delta)$, $y_k = y(k\Delta)$, and

$$A_\delta = \Gamma_\Delta A_\rho/\Delta,\ B_\delta = \Gamma_\Delta B_\rho/\Delta,\ C_\delta = C_\rho,\ D_\delta = D_\rho,\ \Gamma_\Delta = \int_0^\Delta e^{\tau A_\rho} d\tau. \tag{6}$$

The q-domain model takes the form of (A_q, B_q, C_q, D_q) with

Hence, if $\Delta \to 0$, then $(A_\delta \to A_\rho, B_\delta \to B_\rho)$ while $(A_q \to I_n, B_q \to 0)$. Let

$$S_G(\zeta) = \begin{bmatrix} A - \zeta I_n & B \\ C & D \end{bmatrix} \tag{8}$$

denote the system matrix associated with $G(\zeta)$. The normal rank of $S_G(\zeta)$, denoted normrank $(S_G(\zeta))$, is the maximally possible rank of $S_G(\zeta)$ for at least one $\zeta \in C$. A complex $\zeta_0 \in C$ is called an invariant zero of $G(\zeta)$ if it satisfies rank $(S_G(\zeta_0)) <$ normrank $S_G(\zeta)$ (see Lemma 10 given in Appendix 2).

2. Discrete-time Riccati equations

Consider the discrete-time Riccati equation

$$\begin{aligned} & P_q^T X_q P_q - X_q + \\ & \quad -(P_q^T X_q Q_q + S_q)(T_q + Q_q^T X_q Q_q)^{-1}(P_q^T X_q Q_q + S_q)^T + R_q \\ & \quad = 0_{n\times n} \end{aligned} \tag{9}$$

where $P_q, R_q = R_q^T \in R^{n\times n}$, $Q_q, S_q \in R^{n\times m}$ and $T_q = T_q^T \in R^{m\times m}$. Assuming that

$$P_q = I_n + \Delta P,\ Q_q = Q,\ R_q = \Delta^2 R,\ S_q = \Delta S,\ T_q = T \tag{10}$$

where $P \in R^{n\times n}$, $R = R^T \in R^{n\times n}$, $Q, S, \in R^{n\times m}$ and $T = T^T \in R^{m\times m}$ we get the corresponding δ-domain Riccati equation (δARE)

$$\begin{aligned} & P^T X + XP + \Delta P^T XP + \\ & \quad -((I_n + \Delta P^T)XQ + S)(T + \Delta Q^T XQ)^{-1}((I_n + \Delta P^T)XQ + S)^T + R \\ & \quad = 0_{n\times n} \end{aligned} \tag{11}$$

where $X = X_q/\Delta$.

Let (U, W) denote a pair of real matrices associated with (11)

$$(U, W) = \left(\begin{bmatrix} P & 0_{n\times n} & Q \\ -R & -P^T & -S \\ S^T & Q^T & T \end{bmatrix}, \begin{bmatrix} I_n & 0_{n\times n} & 0_{n\times m} \\ 0_{n\times n} & I_n + \Delta P^T & 0_{n\times m} \\ 0_{m\times n} & -\Delta Q^T & 0_{m\times} \end{bmatrix} \right) \tag{12}$$

The set of all matrices of the form $U - \lambda W$ with $\lambda \in C$ is said to be a $(2n+m) \times (2n+m)$ extended matrix pencil. The eigenvalues of the extended pencil are elements of the set $\lambda(U, W)$ defined by $\lambda(U, W) = \{z \in C : \det(U - zW) = 0\}$. If $\lambda \in \lambda(U, W)$ and $Ux = \lambda Wx$ with $x \neq 0$ then x is referred to as an eigenvector of the extended pencil (Golub and Van Loan, 1996; Stewart, 1973, 2001).

Let $X_-(U, W)$ of dimension $n_- = \dim(X_-(U, W)) \le n$ denote the invariant

$R^{(n+n+m)\times n_-}$ be a matrix of full column rank whose columns form a basis for $X_-(U,W)$. This means that $X_-(U,W) = \mathrm{Im}[X_1^T\ X_2^T\ X_3^T]^T$ and

$$U\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = W\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}\Lambda \tag{13}$$

where $\Lambda \in R^{n_-\times n_-}$ is stable, $\lambda(\Delta) \subset D_\Delta$. The domain of δRic, denoted by dom(δRic), consists of all pairs (U,W) such that $n_- = n$ and $X_1 \in R^{n\times n}$ is non-singular. The following lemma can be regarded as a δ-domain version of the standard result (Van Dooren, 1981a; Arnold and Laub, 1984; Laub, 1991; Lancaster and Rodman, 1995) that recasts the δARE of (11) as a generalised eigenvalue problem.

LEMMA 1 *Let* $(U,W) \in \mathrm{dom}(\delta\mathrm{Ric})$ *and* $X = X_2X_1^{-1}$. *Then*

(i) X *is unique (that is denoted as* $X = \delta\mathrm{Ric}(U,W)$*) and symmetric* $(X = X^T)$,

(ii) $T + \Delta Q^TXQ$ *is non-singular and* X *satisfies the* δARE *of (11),*

(iii) $F_\delta = X_3X_1^{-1}$ *is unique and* $F_\delta = -(T+\Delta Q^TXQ)^{-1}((I_n+\Delta P^T)XQ+S)^T$,

(iv) $G_\delta = P + QF_\delta = X_1\Lambda X_1^{-1}$ *is stable,* $\lambda(G_\delta) \subset D_\Delta$.

The matrix T of (12) is often diagonal or even identity, which makes T^{-1} trivial to determine and in such cases a reduced in-order generalised eigenvalue problem treatment based on standard techniques for $2n \times 2n$ matrix pencils can be utilised (Arnold and Laub, 1984; Lancaster and Rodman, 1995; Benner et al., 1997; Ionescu et al., 1997; Suchomski, 2001c). In general, T may instead be non-diagonal and ill-conditioned with respect to inversion, or possibly even singular, in which case the considered technique for $(2n+m)\times(2n+m)$ extended pencils should be used. On the other hand, the use of the notions of extended pencils turns to be an effective and 'natural' tool for obtaining the extended dual J-lossless factorisations being considered in Section 4.2.

Let $(U,W) \in \mathrm{dom}(\delta\mathrm{Ric})$ and P, Q, R, S and T be subject to perturbations $\varepsilon\bar{P}$, $\varepsilon\bar{Q}$, $\varepsilon\bar{R}$, $\varepsilon\bar{S}$, and $\varepsilon\bar{T}$, respectively. It is assumed that $\bar{R}$ and $\bar{T}$ are both symmetric, and $\varepsilon \in R$. Define a directional derivative of $X = \delta\mathrm{Ric}(U,W)$

$$\begin{aligned}&\nabla_\varepsilon X(\bar{P},\bar{Q},\bar{R},\bar{S},\bar{T}|P,Q,R,S,T) = \\ &= \lim_{\varepsilon\to 0}\frac{X(P+\varepsilon\bar{P},Q+\varepsilon\bar{Q},R+\varepsilon\bar{R},S+\varepsilon\bar{S},T+\varepsilon\bar{T}) - X(P,Q,R,S,T)}{\varepsilon}.\end{aligned} \tag{14}$$

This derivative, as an image of $(\bar{P},\bar{Q},\bar{R},\bar{S},\bar{T})$ in a linear and continuous mapping, established by the Fréchet derivative $\nabla X(P,Q,R,S,T)$, can be regarded as the Fréchet differential of X at (P,Q,R,S,T). A norm of $\nabla X(P,Q,R,S,T)$

T). Let

$$\begin{aligned} &\| \nabla X(P,Q,R,S,T) \| = \\ &= \sup_{\| (\bar{P},\bar{Q},\bar{R},\bar{S},\bar{T}) \| \neq 0} \frac{\| \nabla_{\varepsilon} X(\bar{P},\bar{Q},\bar{R},\bar{S},\bar{T} | P,Q,R,S,T) \|_R}{\| (\bar{P},\bar{Q},\bar{R},\bar{S},\bar{T}) \|_D} \end{aligned} \tag{15}$$

where $\| \cdot \|_D$ and $\| \cdot \|_R$ are norms on the domain and the range space of $\nabla X(P,Q,R,S,T)$. It is convenient to use the weighted Frobenius norm on the domain space

$$\| (\bar{P},\bar{Q},\bar{R},\bar{S},\bar{T}) \|_D = \left\| \left(\frac{\bar{P}}{\|P\|_F}, \frac{\bar{Q}}{\|Q\|_F}, \frac{\bar{R}}{\|R\|_F}, \frac{\bar{S}}{\|S\|_F}, \frac{\bar{T}}{\|T\|_F}, \right) \right\|_F \tag{16}$$

and the Frobenius norm $\| \cdot \|_F$ on the range space of $\nabla X(P,Q,R,S,T)$, so that

$$\begin{aligned} &\| \nabla X(P,Q,R,S,T) \| = \sup_{\| (\bar{P},\bar{Q},\bar{R},\bar{S},\bar{T}) \| \neq 0} \frac{1}{\| (\bar{P},\bar{Q},\bar{R},\bar{S},\bar{T}) \|_F} \times \\ &\| \nabla_{\varepsilon} X \left(\|P\|_F \bar{P}, \|Q\|_F \bar{Q}, \|R\|_F \bar{R}, \|S\|_F \bar{S}, \|T\|_F \bar{T}, | P,Q,R,S,T \right) \|_F \end{aligned} \tag{17}$$

A relative condition number of the δARE of (11), which measures the sensitivity of X with respect to perturbations in (P,Q,R,S,T), can be defined in the following way (see Suchomski, 2001c, 2002b)

$$\kappa_{\delta}(P,Q,R,S,T) = \frac{\| \nabla X(P,Q,R,S,T) \|}{\|X\|_F} . \tag{18}$$

Let $\otimes$ denote the Kronecker product of two matrices, $\mathrm{vec}(M)$ denote the vector obtained by stacking the columns of a matrix M into one vector and $\| \cdot \|_s$ be the spectral norm. The surveys of the Kronecker product, the vec operators, and vec-permutation matrices can be found in Graham (1981), Henderson and Searle (1981), Weinmann (1991), Higham (1996). Moreover, let M^+ denote the Moore-Penrose pseudo-inverse of M (Boullion and Odell, 1971; Meyer, 2000).

LEMMA 2 *The relative condition number $\kappa_{\delta}(P,Q,R,S,T)$ of the δARE of (11), which measures the sensitivity of X with respect to perturbations in (P,Q,R,S,T), takes the form*

$$\| (F_P, F_Q, F_R, F_S, F_T) \|$$

where

$$F_P = \| P \|_F \; H_\delta^+ [I_n \otimes (I_n + \Delta G_\delta^T) X + ((I_n + \Delta G_\delta^T) X \otimes I_n) T_{n,n}] \quad (20)$$

$$F_Q = \| Q \|_F \; H_\delta^+ [F_\delta^T \otimes (I_n + \Delta G_\delta^T) X + ((I_n + \Delta G_\delta^T) X \otimes F_\delta^T) T_{n,m}] \quad (21)$$

$$F_R = \| R \|_F \; H_\delta^+ \quad (22)$$

$$F_S = \| S \|_F \; H_\delta^+ [F_\delta^T \otimes I_n + (I_n + F_\delta^T) T_{n,m}] \quad (23)$$

$$F_T = \| T \|_F \; H_\delta^+ (F_\delta^T \otimes F_\delta^T) \quad (24)$$

and

$$H_\delta \;\; = \;\; G_\delta^T \otimes I_n + I_n \otimes G_\delta^T + \Delta G_\delta^T \otimes G_\delta^T \quad (25)$$

while

$$T_{n,m} \;\; = \;\; \sum_{i=1}^{n} \sum_{j=1}^{m} e_{n,i} e_{m,j}^T \otimes e_{m,j} e_{n,i}^T \quad (26)$$

denotes a vec-permutation matrix for $e_{k,l}$ as the l-th unit vector in R^k .

Proof. Proof can be done similarly as in Suchomski (2001c). ■

REMARK 1 *Let $\lambda(G_\delta) = \{\lambda_i\}_{i=1}^n$, hence $\lambda(H_\delta) = \{\lambda_i + \lambda_j + \Delta\lambda_i\lambda_j\}_{i,j=1}^n$. It follows that H_δ is invertible iff H_δ is stable, i.e. X is the stabilising solution to (11) . In this case, a certain δ-domain Lyapunov equation, which corresponds to the definition of $\kappa_\delta(P, Q, R, S, T)$, has a unique solution (Suchomski, 2001c, 2002c). For a non-stabilising X and G_δ having eigenvalues on $\partial \bar{D}_\Delta$ the matrix H_δ is non-invertible and the corresponding δ-domain Lyapunov equation has a set of non-unique solutions from which the one of the minimal norm should be taken.*

A relative condition number of the q-domain ARE of (9), denoted as $\kappa_q(P_q, Q_q, R_q, S_q, T_q)$, can be defined in a similar manner (Suchomski, 2001c, 2002b). Using a first-order-in-Δ analysis one can easily derive the following lemma that completely explains the superiority of δ-domain solutions to their counterparts based on the forward-shift operator q.

LEMMA 3 *For a sufficiently small sampling period Δ there is*

$$\kappa_q(P_q, Q_q, R_q, S_q, T_q) \propto \frac{\kappa_\delta(P, Q, R, S, T)}{\Delta} \, . \quad (27)$$

Hence, the q-domain AREs of the assumed type of parameterisation, (10), become ill-conditioned as $\Delta \to 0$.

It is worth noting that the affine transformation $I_n + \Delta P$ of P and not scaling of R and S turns to be the main reason for which the q-domain solution is

3. H_∞ optimisation in the delta-domain

Let $RL_\infty^{p\times r}$ denote the space of proper real-rational $p\times r$-matrix-valued functions of $\zeta \in C$ which are analytical in $\partial\bar{D}_\Delta$. $RH_\infty^{p\times r}$ is the subspace of $RL_\infty^{p\times r}$ consisting of all stable matrices. The $RH_\infty^{p\times r}$ infinity norm is defined as $\| \Phi \|_\infty = \sup_{\omega\in R} \| \Phi((e^{j\omega\Delta}-1)/\Delta) \|_s$. The set of all unitary bounded matrices in $RH_\infty^{p\times r}$ is defined by $BH_\infty^{p\times r} = \{\Phi \in RH_\infty^{p\times r} : \| \Phi \|_\infty < 1\}$. The group of all units of $RH_\infty^{p\times r}$ is denoted by $GH_\infty^{p} = \{\Phi \in RH_\infty^{p\times p} : \Phi^{-1} \in RH_\infty^{p\times p}\}$. If $\Phi \in GH_\infty^{p}$, it is said to be unimodular in $RH_\infty^{p\times p}$. Moreover, let $J_{mn} \in R^{(m+n)\times(m+n)}$ be a signature matrix defined as $J_{mn} = I_m \oplus (-I_n)$.

3.1. The standard problem

Consider a linear finite-dimensional discrete-time generalised plant

$$P : \begin{bmatrix} w \\ \\ u \end{bmatrix} \to \begin{bmatrix} z \\ \\ y \end{bmatrix} \tag{28}$$

with four vector-valued input/output signals: w is the exogenous input of dimension r, u of dimension p is the controlling input (manipulated variable), z of dimension m is the controlled output (objective) and y is the measured output of dimension q. The plant can be described by its properly dimensioned *scattering matrix* (Kimura, 1995, 1997)

$$P(\zeta) = \begin{bmatrix} P_{zw}(\zeta) & P_{zu}(\zeta) \\ P_{yw}(\zeta) & P_{yu}(\zeta) \end{bmatrix}. \tag{29}$$

A closed-loop system $LF(P,K) : w \to z$ given in Fig. 1 can be described by a *linear fractional transformation* of a filter (controller) $K : y \to u$ with respect to the plant P (Kimura, 1995, 1997),

$$LF(P,K) = P_{zw} + P_{zu}K(I_n - P_{yu}K)^{-1}P_{yw}. \tag{30}$$

The standard H_∞ optimisation problem is to find a causal linear K which

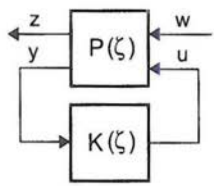


Figure 1. System configuration with generalised plant

internally stabilises the closed-loop system $LF(P,K)$ and enforces the norm bound $\| LF(P,K) \|_\infty < \gamma$ for a prespecified $\gamma > 0$(Francis, 1987; Doyle et al., 1989; Green and Limebeer, 1995; Zhou et al., 1996; Kimura, 1997). Let

$$P(\zeta) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & C_{yw} & C_{yu} \end{array}\right], \quad A \in R^{n\times n} \tag{31}$$

denote a generalised plant. Consider the common conditions for the plant regularity (Stoorvogel, 1992):

(C1) (A, B_u, C_y) is stabilisable and detectable,

(C2) D_{zu} is injective ($D_{zu}^T D_{zu} > 0$) and D_{yw} surjective ($D_{yw} D_{yw}^T > 0$),

(C3) rank $\begin{bmatrix} \bar{A}(\omega) & B_u \\ C_z & D_{zu} \end{bmatrix} = n + p, \quad \forall\omega \geq 0,$

(C4) rank $\begin{bmatrix} \bar{A}(\omega) & B_w \\ C_y & D_{yw} \end{bmatrix} = n + q, \quad \forall\omega \geq 0,$
where $\bar{A}(\omega) = A - \Delta^{-1}(e^{j\omega\Delta} - 1)I_n,$

(C5) $D_{yu} = 0$.

In the case of the dual J-lossless factorisation approach it is assumed that all the above conditions (C1-C5) are satisfied while in the approach based on the extended dual J-lossless factorisation the fourth condition (C4) is not valid.

3.2. H_∞ synthesis with dual chain-scattering representations of the plant

The plant P of (29) with $m = p$ and an invertible $P_{zu}(\zeta)$ can be characterised via its *dual chain-scattering representation*

$$G : \begin{bmatrix} z \\ w \end{bmatrix} \to \begin{bmatrix} u \\ y \end{bmatrix} \tag{32}$$

where

$$G(\zeta) = \begin{bmatrix} G_{uz}(\zeta) & G_{uw}(\zeta) \\ G_{yz}(\zeta) & G_{yw}(\zeta) \end{bmatrix} \tag{33}$$

is called a *dual chain-scattering matrix* (Kimura, 1991, 1992a,b, 1995, 1997). Consider a closed-loop system given in Fig. 2, where $K : y \to u$ being an $m \times q$ transfer function stands for a filter. The system can be characterised as a *dual homographic transformation* $DHM(G,K) : w \to z$ denoted by

$$F_{\bar{x}} = -(T_{\bar{x}} + \Delta Q_{\bar{x}}^T \bar{X} Q_{\bar{x}})^{-1}((I_n + \Delta P_{\bar{x}}^T)\bar{X} Q_{\bar{x}})^T \tag{44}$$

$$C_{\bar{x}} = C + DF_{\bar{x}} \tag{45}$$

while $N_{x\bar{x}} \in R^{(m+q)\times(m+q)}$ *is a non-singular matrix satisfying*

$$N_{x\bar{x}}(D(T_{\bar{x}} + \Delta Q_{\bar{x}}^T \bar{X} Q_{\bar{x}})^{-1} D^T - \Delta C_{\bar{x}}(I_n - X\bar{X})^{-1} X C_{\bar{x}}^T) N_{x\bar{x}}^T = J_{mq}. \tag{46}$$

Let $G_\gamma(\zeta)$ denote the plant model scaled with γ and assume that $G_\gamma(\zeta)$ has a dual (J_{mq}, J_{mr})-lossless factorisation $G_\gamma(\zeta) = \Omega(\zeta)\Psi(\zeta)$. The set of filters $K(\zeta) \in RH_\infty^{m\times q}$, for which $\| DHM(G_\gamma, K)\|_\infty < 1$ holds, is parameterised with an arbitrary transfer matrix $\Phi(\zeta) \in BH_\infty^{m\times q}$

$$K = DHM(\Omega_{-1}, \Phi). \tag{47}$$

The representation $G_\gamma(\zeta) = \Omega(\zeta)\Psi(\zeta)$ implies that all unstable poles and zeros of the system $G_\gamma(\zeta)$ are absorbed in $\Psi(\zeta)$. Therefore, the H_∞ filter $K(\zeta)$ cancels out all the stable poles and zeros of $G_\gamma(\zeta)$ and takes care of only the unstable poles and zeros from the power point of view (Kimura, 1997).

REMARK 2 *From Lemma 1 it follows that* $H_x \in R^{n\times(m+q)}$ *and* $F_{\bar{x}} \in R^{(m+r)\times n}$ *are such that* $A + H_x C$ *and* $A + BF_{\bar{x}}$ *are stable.*

REMARK 3 *Let* X_q *and* $\bar{X}_q$ *denote solutions obtained via employing the q-domain representations of the corresponding discrete-time algebraic Riccati equations. Hence* $X = X_q/\Delta$ *and* $\bar{X} = \Delta\bar{X}_q$.

REMARK 4 *For a stable* A *the zero solution* $\bar{X} = 0_{n\times n}$ *satisfies the second Riccati equation. As a consequence, we have* $F_{\bar{x}} = 0_{(m+r)\times n}$, $C_{\bar{x}} = C$, *and* $N_{x\bar{x}} = M$. *It simplifies the realisation of* $\Omega(\zeta)$ *and its inversion required in (47)*

$$\Omega(\zeta) = \left[\begin{array}{c|c} A & H_x \\ \hline -C & I_{m+q} \end{array}\right] M_x^{-1} \tag{48}$$

$$\Omega(\zeta)^{-1} = M_x \left[\begin{array}{c|c} A + H_x C & H_x \\ \hline C & I_{m+q} \end{array}\right] \tag{49}$$

As a consequence, we obtain the following form of the dual (J_{mq}, J_{mr})*-lossless factor* $\Psi(\zeta) \in RH_\infty^{(m+q)\times(m+r)}$ *of* $G(\zeta)$ *(note that* $\Psi(\zeta)$ *is stable)*

$$\Psi(\zeta) = M_x \left[\begin{array}{c|c} A + H_x C & B + H_x D \\ \hline C & D \end{array}\right]. \tag{50}$$

Since

zeros of $\Psi(\zeta)$ are equal to those of $G(\zeta)$. In particular, all 'unstable' zeros of $G(\zeta)$ (i.e. zeros located outside $\bar{D}_\Delta$) are absorbed in $\Psi(\zeta)$. From Lemma 12 given in Appendix 3 it follows that if $\zeta_0 \in D_\Delta$ is a 'stable' zero of $G(\zeta)$, then $\zeta_0 \in \lambda(A + H_x C)$. Consequently, in such a case, (50) is only a non-minimal realisation of $\Psi(\zeta)$.

4.2. The extended dual J-lossless approach

A necessary condition for the existence of the stabilising solution X of Theorem 1 is that $G(\zeta)$ has no zeros on $\partial\bar{D}_\Delta$. Let us discuss the case in which this assumption about the generalised plant does not hold. The following definition of the so-called *extended* dual J-lossless factorisation is basically analogous to those for continuous-time (Hara et al., 1992) and q-domain discrete-time cases (Hung and Chu, 1995).

DEFINITION 3 *If $G(\zeta) \in RL_\infty^{(m+q)\times(m+r)}$ is represented as a product $G(\zeta) = \Omega(\zeta)\Psi(\zeta)$ where $\Psi(\zeta) \in RL_\infty^{(m+q)\times(m+r)}$ is dual (J_{mq}, J_{mr})-lossless and $\Omega(\zeta) \in RH_\infty^{(m+q)\times(m+q)}$ does not have any zeros outside $\bar{D}_\Delta$, then $G(\zeta)$ is said to have an extended dual (J_{mq}, J_{mr})-lossless factorisation.*

Let $G(\zeta) \in RL_\infty^{(m+q)\times(m+r)}$ have n_z invariant zeros on $\partial\bar{D}_\Delta$. An extended dual (J_{mq}, J_{mr})-lossless factorisation of $G(\zeta)$ (if it exists) can be obtained by using a technique similar to that called 'zero compensation' (Copeland and Safonov, 1992a,b, 1995). Suppose that a left zero compensator $U(\zeta)$ of a minimal realisation of dimension n_z exists, for which $U(\zeta)^{-1} \in RH_\infty^{(m+q)\times(m+q)}$ and

$$\tilde{G}(\zeta) = U(\zeta)G(\zeta) \in RL_\infty^{(m+q)\times(m+r)} \tag{52}$$

with no zeros on $\partial\bar{D}_\Delta$ has a dual (J_{mq}, J_{mr})-lossless factorisation $\tilde{G}(\zeta) = \tilde{\Omega}(\zeta)\Psi(\zeta)$ where $\tilde{\Omega}(\zeta) = GH_\infty^{m+q}$. It follows that $G(\zeta) = U(\zeta)^{-1}\tilde{G}(\zeta) = \Omega(\zeta)\Psi(\zeta)$ with

$$\Omega(\zeta) = U(\zeta)^{-1}\tilde{\Omega}(\zeta) \in RH_\infty^{(m+q)\times(m+q)} \tag{53}$$

can stand for an extended dual (J_{mq}, J_{mr})-lossless factorisation of $G(\zeta)$. On account of the above, we can see that all poles of $U(\zeta)$ are on $\partial\bar{D}_\Delta$ and all zeros are in D_Δ. Moreover, $\Omega(\zeta)$ can be represented by a realisation of dimension of $n + n_z$. Seeking for a minimal realisation of dimension of n, we can observe that the unimodularity of $\tilde{\Omega}(\zeta)$ implies that the only way that allows for such a simplification of $\Omega(\zeta)$ is a stable pole-zero cancellation between poles of $U(\zeta)^{-1}$ and zeros of $\tilde{\Omega}(\zeta)$.

The set of all filters $K(\zeta)$ satisfying $\| DHM(G_\gamma K) \|_\infty < 1$ is given by (47), where $\Phi(\zeta) \in RH_\infty^{m\times q}$ is a parameter such that $K(\zeta) \in RH_\infty^{m\times q}$

$\Omega(\zeta)^{-1} = \tilde{\Omega}(\zeta)^{-1}U(\zeta)$ should be derived without the necessity of obtaining a left zero compensator.

Assume that $G(\zeta) \in RL_\infty^{(m+q)\times(m+r)}$ of a minimal realisation (A, B, C, D) has n_z invariant zeros on the boundary $\partial\bar{D}_\Delta$. According to (39) the following transposed system should be considered

$$G^T(\zeta) = \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array}\right] \in RL_\infty^{(m+r)\times(m+q)} \tag{54}$$

For the corresponding system matrix $S_{G^T}(\zeta)$ we can find a generalised (upper) real Schur QZ-transformation (Emami-Naeini and Van Dooren, 1982; Golub and Van Loan, 1996; Stewart, 2001) with orthogonal (unitary) matrices $Q_z \in R^{(n+m+r)\times(n+m+r)}$ and $Z_z \in R^{(n+m+q)\times(n+m+q)}$ such that

$$Q_z^T S_{G^T}(\zeta) Z_z = \begin{bmatrix} S_z - \zeta T_z & * \\ 0_{(n+m+r-n_z)\times n_z} & * \end{bmatrix} \tag{55}$$

where $S_z - \zeta T_z$ with $S_z, T_z \in R^{n_z\times n_z}$ is a regular pencil containing all the elementary divisors associated with the $\partial\bar{D}_\Delta$ zeros of $G(\zeta)$. Therefore, $\lambda(S_z T_z) = \lambda(T_z^{-1}S_z) \subset \partial\bar{D}_\Delta$. Let Q_z and Z_z be partitioned in conformity with $S_{G^T}(\zeta)$

$$\Omega_z = \begin{array}{c} \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} \begin{array}{l} n \\ m+r \end{array} \\ \quad n_z \quad n+m+r-n_z \end{array} \quad \text{and} \quad Z_z = \begin{array}{c} \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{array}{l} n \\ m+q \end{array} \\ \quad n_z \quad n+m+q-n_z \end{array} \tag{56}$$

From (55) it follows that $Q_{11} = Z_{11}T_z^{-1}$ and $Q_{21} = 0_{(m+r)\times n_z}$ which gives

$$\begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} \begin{bmatrix} Z_{11} \\ Z_{21} \end{bmatrix} = \begin{bmatrix} I_n & 0_{n\times(m+q)} \\ 0_{(m+r)\times n} & 0_{(m+r)\times(m+q)} \end{bmatrix} \begin{bmatrix} Z_{11} \\ Z_{21} \end{bmatrix} T_z^{-1}S_z. \tag{57}$$

Orthogonality of Q_z implies $Q_{11}^T Q_{11} = I_{n_z}$ and $Q_{11}^T Q_{12} = 0_{n_z\times(n+m+r-n_z)}$. Formula (57) gives

$$A^T Z_{11} + C^T Z_{21} = Z_{11}T_z^{-1}S_z \tag{58}$$

$$B^T Z_{11} + D^T Z_{21} = 0_{(m+r)\times n_z}. \tag{59}$$

Hence, considering (12) and (39) yields

$$P_x Z_{11} + Q_x Z_{21} = Q_{11}S_z \tag{60}$$

$$R_x Z_{11} + S_x Z_{21} = 0_{n\times n_z} \tag{61}$$

$$S_x^T Z_{11} + T_x Z_{21} = 0_{(m+q)\times n_z}. \tag{62}$$

This clearly shows that matrices Z_{11} and Z_{21} can be used for establishing the following basis of an invariant subspace of the extended pencil $U_x - \zeta W_x$, associated with the eigenvalues $\lambda(T_z^{-1}S_z) \subset \partial\bar{D}_\Delta$,

$$U_x \begin{bmatrix} Z_{11} \\ 0 \end{bmatrix} = W_x \begin{bmatrix} Z_{11} \\ 0 \end{bmatrix} T_z^{-1}S_z \tag{63}$$

Assumption 1 *Let there exist a basis* $[S_1^T\ S_2^T\ S_3^T]^T \in R^{(n+n+(m+q))\times(n-n_z)}$ *of a stable invariant subspace of the extended pencil* $U_x - \zeta W_x$

$$U_x \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} = W_x \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} \Sigma_{11} \tag{64}$$

where $\Sigma_{11} \in R^{(n-n_z)\times(n-n_z)}$ *and* $\lambda(\Sigma_{11}) \subset D_\Delta$*, such that*
(A1) $[S_1 \quad Z_{11}] \in R^{n\times n}$ *is non-singular,*
(A2) $X = [S_2 \quad 0_{n\times n_z}][S_1 \quad Z_{11}]^{-1} \in R^{n\times n}$ *is positive semidefinite,* $X \geq 0$*,*
(A3) a non-singular $M_x \in R^{(m+q)\times(m+q)}$ *can be find such that (41) is satisfied.*

From (12) and (64) it follows that

$$P_x S_1 + Q_x S_3 = S_1 \Sigma_{11} \tag{65}$$
$$-R_x S_1 - P_x^T S_2 - S_x S_3 = S_2 \Sigma_{11} + \Delta P_x^T S_2 \Sigma_{11} \tag{66}$$
$$S_x^T S_1 + Q_x^T S_2 + T_x S_3 = -\Delta Q_x^T S_2 \Sigma_{11} \tag{67}$$

Using (60)-(62) together with (66) and (67) we obtain

$$(T_z^{-1} S_z)^T (Z_{11}^T S_2)(I_{n-n_z} + \Delta \Sigma_{11}) + (Z_{11}^T S_2)\Sigma_{11} = 0_{n_z\times(n-n_z)} \tag{68}$$

which can be interpreted as a δ-domain Sylvester equation with respect to $Z_{11}^T S_2$. Since $\lambda(T_z^{-1}S_z) \cap \lambda(\Sigma_{11}) = \emptyset$, we conclude that this equation has the unique zero 'solution' $Z_{11}^T S_2 = 0_{n_z\times(n-n_z)}$. Note that from the assumed minimality of the realisation (A, B, C, D) it follows that the pair $(T_z^{-1}S_z, Z_{21})$ is observable. For this reason, eigenvalues of $\Sigma_{22} = T_z^{-1}S_z - T_z^{-1}\tilde{K}Z_{21} \in R^{n_z\times n_z}$ can be placed arbitrarily by a suitable tuning of an auxiliary matrix $\tilde{K} \in R^{n_z\times(m+q)}$. Let $\tilde{K}$ be chosen in a manner such that Σ_{22} is stable, $\lambda(\Sigma_{22}) \subset D_\Delta$. By defining $\Sigma_{21} = -T_z^{-1}\tilde{K}S_3 \in R^{n_z\times(n-n_z)}$ and taking into account (60)-(62) and (65)-(67) we can observe that

$$\tilde{U}_x \begin{bmatrix} S_1 & Z_{11} \\ S_2 & 0_{n\times n_z} \\ S_3 & Z_{21} \end{bmatrix} = \tilde{W}_x \begin{bmatrix} S_1 & Z_{11} \\ S_2 & 0_{n\times n_z} \\ S_3 & Z_{21} \end{bmatrix} \begin{bmatrix} \Sigma_{11} & 0_{(n-n_z)\times n_z} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \tag{69}$$

where a pair $(\tilde{U}_x, \tilde{W}_x)$ is obtained from (U_x, W_x) of (39) after replacing Q_x by $\tilde{Q}_x = Q_x - Q_{11}\tilde{K}$. It follows that $(\tilde{U}_x, \tilde{W}_x) \in \mathrm{dom}(\delta\mathrm{Ric})$ and for X given in (A2) we have $X = \delta\mathrm{Ric}(\tilde{U}_x, \tilde{W}_x)$. Note that X does not depend on $\tilde{K}$.

Lemma 5 *Let* $G(\zeta) \in RL_\infty^{(m+q)\times(m+r)}$ *has* n_z *zeros on* $\partial\bar{D}_\Delta$*. The system*

$$\tilde{G}(\zeta) = \left[\begin{array}{c|c} A & B \\ \hline \end{array}\right] \in RL_\infty^{(m+q)\times(m+r)} \tag{70}$$

in which $\tilde{K} \in R^{n_z \times (m+q)}$ *stabilises* $\Sigma_{22} = T_z^{-1} S_z - T_z^{-1} \tilde{K} Z_{21}$, *can be represented as* $\tilde{G}(\zeta) = U(\zeta) G(\zeta)$, *where*

$$U(\zeta) = \left[\begin{array}{c|c} S_z^T T_z^{-T} & Z_{21}^T \\ \hline \tilde{K}^T T_z^{-T} & I_{m+q} \end{array} \right] \tag{71}$$

is a left zero compensator and $U(\zeta)^{-1} \in RH_\infty^{(m+q)\times(m+q)}$.

Proof. Using previously defined Q_z and Z_z of (56), together with (58) and (59) we obtain

$$Z_z^T S_{\tilde{G}}(\zeta) Q_z = \begin{bmatrix} (S_z^T - Z_{21}^T \tilde{K}^T) - \zeta T_z^T & 0_{n_z \times (n+m+r-n_z)} \\ * & * \end{bmatrix}. \tag{72}$$

Since $\lambda(S_z^T - Z_{21}^T \tilde{K}^T, T_z^T) = \lambda(\Sigma_{22}) \subset D_\Delta$ we can conclude that $\tilde{G}(\zeta)$ has n_z stable zeros. A left zero compensator $U(\zeta)$ must have n_z poles on $\partial \bar{D}_\Delta$. Assuming

$$U(\zeta) = \left[\begin{array}{c|c} S_z^T T_z^{-T} & B_u \\ \hline \tilde{K}^T T_z^{-T} & I_{m+q} \end{array} \right] \tag{73}$$

where $B_u \in R^{n_z \times (m+q)}$ gives

$$U(\zeta) G(\zeta) = \left[\begin{array}{cc|c} S_z^T T_z^{-T} & B_u C & B_u D \\ 0_{n \times n_z} & A & B \\ \hline \tilde{K}^T T_z^{-T} & C & D \end{array} \right] \tag{74}$$

It suffices to show that the modes of (74) that correspond to $S_z^T T_z^{-T}$ are uncontrollable and $U(\zeta)^{-1} \in RH_\infty^{(m+q)\times(m+q)}$. From (59) we have that for $B_u = Z_{21}^T$ there is

$$\begin{bmatrix} I_{n_z} & Z_{11}^T \\ 0_{n \times n_z} & I_n \end{bmatrix} \begin{bmatrix} B_u D \\ B \end{bmatrix} = \begin{bmatrix} 0_{n_z \times (m+r)} \\ B \end{bmatrix}. \tag{75}$$

From this we can conclude that $\begin{bmatrix} I_{n_z} & Z_{11}^T \\ 0_{n \times n_z} & I_n \end{bmatrix}$ can be used as a suitable similarity matrix (see (58)),

$$\begin{bmatrix} I_{n_z} & Z_{11}^T \\ 0_{n \times n_z} & I_n \end{bmatrix} \begin{bmatrix} S_z^T T_z^{-T} & Z_{21}^T C \\ 0_{n \times n_z} & A \end{bmatrix} \begin{bmatrix} I_{n_z} & -Z_{11}^T \\ 0_{n \times n_z} & I_n \end{bmatrix} = \begin{bmatrix} S_z^T T_z^{-T} & 0_{n_z \times n} \\ 0_{n \times n_z} & A \end{bmatrix} \tag{76}$$

Hence

$$\tilde{G}(\zeta) = U(\zeta)G(\zeta) = \left[\begin{array}{c|c} A & B \\ \hline \tilde{C} & D \end{array}\right] \tag{78}$$

where $\tilde{C} = C - \tilde{K}^T Q_{11}^T$. It is obvious that $(\tilde{U}_x, \tilde{W}_x)$ corresponds to $\tilde{G}(\zeta)$. Moreover, from

$$U(\zeta)^{-1} = \left[\begin{array}{c|c} S_z^T T_z^{-T} - Z_{21}^T \tilde{K}^T T_z^{-T} & Z_{21}^T \\ \hline -\tilde{K}^T T_z^{-T} & I_{m+q} \end{array}\right] = \left[\begin{array}{c|c} \Sigma_{22}^T & Z_{21}^T \\ \hline -\tilde{K}^T T_z^{-T} & I_{m+q} \end{array}\right] \tag{79}$$

it follows that $U(\zeta)^{-1}$ is stable and $U(\zeta)$ does not introduce any zeros on $\partial\bar{D}_\Delta$. ■

Assume that Theorem 1 applied to the transfer matrix $\tilde{G}(\zeta)$ of (78) yields $\tilde{X}, \bar{X}, \tilde{M}_x$ $\tilde{H}_x$ and $\tilde{N}_{x\bar{x}}$ (note that $\bar{X}$ does not depend on $\tilde{K}$). Clearly, $\tilde{X} = X$. From (A2) it follows that $XZ_{11} = 0_{n\times n_z}$. Consequently, we have $\tilde{Q}_x^T X = Q_x^T X$ and $\tilde{Q}_x^T X \tilde{Q}_x = Q_x^T X Q_x$. Hence $\tilde{M}_x = M_x$ and $\tilde{H}_x = H_x$. Since $\tilde{C}_{\bar{x}} = \tilde{C} + DF_{\bar{x}} = C_{\bar{x}} - \tilde{K}^T Q_{11}^T$, which implies $\tilde{C}_{\bar{x}} X = C_{\bar{x}} X$. Taking into account the fact that $(I_n - X\bar{X})^{-1} X = X(I_n - \bar{X}X)^{-1}$ we conclude that $\tilde{N}_{x\bar{x}} = N_{x\bar{x}}$. It remains to prove that $\Omega(\zeta) = U(\zeta)^{-1}\tilde{\Omega}(\zeta)$ having n_z zeros on $\partial\bar{D}_\Delta$ can be derived without obtaining a left zero compensator.

LEMMA 6 *The factor $\Omega(\zeta) \in RH_\infty^{(m+q)\times(m+q)}$ of the extended dual (J_{mq}, J_{mr}) -lossless factorisation $G(\zeta) = \Omega(\zeta)\Psi(\zeta)$ of $G(\zeta) \in RL_\infty^{(m+q)\times(m+r)}$ with n_z zeros on $\partial\bar{D}_\Delta$ takes (if it exists) the form of (42).*

Proof. Theorem 1 yields

$$\tilde{\Omega}(\zeta) = \left[\begin{array}{c|c} A + BF_{\bar{x}} & (I_n - X\bar{X})^{-1} H_x \\ \hline -C_{\bar{x}} + \tilde{K}^T Q_{11}^T & I_{m+q} \end{array}\right] N_{x\bar{x}}^{-1}. \tag{80}$$

By virtue of (79), we have

$$U(\zeta)^{-1}\tilde{\Omega}(\zeta) = \left[\begin{array}{cc|c} \Sigma_{22}^T & -Z_{21}^T(C_{\bar{x}} - \tilde{K}^T Q_{11}^T) & Z_{21}^T \\ 0_{n_z\times n} & A + BF_{\bar{x}} & (I_n - X\bar{X})^{-1} H_x \\ \hline -\tilde{K}^T T_z^{-T} & -C_{\bar{x}} + \tilde{K}^T Q_{11}^T & I_{m+q} \end{array}\right] N_{x\bar{x}}^{-1}. \tag{81}$$

Using similarity transformation with matrix $\begin{bmatrix} I_{n_z} & -Z_{11}^T \\ 0_{n\times n_z} & I_n \end{bmatrix}$ and taking into

$$U(\zeta)^{-1}\tilde{\Omega}(\zeta) = \left[\begin{array}{cc|c} \Sigma_{22}^T & 0_{n_z\times n} & Z_{21}^T - Z_{11}^T(I_n - X\bar{X})^{-1}H_x \\ 0_{n\times n_z} & A + BF_{\bar{x}} & (I_n - X\bar{X})^{-1}H_x \\ \hline -\tilde{K}^T T_z^{-T} & -C_{\bar{x}} & I_{m+q} \end{array}\right] N_{x\bar{x}}^{-1}. \tag{82}$$

On the other hand, the use of Lemma 1 shows that H_x of (43) can be represented as

$$H_x = [S_1 \quad Z_{11}]^{-T}[S_3 \quad Z_{21}]^T. \tag{83}$$

This clearly forces $(I_n - X\bar{X})^{-1}H_x = [S_1 - \bar{X}S_2 \quad Z_{11}]^{-T}[S_3 \quad Z_{21}]^T$ and consequently $Z_{11}^T(I_n - X\bar{X})^{-1}H_x = Z_{21}^T$. The above implies that the stable modes of $U(\zeta)^{-1}\tilde{\Omega}(\zeta)$ associated with Σ_{22} are uncontrollable,

$$\Omega(\zeta) = U(\zeta)^{-1}\tilde{\Omega}(\zeta) = \left[\begin{array}{c|c} A + BF_{\bar{x}} & (I_n - X\bar{X})^{-1}H_x \\ \hline -C_{\bar{x}} & I_{m+q} \end{array}\right] N_{x\bar{x}}^{-1}. \tag{84}$$

which finishes the proof. ∎

Consequently, the following theorem, being the main result of this section, can be stated.

THEOREM 2 *Let (A, B, C, D) be a minimal realisation of the transfer matrix $G(\zeta) \in RL_\infty^{(m+q)\times(m+r)}$ having n_z zeros on $\partial\bar{D}_\Delta$. Let $[S_1^T \quad S_2^T \quad S_3^T] \in R^{(n+n+(m+q))\times(n-n_z)}$ denote a basis of a stable invariant $(n-n_z)$–dimensional subspace of the extended pencil $U_x - \zeta W_x$ of (39). $G(\zeta)$ has an extended dual (J_{mq}, J_{mr})-lossless factorisation $G(\zeta) = \Omega(\zeta)\Psi(\zeta)$ iff the following conditions hold:*

- *(i)* $[S_1 \quad Z_{11}] \in R^{n\times n}$ *is non-singular and* $X = [S_2 \quad 0_{n\times n_z}][S_1 \quad Z_{11}]^{-1} \geq 0$, $X \in R^{n\times n}$;
- *(ii)* $(U_{\bar{x}}, W_{\bar{x}}) \in \mathrm{dom}(\delta\mathrm{Ric})$ *and* $\bar{X} = \delta\mathrm{Ric}(U_{\bar{x}}, W_{\bar{x}}) \geq 0$ *for* $(U_{\bar{x}}, W_{\bar{x}})$ *defined by (40);*
- *(iii)* $\| X\bar{X} \|_s < 1$;
- *(iv) there exists a non-singular* $M_x \in R^{(m+q)\times(m+q)}$ *satisfying (41).*

REMARK 5 *By virtue of Theorem 1, $\lambda(A + H_x\tilde{C}) = \lambda(\Sigma_{11}) \cup (\Sigma_{22}) \subset D_\Delta$. On the other hand, X of Theorem 2 is not a stabilising solution to the δ-domain Riccati equation corresponding to (U_x, W_x). Since $\Omega(\zeta)$ has n_z zeros on $\partial\bar{D}_\Delta$, the matrix $A + BF_{\bar{x}} + (I_n - X\bar{X})^{-1}X_xC_{\bar{x}}$ is not stable: $\lambda(T_z^{-1}S_z) \subset \lambda(A +$*

REMARK 6 *For a stable A we observe that $A + H_x C$ is unstable and zeros of $\Omega(\zeta)$ are equal to $\lambda(A + H_x C) = \lambda(\Sigma_{11}) \cup \lambda(T_z^{-1} S_z)$. Note, however, that $\Psi(\zeta)$ is stable since its realisation of (50) is not minimal because of the pole-zero cancellations on $\partial\bar{D}_\Delta$. Hence, if $G(\zeta)$ has no 'stable' zeros, then poles of $\Psi(\zeta)$ are equal to $\lambda(\Sigma_{11}) \subset D_\Delta$ and zeros of $\Psi(\zeta)$ are equal to those zeroes of $G(\zeta)$, which are located outside $\partial\bar{D}_\Delta$ (see Remark 4).*

5. Delta-domain H_∞-suboptimal estimation

This section demonstrates the use of the offered methodology for solving the H_∞ estimation problem. Consider a linear discrete-time model of a plant (Fig. 3) with three vector-valued input/output signals: w_1 and w_2 are the exogenous inputs (disturbances) of dimensions r_1 and r_2, respectively, and y is the measured output of dimension q (Suchomski, 2002a). Let x denote the observed

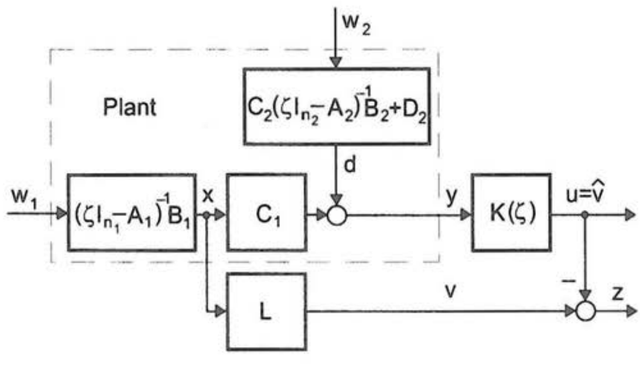


Figure 3. Formulation of the estimation problem

state vector of dimension n_1 and a reference signal $v = Lx$ of dimension m be a weighted state vector, where $L \in R^{m \times n_1}$ stands for a weighting matrix. The measurement noise channel $w_2 \to d$ is represented by the transfer matrix $C_2(\zeta I_{n_2} - A_2)^{-1} B_2 + D_2$. An approximate weighted state vector $\hat{v}$ is generated by employing the filter (estimator) described by the transfer matrix $K(\zeta)$. By defining a residue $z = v - \hat{v}$ as the controlled output (objective) we can easily obtain the corresponding generalised plant of the previously considered structure (Fig. 2) with an extended state vector of dimension $n = n_1 + n_2$ and suitably defined signals $u = \hat{v}$ and $w = [w_1^T \; w_2^T]^T \in R^r$ with $r = r_1 + r_2$. It is easily seen that we are really faced with two conflicting requirements: small $\| K(\zeta) \|_\infty$ yields good attenuation of the measurement noises but may degrade chain-scattering representation takes the form

$$G(\zeta) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|cc} A_1 & 0_{n_1\times n_2} & 0_{n\times m} & \left[\begin{array}{cc} B_1 & 0_{n_1\times r_2} \\ 0_{n_2\times r_1} & B_2 \end{array}\right] \\ 0_{n_2\times n_1} & A_2 & & \\ \hline L & 0_{m\times n_2} & -\gamma I_m & 0_{m\times r} \\ C_1 & C_2 & 0_{q\times m} & \left[\begin{array}{cc} 0_{q\times r_1} & D_2 \end{array}\right] \end{array}\right]. \tag{85}$$

The standard problem of the suboptimal H_∞ estimation is to find a causal and stable $K(\zeta)$ that enforces the H_∞ norm bound $\| T_{zw} \|_\infty < \gamma$, where $T_{zw} : w \to z$ denotes the $m \times r$ transfer matrix and $\gamma > 0$ is a prespecified parameter (Doyle et al., 1989; Zhou et al., 1996). This problem can thus be reformulated as (Suchomski, 2002a)

$$\text{find}_{K(\zeta)\in RH_\infty^{m\times q}} \; \| DHM(G_\gamma, K) \|_\infty < 1. \tag{86}$$

It follows from (34) that $DHM(G_\gamma, K)$ is an affine function of $K(\zeta)$

$$\begin{aligned} DHM(G_\gamma, K) = {} & \left[\begin{array}{cc} \gamma^{-1} L(\zeta I_{n_1} - A_1)^{-1} B_1 & 0_{m\times r_2} \end{array}\right] + \\ & -\gamma^{-1} K(\zeta) \left[\begin{array}{cc} C_1(\zeta I_{n_1} - A_1)^{-1} B_1 & C_2(\zeta I_{n_2} - A_2)^{-1} B_2 + D_2 \end{array}\right] \end{aligned} \tag{87}$$

so that we have the standard model matching problem in H_∞ (Francis, 1987; Hung, 1989; Liu and Mita, 1989; Doyle et al., 1992; Green and Limebeer, 1995; Dullerud and Paganini, 2000).

REMARK 7 *From (85) it follows that condition (C1) holds iff both A_1 and A_2 are stable. So, in order to satisfy (C2) it should be assumed that* $\operatorname{rank}(D_2) = q$. *Hence, for an asymptotically stable A only conditions (i) and (iv) of Theorem 1 or Theorem 2 are to be satisfied so as to establish the existence of a suitable estimators. Note that, in general, the system considered is not stabilisable, because the output of the estimator does not affect the signal generator (Zhou et al., 1996). Such a more general case, in which there is no requirement for the estimator to be internally stabilising and 'we do not care what happens to the state x, and indeed can do nothing about it, but our aim is to ensure that our estimate of Lx is a good one' is discussed by Green and Limebeer (1995).*

REMARK 8 *Assume that M_x is partitioned as*

$$M_x = \begin{array}{cc} \left[\begin{array}{cc} M_{11} & M_{12} \\ M_{21} & M_{22} \end{array}\right] & \begin{array}{c} m \\ q \end{array} \\ \begin{array}{cc} m \quad & \quad q \end{array} & \end{array} \tag{88}$$

Considering (41) as a simple Riccati-type equation we can seek a solution M_x

sufficiently large γ we can try to find a different M_x of a block lower triangular form. An easy algebra shows that in such a case, we have

$$M_{11} = L_{11}^{-T},\ M_{12} = 0_{m\times q},\ M_{22} = L_{22}^{-T},\ M_{21} = -M_{22}E_{12}^{T}M_{11}^{T}M_{11} \tag{89}$$

where

$$E = \begin{bmatrix} \gamma^2 I_m - \Delta\bar{L}X\bar{L}^T & -\Delta\bar{L}X\bar{C}^T \\ -\Delta\bar{C}X\bar{L}^T & -D_2D_2^T - \Delta\bar{C}X\bar{C}^T \end{bmatrix} = \begin{array}{c} \begin{bmatrix} E_{11} & E_{12} \\ E_{12}^T & E_{22} \end{bmatrix}\begin{matrix} m \\ q \end{matrix} \\ \begin{matrix} m & \quad q \end{matrix} \end{array} \tag{90}$$

$$\bar{L} = [L \quad 0_{m\times n_2}] \in R^{m\times n}, \quad \bar{C} = [C_1 \quad C_1] \in R^{q\times n} \tag{91}$$

and $L_{11} \in R^{m\times m}$ is the Cholesky factor of E_{11}, $L_{22} \in R^{q\times q}$ is the Cholesky factor of $-E_{11}^s$, while $E_{11}^s = E_{22} - E_{12}^T E_{11}^{-1} E_{12}$ denotes the Schur complement of E_{11}. Hence, M_x of the assumed structure exists iff $E_{11} > 0$ and $E_{11}^s < 0$.

We will also study the unweighted (straight) modelling of the measurement disturbances d acting directly on the output (Fig. 3). This type of modelling can be interpreted as a consequence of the lack of any prior knowledge about the nature of measurement noises. The suitable scaled dual chain-scattering representation of dimension of $n = n_1$ has the following form, with $q = r_2$ (Suchomski, 2002a),

$$G_\gamma(\zeta) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|cc} A_1 & 0_{n_1\times m} & \begin{bmatrix} B_1 & 0_{n_1\times r_2} \end{bmatrix} \\ \hline L & -\gamma I_m & 0_{m\times r} \\ C_1 & 0_{q\times m} & \begin{bmatrix} 0_{q\times r_1} & I_q \end{bmatrix} \end{array}\right]. \tag{92}$$

5.1. The dual J-lossless approach

To start with, consider a stable regular system $G_\gamma(\zeta)$ with no zeros on $\partial\bar{D}_\Delta$. Assume that conditions (i) and (iv) specified in Theorem 1 are valid. Let $\Omega(\zeta)^{-1} \in GH_\infty^{m+q}$ of (49) be partitioned as

$$\Omega(\zeta)^{-1} = \begin{array}{c} \begin{bmatrix} \overline{\Omega}_{11}(\zeta) & \overline{\Omega}_{12}(\zeta) \\ \overline{\Omega}_{21}(\zeta) & \overline{\Omega}_{22}(\zeta) \end{bmatrix}\begin{matrix} m \\ q \end{matrix} \\ \begin{matrix} m & \qquad q \end{matrix} \end{array} \tag{93}$$

For M_x of (88), zeroing the termination in (47), i.e. setting $\Phi = 0_{m\times q}$, gives the so-called central solution $K(\zeta) = -\bar{\Omega}_{11}(\zeta)^{-1}\bar{\Omega}_{12}(\zeta)$ of the following general form (see Appendix 1)

$$K(\zeta) = \left[\begin{array}{c|c} A + H_q\bar{C} - H_m M_{11}^{-1}M_{12}\bar{C} & H_m M_{11}^{-1}M_{12} - H_q \\ \hline \end{array}\right] \tag{94}$$

where $H_m \in R^{n\times m}$ and $H_q \in R^{n\times q}$ are suitable submatrices of H_x

$$H_x = \begin{array}{c} \begin{bmatrix} H_m & H_q \end{bmatrix} \\ \begin{matrix} \;\; m & \;\; q \end{matrix} \end{array} \tag{95}$$

LEMMA 7 *If a block lower triangular M_x of (88) exists ($M_{12} = 0_{m\times q}$), then the central estimator has the simplest strictly proper form*

$$K(\zeta) = \left[\begin{array}{c|c} A + H_q\bar{C} & -H_q \\ \hline \bar{L} & 0_{m\times q} \end{array}\right] \tag{96}$$

The structure of this estimator is illustrated in Fig. 4.

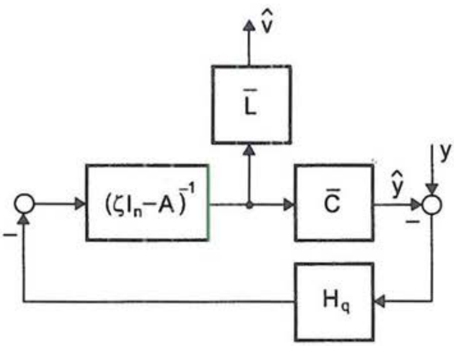


Figure 4. Central estimator

5.2. The extended dual J-lossless approach

It is assumed that for a stable $G_\gamma(\zeta)$ with n_z invariant zeros on $\partial\bar{D}_\Delta$ the conditions (i) and (iv) of Theorem 2 are fulfilled. Let us start from the following partition

$$Z_{21} = \begin{array}{cc} \begin{bmatrix} \overline{Z}_{21} \\ \underline{Z}_{21} \end{bmatrix} & \begin{matrix} m \\ q \end{matrix} \end{array} \tag{97}$$

By virtue of (39) and (85), we have

$$S_x = \begin{bmatrix} 0_{n\times m} & \begin{matrix} 0_{n_1\times q} \\ B_2 D_2^T \end{matrix} \end{bmatrix} \quad \text{and} \quad T_x = \begin{bmatrix} -\gamma^2 I_m & 0_{m\times q} \\ 0_{q\times m} & D_2 D_2^T \end{bmatrix}. \tag{98}$$

From this and (62) it is easily seen that $\bar{Z}_{21} = 0_{m\times n_z}$. Hence, formula (83) shows that

$$H_m = [S_1 \quad Z_{11}]^{-T} \begin{bmatrix} S_3^T \begin{bmatrix} I_m \\ 0_{q\times m} \end{bmatrix} \end{bmatrix}. \tag{99}$$

Now, consider $\Omega(\zeta)^{-1}$ of (49) expressed in terms of the following similar model

$$\Omega(\zeta)^{-1} = M_x \left[\begin{array}{c|c} [S_1\ Z_{11}]^T(A+H_xC)[S_1\ Z_{11}]^{-T} & [S_1\ Z_{11}]^T[H_m\ H_q]^T \\ \hline C[S_1\ Z_{11}]^{-T} & I_{m+q} \end{array} \right]. \quad (100)$$

From (63), (64) , (95) and (99) it follows that

$$\Omega(\zeta)^{-1} = M_x \left[\begin{array}{cc|c} \Sigma_{11} & 0_{(n-n_z)\times n_z} & S_3^T \\ 0_{n_z\times(n-n_z)} & T_z^{-1}S_z & [0_{n_z\times m}\ \underline{Z}_{21}^T] \\ \hline \multicolumn{2}{c|}{C\left[S_1\ Z_{11}\right]^{-T}} & I_{m+q} \end{array} \right] \quad (101)$$

which implies that the modes $\lambda(T_z^{-1}S_z) \subset \partial\bar{D}_\Delta$ of

$$\left[\begin{array}{c} \bar{\Omega}_{11}(\zeta) \\ \bar{\Omega}_{21}(\zeta) \end{array} \right] = M_x \left[\begin{array}{c|c} A+H_xC & H_m \\ \hline C & \left[\begin{array}{c} I_m \\ 0_{q\times m} \end{array} \right] \end{array} \right] \quad (102)$$

are non-controllable. Let $x_0 \in R^n$ denote a left eigenvector of $A+H_xC$ associated with a given $\zeta_0 \in \lambda(T_z^{-1}S_z)$. Assuming the following partition

$$[S_1 \quad Z_{11}]^{-1}x_0 = \left[\begin{array}{c} \overline{x}_0 \\ \underline{x}_0 \end{array} \right] \begin{array}{l} n-n_z \\ n_z \end{array} \quad (103)$$

and taking into account the definitional equality $x_0^T(A+H_xC-\zeta_0I_n) = 0_{1\times n}$, we conclude that from (100) and (101) $\bar{x}_0 = 0_{n-n_z}$. It follows that

$$[x_0^T \quad 0_{1\times(m+q)}] \left[\begin{array}{c|c} A+H_xC-\zeta_0I_n & H_m \\ \hline C & \left[\begin{array}{c} I_m \\ 0_{q\times m} \end{array} \right] \end{array} \right] = 0_{1\times(n+m)}\,. \quad (104)$$

Therefore, ζ_0 is also an invariant zero of (102) by Lemma 10 given in Appendix 2. Note that zeroing of $\underline{Z}_{21} \in R^{q\times n_z}$ must be excluded. Otherwise, not only $x_0^TH_m = 0_{1\times m}$ but also $x_0^TH_q = 0_{1\times q}$, and consequently $\zeta_0 \in \partial\bar{D}_\Delta$ turns out to be an invariant zero of $\Omega(\zeta)^{-1}$, which is a contradiction, since $\Omega(\zeta) \in RH_\infty^{(m+q)\times(m+q)}$. From what has been shown, another simple lemma immediately follows.

LEMMA 8 *In the case of $G_\gamma(\zeta)$ with zeros on $\partial\bar{D}_\Delta$, using of central estimators is to be excluded since $K(\zeta) = -\overline{\Omega}_{11}(\zeta)^{-1}\overline{\Omega}_{12}(\zeta)$ always has poles (a pole) on $\partial\bar{D}_\Delta$.*

Let $K(\zeta) = DHM(\Omega(\zeta)^{-1}, \Phi)$ with a non-zero static termination $\Phi \in R^{m\times q}$

$$K(\zeta) = \left[\begin{array}{c|c} A + H_x C - H_m D_1^{-1}\hat{C} & H_m D_1^{-1} D_2 - H_q \\ \hline D_1^{-1}\hat{C} & -D_1^{-1} D_2 \end{array}\right] \tag{105}$$

where

$$\hat{C} = M_{11}\overline{L} + M_{12}\overline{C} - \Phi(M_{21}\overline{L} + M_{22}\overline{C}) \in R^{m\times n} \tag{106}$$

$$D_1 = M_{11} - \Phi M_{21} \in R^{m\times m} \quad \text{and} \quad D_2 = M_{12} - \Phi M_{22} \in R^{m\times q}. \tag{107}$$

Considering an equivalent model of (105) in which $[S_1 \quad Z_{11}]^T$ is employed as a similarity matrix and taking into account (99)-(101) we conclude that $\lambda(T_z^{-1}S_z) \subset \lambda(A + H_x C - H_m D_1^{-1}\hat{C})$. Hence, making modes $\lambda(T_z^{-1}S_z)$ non-observable is a necessary condition for $K(\zeta)$ to be stable. From (99) and (105) it follows that choosing Φ in such a way that

$$\hat{C}\,[S_1 \;\; Z_{11}]^{-T} \;=\; [\underset{n-n_z}{*} \;\; \underset{n_n}{0_{m\times n_z}}] \in R^{m\times n} \tag{108}$$

ensures both the required block-diagonal structure of the corresponding state matrix of $K(\zeta)$ and the zeroing of the suitable part of the output matrix of this model. Formula (106) shows that for this to happen, the following linear equation in Φ should be solved

$$\Phi\underline{V} = \overline{V} \tag{109}$$

where $\overline{V} \in R^{m\times n_z}$ and $\underline{V} \in R^{q\times n_z}$ are defined by

$$M_x C[S_1 \quad Z_{11}]^{-T}\begin{bmatrix} 0_{(n-n_z)\times n_z} \\ I_{n_z}\end{bmatrix} = \begin{array}{c} \begin{bmatrix} \overline{V} \\ \underline{V} \end{bmatrix} \end{array}\begin{array}{l} m \\ q \end{array}. \tag{110}$$

Since only unitary bounded solutions are admissible (i.e. $\| \Phi \| < 1$ is obligatory) it is a rational choice to examine the minimum-norm solution $\Phi = \overline{V}\underline{V}^+$, where $\underline{V}^+ \in R^{n_z\times q}$.

The following lemma summarises the above development.

LEMMA 9 *Any unitary bounded solution Φ to (109), leading to a stable $K(\zeta) = DHM(\Omega(\zeta)^{-1}, \Phi) \in RH_\infty^{m\times q}$ of the minimal order $(n - n_z)$ is satisfying with respect to the problem of (86). For (109) to be solved we claim $n_z \le q$ and $\mathrm{Im}(\overline{V}^T) \subset \mathrm{Im}(\underline{V}^T)$.*

REMARK 9 *From (12) it follows that* $\det(U-\lambda W)|_{\lambda=-1/\Delta} \propto \det(I_n+\Delta P)\det(T-\Delta S^T(I_n+\Delta P)^{-1}Q)$, *hence* $\det(U_x-\lambda W_x)|_{\lambda=-1/\Delta} \propto \det(I_n+\Delta A)\det(DJ_{mr}G_\gamma(-1/\Delta)^T)$ *by (39). In consequence, for an inadequate modelling of the measure-*

that $-1/\Delta \in \lambda(U_x, W_x)$ *and* $-1/\Delta \in \lambda(A + H_x C)$. *Such a singular filtering (see Hautus and Silverman, 1983; Stoorvogel, 1992; Willems, 1981; Willems et al., 1986) can give a bounded* H_∞ *norm for the objective transfer matrix, but in general should be avoided, especially for small* Δ *s.*

REMARK 10 *To avoid complex (non-real) arithmetic manipulations the following standard procedure for cancellation of the non-observable modes of (105) can be applied: after performing the singular value decomposition of the observability matrix* $M_o = U_o \Sigma_o V_o^T \in R^{mn\times n}$ *corresponding to the pair* $(A + H_x C - H_m D_1^{-1}\hat{C}, D_1^{-1}\hat{C})$, *we use* $V_o \in R^{n\times n}$ *as a unitary similarity matrix leading to the following non-minimal realisation of (105):* $V_o^T(A+H_x C-H_m D_1^{-1}\hat{C})V_0, V_o^T (H_m D_1^{-1} D_2 - H_q), D_1^{-1}\hat{C}V_0, -D_1^{-1}D_2)$, *and finally, a minimal realisation is obtained via taking suitable upper-left* $(n - n_z)$ *submatrices.*

6. Numerical examples

Two examples of the δ-domain H_∞ estimation are given (see Fig. 3). The first example concerns a regular system having no zeros on $\partial\bar{D}_\Delta$. In the second example, the developed mechanism for dealing with such zeros is illustrated. A stable plant is described by the following continuous-time model

$$P_c(s) = \left[\begin{array}{ccc|c} -2.3 & -0.4 & -1.3 & 12 \\ -1 & -2 & -1.15 & 12 \\ -1.7 & 0.4 & -2.7 & 4 \\ \hline 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]. \tag{111}$$

Assumption of $\Delta = 0.02s$ gives the corresponding δ-domain discrete-time model $(P : w_1 \to y)$

$$P(\zeta) = \left[\begin{array}{ccc|c} -2.2230 & -0.3882 & -1.2324 & 11.6330 \\ -0.9394 & -1.9611 & -1.0850 & 11.6038 \\ -1.6211 & 0.3882 & -2.6118 & 3.7438 \\ \hline 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \tag{112}$$

Two different continuous-time models of the measurement noise channel will be considered

$$P_{dc}(s) = \left[\begin{array}{cc|cc} -0.04 & 0 & 1 & 0 \\ 0 & -0.02 & 0 & 1 \\ \hline 0.04 & 0 & 0.1 & 0 \end{array}\right. \tag{113}$$

and

$$P_{dc}(s) = \left[\begin{array}{cc|cc} -0.5 & 0 & 1 & 0 \\ 0 & -0.03 & 0 & 1 \\ \hline -0.05 & 0 & 0.1 & 0 \\ 0 & -0.018 & 0 & 0.6 \end{array}\right]$$

Note that in the second case we are faced with a problem of measurement noises of two significantly different time scales. It follows that $n_1 = 3$, $n_2 = 2$, $r_1 = 1$, $r_2 = 2$, $q = 2$, $m = 1$, and consequently $n = 5$, and $r = 3$. Moreover, an exemplary $L = [1.25 \quad -1 \quad 0]$ is taken.

6.1. First example: regular case

In this regular case, the δ-domain model of the measurement noise channel $(P_d : w_2 \to y)$ has no zeros on $\partial\bar{D}_\Delta$

$$\begin{aligned} P_d(\zeta) &= \left[\begin{array}{cc|cc} -0.0400 & 0 & 0.9996 & 0 \\ 0 & -0.0200 & 0 & 0.9998 \\ \hline 0.04 & 0 & 0.1 & 0 \\ 0 & 0.005 & 0 & -0.1 \end{array}\right] = \begin{bmatrix} P_{d1}(\zeta) & 0 \\ 0 & P_{d2}(\zeta) \end{bmatrix} = \\ &= \begin{bmatrix} \dfrac{0.1\zeta + 0.044}{\zeta + 0.0400} & 0 \\ 0 & \dfrac{-0.1\zeta + 0.003}{\zeta + 0.0200} \end{bmatrix}. \end{aligned} \tag{114}$$

For an admissible $\gamma = 1.4$ we have

$$X = \begin{bmatrix} 8.3303 & 7.9107 & 1.3424 & -2.1299 & 0.0452 \\ 7.9107 & 7.5364 & 1.3526 & -1.8750 & 0.0289 \\ 1.3424 & 1.3526 & 0.4697 & 0.1106 & -0.0249 \\ -2.1299 & -1.8750 & 0.1106 & 1.9027 & -0.4586 \\ 0.0452 & 0.0289 & -0.0249 & -0.4586 & 25.0000 \end{bmatrix} \geq 0 \tag{115}$$

$$H_x = \begin{bmatrix} 0.9590 & -30.9915 & -65.4913 \\ 0.8810 & -27.5754 & -65.6203 \\ 0.0564 & 1.0790 & -21.2502 \\ -0.4135 & 7.4814 & -5.3961 \\ 0.0144 & 0.1917 & -0.0103 \end{bmatrix}, \quad M_x = \begin{bmatrix} 0.7171 & 0 & 0 \\ 0.0119 & 9.7308 & 0 \\ 0.0241 & 0.0730 & 7.1753 \end{bmatrix}$$

which gives the following algorithms for the H_∞ estimation:

$$K(\zeta) \ = \left[\begin{array}{rrrrr|rr} -33.2145 & 30.6033 & -66.7237 & -1.2397 & -0.3275 & 30.9915 & 65.4913 \\ -28.5148 & 25.6143 & -66.7052 & -1.1030 & -0.3281 & 27.5754 & 65.6203 \\ -0.5421 & -0.6909 & -23.8620 & 0.0432 & -0.1063 & -1.0790 & 21.2502 \\ 7.4814 & -7.4814 & -5.3961 & 0.2593 & -0.0270 & -7.4814 & 5.3961 \\ 0.1917 & -0.1917 & -0.0103 & 0.0077 & -0.0200 & -0.1917 & 0.0103 \\ \hline 1.25 & -1 & 0 & 0 & 0 & 0 & 0 \end{array}\right] =$$

$$= [K_1(\zeta) \quad K_2(\zeta)] =$$

$$= \left[\begin{array}{c} \dfrac{11.1640(\zeta+0.0202)(\zeta+0.0400)(\zeta+1.7488)(\zeta+26.4519)}{(\zeta+0.0201)(\zeta+0.3371)(\zeta^2+3.6800\zeta+3.6302)(\zeta+27.1858)} \\ \dfrac{16.24380(\zeta+0.0200)(\zeta-0.1541)(\zeta+2.9903)(\zeta+5.0070)}{(\zeta+0.0201)(\zeta+0.3371)(\zeta^2+0.3.6800\zeta+3.6302)(\zeta+27.1858)} \end{array}\right]^T . \qquad (117)$$

In the case of the unweighted modelling of measurement noises ($d = w_2$ and $n = n_1 = 3$), we obtain

$$X = \begin{bmatrix} 65.8531 & 55.4485 & -9.2417 \\ 55.4485 & 47.1946 & -6.4509 \\ -9.2417 & -6.4509 & 4.8804 \end{bmatrix} \geq 0 \qquad (118)$$

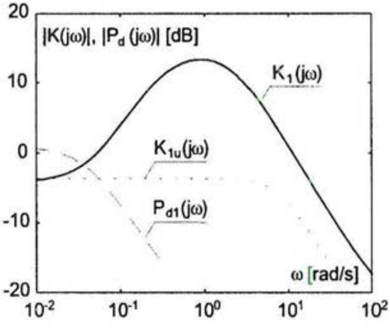
$$H_x = \begin{bmatrix} 13.7195 & -10.3125 & 8.8603 \\ 11.2356 & -8.1650 & 6.2141 \\ -2.7750 & 2.8069 & -4.5567 \end{bmatrix}, \quad M_x = \begin{bmatrix} 0.7601 & 0 & 0 \\ 0.0536 & 0.9767 & 0 \\ -0.0531 & 0.0559 & 0.9535 \end{bmatrix} \qquad (119)$$

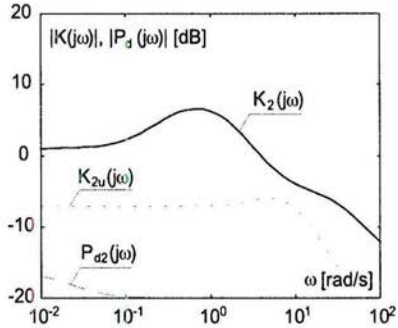
and consequently

$$K(\zeta) = \left[\begin{array}{rrr|rr} -12.5355 & 9.9243 & 7.6279 & 10.3125 & -8.8603 \\ -9.1044 & 6.2039 & 5.1292 & 8.1650 & -6.2141 \\ 1.1858 & -2.4188 & -7.1685 & -2.8069 & 4.5567 \\ \hline 1.25 & -1 & 0 & 0 & 0 \end{array}\right] =$$

$$= [K_{1u}(\zeta) \ \ K_{2u}(\zeta)] = \left[\begin{array}{c} \dfrac{4.7256(\zeta+1.8785)(\zeta+5.3933)}{(\zeta+1.8463)(\zeta^2+11.6538\zeta+39.8184)} \\ \dfrac{4.8612(\zeta+2.0099)(\zeta+3.3222)}{(\zeta+1.8463)(\zeta^2+11.6538\zeta+39.8184)} \end{array}\right]^T . \qquad (120)$$

Frequency responses of $K(\zeta)$ and $P_d(\zeta)$ in $\zeta = (e^{j\omega\Delta} - 1)/\Delta$ are plotted in Fig. 5. As we can observe, the additional knowledge about the measurement noise that follows from the model $P_d(\zeta)$ gives a possibility for improving the estimator: if the gain of the measurement noise channel is relatively high (Fig. 5a - low frequencies) the considered estimators $K_1(\zeta)$ and $K_{1u}(\zeta)$ have similar characteristics. On the other hand, if the gains of the measurement noise channels are low (Fig. 5a - high frequencies, and Fig. 5b) the estimator $K(\zeta)$ is





a) b)

Figure 5. Frequency responses of estimators and disturbance channels

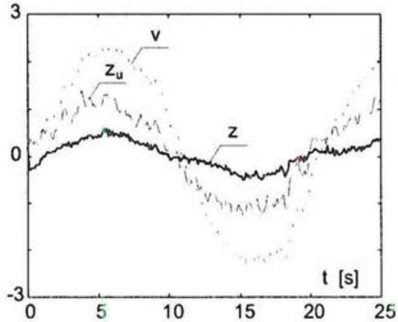


Figure 6. Time characteristics: reference and residues

exemplary reference signal $v = Lx$ as well as the corresponding residues z and z_u are represented.

The following two pairs of conditioning measures are obtained: ($\kappa_\delta = 991.5$, $\kappa_q = 4582.6$) if the exact modelling of the measurement noise channel is employed and ($\kappa_{\delta u} = 123.4$, $\kappa_{qu} = 683.0$) for the case in which no weighting is assumed. It can be observed that the δ-operator approach turns out to be far superior to the q-operator methodology while the reliability of computations is taken into account (this claim is clearly confirmed by plots given in Fig. 7, where smaller samplings periods are also examined). Computations concerning the unweighted model of the measurement noise cannel are numerically more robust mainly because of the lower dimensionality of the corresponding problem. Note, however, that this is not always the case (see the next numerical

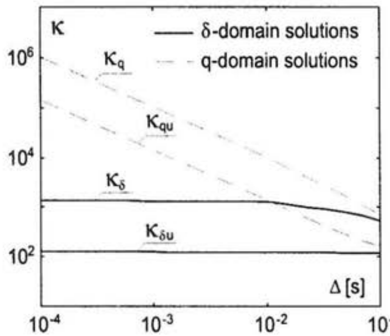


Figure 7. Comparison of Riccati equation conditioning

6.2. Second example: irregular case

The considered model of the measurement noise channel ($P_d : w_2 \to y$) has the δ-domain zero $0 = \zeta_0 \in \partial\bar{D}_\Delta$

$$P_d(\zeta) = \left[\begin{array}{cc|cc} -0.4975 & 0 & 0.9950 & 0 \\ 0 & -0.0300 & 0 & 0.9997 \\ \hline -0.05 & 0 & 0.1 & 0 \\ 0 & -0.018 & 0 & 0.6 \end{array}\right] = $$

$$= \begin{bmatrix} P_{d1}(\zeta) & 0 \\ 0 & P_{d2}(\zeta) \end{bmatrix} = \begin{bmatrix} \dfrac{0.1\zeta}{\zeta + 0.4975} & 0 \\ 0 & \dfrac{0.6\zeta}{\zeta + 0.0300} \end{bmatrix} \quad (121)$$

The numerically reliable technique for computing zeros is based on the Kronecker canonical form of the system matrix and on recent methods for computing it (Van Dooren, 1979; Van Dooren, 1981b; Emami-Naeini and Van Dooren, 1982; Boley, 1987; Varga, 1996). Since the considered generalised plant $G_\gamma(\zeta)$ of (85) has only one $\partial\bar{D}_\Delta$ zero located at the origin, extraction of the entire Kronecker structure of the pencil $S_{G_\gamma^T}(\zeta)$ is not necessary and the following simple algorithm can be applied to obtaining Q_z, Z_z and T_z (it is obvious that $S_z = 0$).

The algorithm contains three main steps:

1) Perform the singular value decomposition (Demmel, 1997; Golub and Van Loan, 1996)

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = U\Sigma V^T \tag{122}$$

where $U \in R^{(n+m+q)\times(n+m+q)}$ and $V \in R^{(n+m+r)\times(n+m+r)}$ are unitary matrices, while $\Sigma \in R^{(n+m+q)\times(n+m+r)}$ is a diagonal matrix with non-negative diagonal elements in increasing order.

2) Compute a Householder matrix $H \in R^{(n+m+r)\times(n+m+r)}$ (Demmel, 1997; Golub and Van Loan, 1996; Meyer, 2000) associated with the first column of $V^T\bar{I}_nU \in R^{(n+m+r)\times(n+m+q)}$, where

$$\bar{I}_n = \begin{bmatrix} I_n & 0_{n\times(m+q)} \\ 0_{(m+r)\times n} & 0_{(m+r)\times(m+q)} \end{bmatrix}. \tag{123}$$

3) Compute

$$Q_z = VH^T, \quad Z_z = U, \quad \begin{bmatrix} T_z & * \\ * & * \end{bmatrix} = HV^T\bar{I}_nU. \tag{124}$$

Consequently, taking an admissible $\gamma = 1.4$, we obtain $T_z = -0.5660$ and

$$X = \begin{bmatrix} 16.7650 & 16.2171 & 3.3359 & -0.2735 & -0.9115 \\ 16.2171 & 15.7124 & 3.3161 & -0.2230 & -0.8507 \\ 3.3359 & 3.3161 & 0.9875 & 0.0580 & -0.0596 \\ -0.2735 & -0.2230 & 0.0580 & 0.9981 & -0.1254 \\ -0.9115 & -0.8507 & -0.0596 & -0.1254 & 0.1308 \end{bmatrix} \geq 0 \tag{125}$$

$$H_x = \begin{bmatrix} 2.1457 & -48.7690 & -8.3531 \\ 2.0517 & -44.4264 & -8.2563 \\ 0.3370 & -0.2035 & -2.2640 \\ -0.0623 & 0.0213 & -0.1542 \\ -0.1521 & 5.0423 & -1.4142 \end{bmatrix}, \quad M_x = \begin{bmatrix} 0.7193 & 0 & 0 \\ 0.0183 & 9.5247 & 0 \\ 0.0143 & -0.0575 & 1.6224 \end{bmatrix} \tag{126}$$

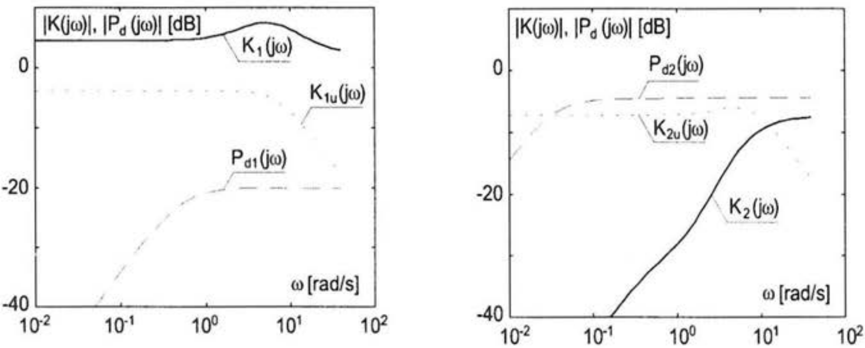
Accounting for (109) gives the following simple linear equation $[\,0.0473\ \ \text{-}0.0780\,]\Phi^T$. The corresponding minimum-norm solution $\Phi = [\,0.1062\ \ \text{-}0.1752\,]$ turns out to be feasible and leads to the H_∞ estimator

$$K(\zeta) = \left[\begin{array}{ccccc|c} -6.9157 & -0.0286 & 0.0020 & -0.0004 & -3.5963 & -0.7224 \\ 42.6596 & -2.9973 & 0.1853 & -0.0351 & 28.3369 & 8.0911 \\ -68.3104 & -3.8323 & -3.2145 & 0.5815 & -48.1174 & -8.8409 \\ 37.7983 & 1.6674 & 1.3342 & -0.7745 & 26.4595 & 5.1847 \end{array}\right. =$$

$$= [K_1(\zeta)\ \ K_2(\zeta)] = \begin{bmatrix} \dfrac{1.4192(\zeta+0.9286\zeta+6.3894)(\zeta+15.2383)}{(\zeta+0.4901)(\zeta^2+6.8668\zeta+12.6031)(\zeta+6.5451)} \\ \dfrac{-0.3947(\zeta+0.0300)(\zeta+1.8698)(\zeta+2.5522)}{(\zeta+0.4901)(\zeta^2+6.8668\zeta+12.6031)(\zeta+6.5451)} \end{bmatrix}^T \tag{127}$$

As is the case in Section 6.1, the simplified model of the measurement noise channel with no weighting has been considered. i.e. (118)-(120).

The results of computations are shown in Figs. 8-10. Frequency responses of $K(\zeta)$ and $P_d(\zeta)$ are illustrated in Fig. 8. Time characteristics computed for



a) b)

Figure 8. Frequency responses of estimators and disturbance channels

various disturbances w are illustrated in Fig. 9: disturbances $d = [d_1\ \ d_2]^T$ are presented in Figs. 9a,c,e, while plots given in Figs. 9b,d,f represent references and the corresponding residues. Conditioning of the considered Riccati equations is illustrated in Fig. 10.

7. Conclusions

The dual and extended dual J-lossless factorisation approach to the suboptimal H_∞ estimation has been presented. The approach is based on discrete-time dual chain-scattering representations of processes being estimated. For both the regular and the irregular processes the fundamental solvability conditions have been derived and represented in terms of the δ-domain state-space setting. The structures of these conditions are much more complicated than in the continuous-time cases and can not be exposed based on the direct bilinear transformation approach. Estimators are obtained via performing two J-lossless factorisations of the corresponding rational transfer matrices. In order to per-

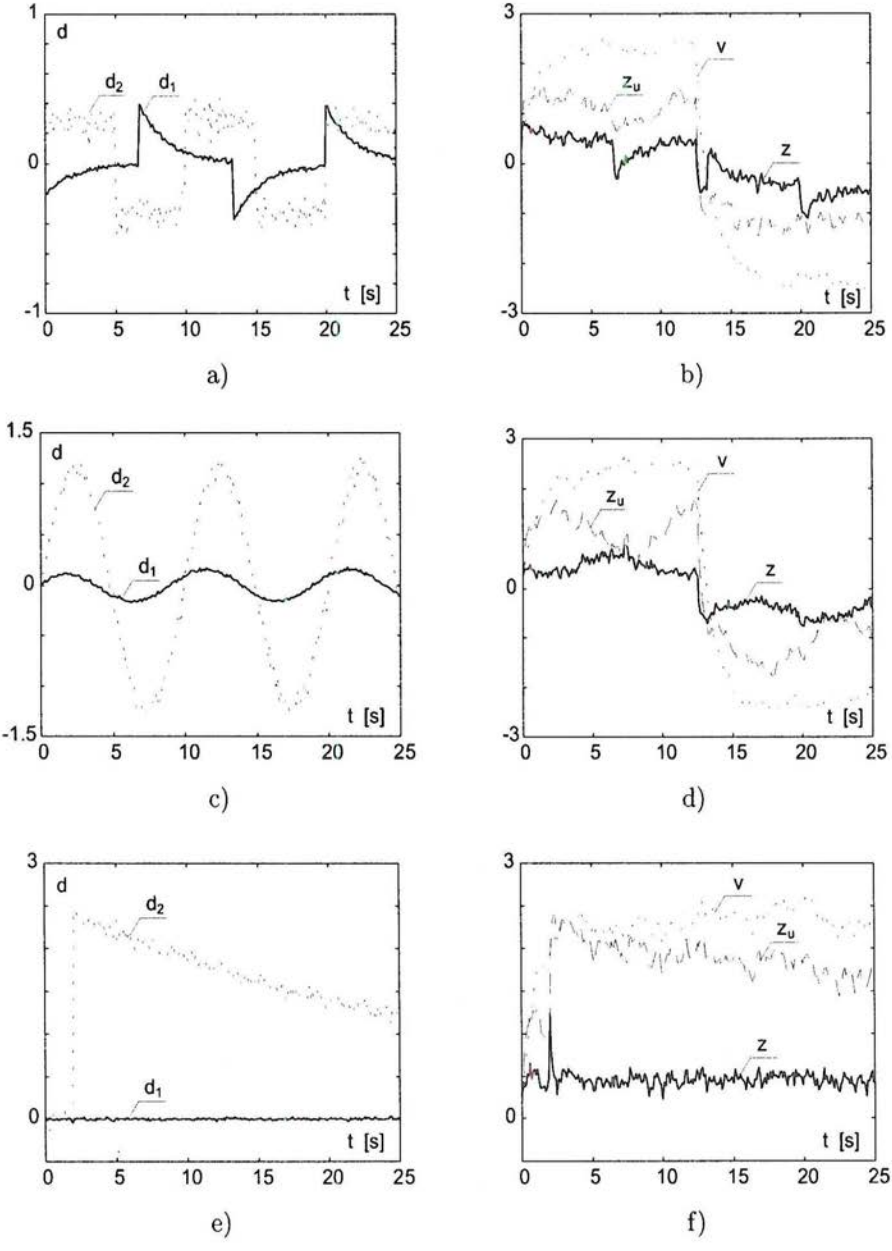


Figure 9. Time characteristics: a,c,e) measurement noise, b,d,f) references and residues

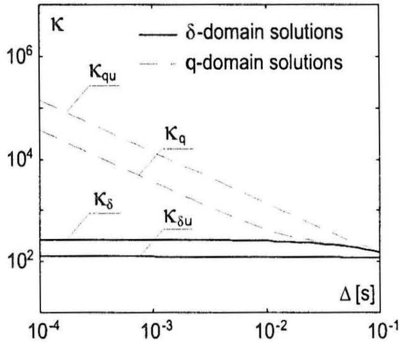


Figure 10. Comparison of the Riccati equation conditioning

Riccati equations be solved. If the estimated process is asymptotically stable only one Riccati equation is to be solved. The properly defined relative condition number has been used as a measure of the numerical sensitivity of these δ-domain Riccati equations and it has been shown that the δ-domain J-lossless factorisation approach to the H_∞ estimation is far superior to the standard q-domain approach. Especially, it has been demonstrated that the δ-domain generalised eigenproblem formulation provides a unified methodology, which facilitates the reliable numerical solution of the considered H_∞ optimisation.

Appendix 1

Consider two systems with properly dimensioned state-space models

$$G : \begin{bmatrix} z \\ w \end{bmatrix} \to \begin{bmatrix} u \\ y \end{bmatrix}, \quad G(\zeta) = \left[\begin{array}{c|cc} A & B_z & B_w \\ \hline C_u & D_{uz} & D_{uw} \\ C_y & D_{yz} & D_{yw} \end{array}\right] \tag{128}$$

$$K : y \to u, \quad K(\zeta) = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right]. \tag{129}$$

A state-space realisation of the dual homographic transformation $DHM(G,K)$ of (34), which represents the transfer function from w to z (Fig. 2), can easily be derived following the development given by Kimura (1997) for homographic transformations. The $DHM(G,K)$ takes the form

$$DHM(G,K) : w \to z, \quad DHM(G,K) = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] \tag{130}$$

$$A_c = \begin{bmatrix} A & 0 \\ B_k C_y & A_k \end{bmatrix} + \begin{bmatrix} B_z \\ B_k D_{yz} \end{bmatrix} D_1^{-1} [-\hat{C} \quad C_k] \tag{131}$$

$$B_c = \begin{bmatrix} B_w - B_z D_1^{-1} D_2 \\ B_k (D_{yw} - D_{yz} D_1^{-1} D_2) \end{bmatrix},$$

$$C_c = D_1^{-1}[-\hat{C} \quad C_k], \; D_c = -D_1^{-1} D_2 \tag{132}$$

where

$$\begin{bmatrix} \hat{C} & D_1 & D_2 \end{bmatrix} = \begin{bmatrix} I & -D_k \end{bmatrix} \begin{bmatrix} C_u & D_{uz} & D_{uw} \\ C_y & D_{yz} & D_{yw} \end{bmatrix} \tag{133}$$

subject to the condition that $D_1 = D_{uz} - D_k D_{yz}$ is invertible. This is the condition for the well-posedness of the feedback scheme of Fig. 2. In the case of a static termination $u = D_k y$ we then obtain

$$DHM(G,K) = \left[\begin{array}{c|c} A - B_z D_1^{-1} \hat{C} & B_w - B_z D_1^{-1} D_2 \\ \hline -D_1^{-1} \hat{C} & -D_1^{-1} D_2 \end{array}\right]. \tag{134}$$

Appendix 2

Consider a system $G(\zeta)$ corresponding to a realisation ($A \in R^{n\times n}$, $B \in R^{n\times r}$, $C \in$

LEMMA 10 *Suppose $S_G(\zeta)$ has full row normal rank. Then ζ_0 is a (left) invariant zero of (A, B, C, D) iff there exist a non-zero $0_n \neq x \in R^n$ and $v \in R^q$ such that $[x^T\ v^T]S_G(\zeta_0) = 0_{1\times(n+r)}$. Moreover, if $v = 0_q$, then ζ_0 is also a non-controllable mode.*

Proof. Based on the definition of invariant zeros we conclude that ζ_0 is an invariant zero if there is a non-zero vector $[x^T\ v^T]^T \in R^{n+q}$ such that $[x^T\ v^T]S_G(\zeta_0) = 0_{1\times(n+r)}$ since $S_G(\zeta)$ has full row normal rank (Douglas and Athans, 1996; Weinmann, 1991). On the other hand, assume that ζ_0 is an invariant zero, then there exists a vector $[x^T\ v^T]^T \neq 0_{n+q}$ such that $[x^T\ v^T]S_G(\zeta_0) = 0_{1\times(n+r)}$. We should assert that $x \neq 0_n$. Otherwise, $v^T[C\ D] = 0_{1\times(n+r)}$ or $v = 0_q$ since $S_G(\zeta)$ has full row normal rank. Consequently, $[x^T\ v^T]^T = 0_{n+q}$ which is a contradiction. Moreover, if $v = 0_q$ we have $x^T[A - \zeta_0 I_n\ B] = 0_{1\times(n+r)}$ which means that ζ_0 is a non-controllable mode by the Popov-Belevitch-Hautus test (Petkov et al., 1991; Zhou et al., 1996; Dullerud and Paganini, 2000). ■

Appendix 3

Two important properties of dual (J_{mq}, J_{mr})-lossless matrices can be stated.

LEMMA 11 *Any dual (J_{mq}, J_{mr})-lossless transfer matrix $G(\zeta) \in RL_\infty^{(m+q)\times(m+r)}$ can be represented as*

$$G(\zeta) = D\left[\begin{array}{c|c} A & B \\ \hline J_{mr}B^T I_A X^{-1} & I_{m+r} \end{array}\right] \tag{135}$$

where $X > 0$ and $D \in R^{(m+q)\times(m+r)}$ is a constant matrix.

Proof. Let (A, B, C, D) be a controllable realisation of $G(\zeta)$ and $X > 0$ denote a matrix satisfying (35)-(37). From (36), $C = DJ_{mr}B^T I_A X^{-1}$, which gives (135). Note that letting $\Delta \to 0$ gives a dual (J_{mq}, J_{mr})−unitary constant matrix as the first factor of the corresponding continuous-time model and a dual J_{mr}-lossless transfer matrix as the second factor of this model. ■

LEMMA 12 *If ζ_0 is an invariant zero of a dual (J_{mq}, J_{mr})-lossless matrix $G(\zeta) \in RL_\infty^{(m+q)\times(m+r)}$, then, $\zeta_0^\sim = -\zeta_0/(1 + \Delta\zeta_0)$ is a pole of $G(\zeta)$.*

Proof. Let (A, B, C, D) be a realisation of $G(\zeta)$. From Lemmas 10 and 11, it follows that if ζ_0 is a zero of $G(\zeta)$, then there exist vectors $0_n \neq x \in R^n$ and $v \in R^{m+q}$ such that $x^T(A - \zeta_0 I_n) + v^T DJ_{mr}B^T I_A X^{-1} = 0_{1\times n}$ and $x^T B + v^T D = 0_{1\times(m+r)}$. Eliminating v yields $x^T(A - BJ_{mr}B^T I_A X^{-1} - \zeta_0 I_n) = 0_{1\times n}$.

References

ARNOLD, W.F. and LAUB, A.J. (1984) Generalized eigenproblem algorithms and software for algebraic Riccati equations. *Proc. IEEE* **72** (12), 1746–1754.

BENNER, P., LAUB, A. J. and MEHRMANN, V. (1997) Benchmarks for the numerical solution of algebraic Riccati equations. *IEEE Control Systems Magazine* **17** (5), 18–28.

BOLEY, D. (1987) Computing rank-deficiency of rectangular matrix pencils. *Systems and Control Letters* **9**, 207–214.

BOULLION, T.L. and ODELL, P.L. (1971) *Generalized Inverse Matrices.* New York, Wiley-Interscience, John Wiley and Sons.

CHEN, J. and PATTON, R. J. (1999) *Robust Model-based Fault Diagnosis for Dynamic Systems.* Dordrecht, London, Kluwer Academic Press.

CHEN, S., WU, J., ISTEPANIAN, R. H., CHU, J. and WHIDBORNE, J. F. (1999) Optimising stability bounds of finite-precision controller structures for sampled-data systems in the δ-operator domain. *IEE Proc., Control Theory and Applications* **146** (6), 517–526.

COLLINS JR., E.G., HADDAD, W.M. and CHALLEBOINA, V. (1997) Robustness analysis in the delta-domain using fixed-structure multipliers. *Proc of the IEEE Conf. Decision and Control,* San Diego, CA, 3286-3291.

COLLINS JR., E. G. and SONG, T. (1999) A delta operator approach to discrete-time H_δ control. *Int. J. Control* **72** (4), 315–320.

COPELAND, R.B. and SAFONOV, M.G. (1992A) A generalized eigenproblem solution for singular H_2 and H_∞ problems. *Control and Dynamic Systems* **50**, 331–394.

COPELAND, R.B. and SAFONOV, M.G. (1992B) Zero cancelling compensation for singular control problems and their application to the inner-outer factorization problem. *Int. J. Robust and Nonlinear Control* **2** (2), 139–164.

COPELAND, R.B. and SAFONOV, M.G. (1995) A zero compensation approach to singular H_2 and H_∞ problems. *Int. J. Robust and Nonlinear Control* **5** (2), 71–106.

DEMMEL, J.W. (1997) *Applied Numerical Linear Algebra.* Philadelphia, SIAM.

DOUGLAS, J. and ATHANS, M. (1996) Multivariable poles, zeros, and pole-zero cancellations. In: W.S.Levine, ed., *The Control Handbook.* Boca Raton, CRC Press. IEEE Press.

DOYLE, J.C., FRANCIS, B.A. and TANNENBAUM, A.R. (1992) *Feedback Control Theory.* New York, Macmillan Publishing Company.

DOYLE, J. C., GLOVER, K., KHARGONEKAR, P. P. and FRANCIS, B. A. (1989) State-space solutions to standard H_2 and H_∞ control problems. *IEEE Trans. Autom. Control* **34** (8), 831–847.

DULLERUD, G. E. and PAGANINI, F. (2000) A course in Robust Control Theory. *A Convex Approach.* New York-Berlin, Springer Verlag.

EMAMI-NAEINI, A. and VAN DOOREN, P. (1982) Computation of Zeros of Linear Multivariable Systems. *Automatica* **18** (4), 415–430.

FEUER, A. and MIDDLETON, R. H. (1995) Conditioning of LMS algorithms with fast sampling. *IEEE Trans. on Automatic Control* **43** (8), 1978–1981.

FRANCIS, B. A. (1987) *A Course in H_∞ Control Theory.* Berlin, New York, Springer Verlag.

GENIN, Y., VAN DOOREN, P., KAILATH, T., DELOSME, J. and MORF, M. (1983) On Σ-lossless transfer functions and related questions. *Linear Algebra and Applics.* **50**, 251–275.

GEVERS, M. and LI, G. (1993) *Parametrizations in Control, Estimation and Filtering Problems.* London, Springer Verlag.

GOLUB, G. H. and VAN LOAN, C. F. (1996) *Matrix Computations.* Baltimore, London, The Johns Hopkins University Press.

GOODWIN, G. C., MIDDLETON, R. H. and POOF, H. V. (1992) High-speed digital signal processing and control. *Proc. IEEE* **80** (2), 240–259.

GRAHAM, A. (1981) *Kronecker products and matrix calculus with applications.* New York, John Wiley and Sons.

GREEN, M. (1992) H_∞controller synthesis by J-lossless coprime factorisation. *SIAM J. Control Optim.* **30** (3), 522–547.

GREEN, M., and LIMEBEER, D. J. N. (1995) *Linear Robust Control.* Englewood Cliffs, N.J., Prentice Hall, Inc.

HARA, S. and SUGIE, T. (1991) Inner-outer factorization for strictly proper functions with $j\omega-$axis zeros. *Syst. Control Lett.* **16**, 179-185.

HARA, S., SUGIE, T. and KONDO, R. (1992) H_∞ control problem with $j\omega$-axis zeros. *Automatica* **28** (1), 55–70.

HASSIBI, B., SAYED, A. H. and KAILTH, T. (1999) *Indefinite-Quadratic Estimation and Control. A Unified Approach to H_2 and H_∞ Theories.* Philadelphia, SIAM.

HAUTUS, M. L. J. and SIVERMAN, L. M. (1983) System structure and singular control. *Linear Algebra and Applics.* **50**, 369–402.

HENDERSON, H. V. and SEARLE, S. S. (1981) The vec-permutation matrix, the vec operator and Kronecker products: A review. *Linear and Multilinear Algebra* **9**, 271–288.

HIGHAM, N. J. (1996) *Accuracy and Stability of Numerical Algorithms.* Philadelphia, SIAM.

HOU, M. and MÜLLER, P. C. (1992) Design of observers for linear systems with unknown inputs. *IEEE Trans. Automatic Control* **37** (6), 871–875.

HUNG, Y. S. (1989) H_∞ optimal control. Part 1. Model matching. *Int. J. Control* **49** (4), 1291–1330.

HUNG, Y. S. and CHU, D. (1995) A simple and unified approach for analyzing discrete-time algebraic Riccati equation and spectral factorisation related to discrete-time H_∞−control problem. *Technical Report*, University of Hong Kong.

IONESCU, V., OARĂ, C. and WEISS, M. (1997) General matrix pencil techniques for the solution of algebraic Riccati equations: a unified approach. *IEEE Trans. Automatic Control* **42** (8), 1085–1097.

KIMURA, H. (1984) Robust stabilizability for a class of transfer functions. *IEEE Trans. Automatic Control* **29** (9), 788–793.

KIMURA, H. (1991) Generalized chain-scattering approach to H_∞ control problems. In: S.P. Bhattacharyya, L.H. Keel, eds., *Control of Uncertain Dynamic Systems.* Boca Raton, CRC Press.

KIMURA, H. (1992A) (J, J')-lossless factorisation using conjugations of zero and pole extractions. In: S. Hosoe, ed., *Robust Control.* Berlin, Heidelberg, Springer Verlag.

KIMURA, H. (1992B) (J, J')-lossless factorization based on conjugation. *Systems Control Letters* **19**, 95–109.

KIMURA, H. (1995) Chain-scattering representation, J-lossless factorization and H_∞ control. *J. Mathematical Systems Estimation and Control* **5**, 203–255.

KIMURA, H. (1997) *Chain-scattering Approach to H_∞ Control.* Boston, Basel, Birkhäuser.

KONGPRAWECHNON, W. and KIMURA, H. (1996) J-lossless conjugation and factorization for discrete-time systems. *Int. J. Control* **65** (5), 867–884.

KONGPRAWECHNON, W. and KIMURA, H. (1998) J-lossless factorization and H_∞ control for discrete-time systems. *Int. J. Control* **70** (3), 423–446.

LANCASTER, P. and RODMAN, L. (1995) *Algebraic Riccati Equations.* Oxford, Clarendon Press.

LAUB, A. J. (1991) Invariant subspace methods for the numerical solution of Riccati equations. In: S. Bittani, A.J. Laub, J.C. Willems, eds., *The Riccati Equation.* Berlin, Heidelberg, Springer Verlag.

LI, Q. and FAN, H. H. (1997) On properties of information matrices of delta operator based adaptive signal processing algorithms. *IEEE Trans. on Signal Processing* **45** (10), 2454–2467.

LIU, K. Z. and MITA, T. (1989) Conjugation and control of discrete-time systems. *Int. J. Control* **50** (4), 1435–1460.

MEYER, C. D. (2000) *Matrix Analysis and Applied Linear Algebra.* Philadelphia, SIAM.

MIDDLETON, R. H. and GOODWIN, G. C. (1986) Improved finite word length characteristics in digital control using delta operators. *IEEE Trans. Automatic Control* **31** (11), 1015–1021.

MIDDLETON, R. H. and GOODWIN, G. C. (1990) *Digital control and estimation.* Englewood Cliffs, N.J., Prentice Hall Int., Inc.

NEUMAN, C. P. (1993) Transformations between delta and forward shift operator transfer function models. *IEEE Trans. on Systems, Man, and Cybernetics* **23** (1), 295–296.

NINNESS, B. M. and GOODWIN, G. C. (1991) The relationship between discrete time and continuous time linear estimation. In: N.K. Sinha, G.P. Rao, eds., *Identification of Continuous-Time Systems.* Dordrecht, Kluwer Academic Publishers.

O'YOUNG, S. D., POSTLETHWAITE, I. and GU, D. W. (1989) A treatment of axis model-matching transformation zeros in the optimal H_2 and H_∞ control design. *IEEE Trans. Automatic Control* **34** (5), 551–553.

PREMARATNE, K., SALVI, R., HABIB, N.R. and LE GALL, J.P. (1994) Delta-operator formulated discrete-time approximations of continuous-time systems. *IEEE Trans. on Automatic Control* **39** (3), 581–585.

SAFONOV, M. G. (1987) Imaginary-axis zeros in multivariable H_∞−optimal control. In: R.F.Curtain, ed., *Modelling, robustness and sensitivity reduction in control systems.* NATO ASI Series, 34, Berlin, Heidelberg, Springer Verlag.

SCHERER, C. (1992A) H_∞−control by state-feedback for plants with zeros on the imaginary axis. *SIAM J. Control and Optimization* **30** (1), 123–142.

SCHERER, C. (1992B) H_∞−optimization without assumption on finite or infinite zeros. *SIAM J. Control and Optimization* **30** (1), 143–166.

STEWART, G. W. (1973) Error and perturbation bounds for subspaces associated with certain eigenvalue problems. *SIAM Review* **15** (4), 727–764.

STEWART, G. W. (2001) *Matrix algorithms, Vol II: Eigensystems.* Philadelphia, SIAM.

STOORVOGEL, A. (1992) *The H_∞ control problem. A state space approach.* New York, London, Prentice Hall Int., Inc.

SUCHOMSKI, P. (2001A) Robust PI and PID controller design in delta domain. *IEE Proc. Control Theory and Applic.* **148** (5), 350–354.

SUCHOMSKI, P. (2001B) A lossless factorisation approach to H_∞ control in delta domain. *Proc. European Control Conf., ECC'01,* Porto, Portugal, FR-IS01-18, 3422-3427.

SUCHOMSKI, P. (2001C) Numerical conditioning of delta-domain Lyapunov and Riccati equations. *IEE Proc. Control Theory and Applic.* **148** (6), SUCHOMSKI, P. (2002A) A dual J-lossless factorisations for suboptimal H_∞ estimation in delta domain. *Proc. of the 15th Triennial World Congress of the International Federation of Automatic Control,* Barcelona, Spain, T-Fr-M05-4 (CD ROM).

SUCHOMSKI, P. (2002B) A J-lossless coprime factorisation approach to H_∞ control in delta domain. *Automatica* **38** (10), 1807–1814.

TSAI, M. C. and POSTLETHWAITE, I. (1991) On J-lossless co-prime factorizations and H_∞ control. *Int. J. Robust and Nonlin. Control* **1** (1), 47–68.

TSAI, M. C. and TSAI, C. S. (1992) Formulations of the H_∞ control problem by using chain-scattering matrix description. *Proc. of the American Control Conf.,* Chicago, 1870-1871.

TSAI, M. C. and TSAI, C. S. (1993) A chain-scattering matrix description approach to H_∞ control. *IEEE Trans. Autom. Control* **38** (9), 1416–1421.

TSAI, M. C. and TSAI, C. S. (1995) A transfer matrix framework approach to the synthesis of H_∞ controllers. *Int. J. Robust and Nonlin. Control* **5** (2), 155–173.

TSAI, M. C, TSAI, C. S. and SUN, Y. Y. (1993) On discrete-time H_∞ control: A lossless coprime factorization approach. *IEEE Trans. Autom. Control* **38** (7), 1143–1147.

VAN DOOREN, P. (1979) The computation of Kronecker's canonical form of a singular pencil. *Linear Algebra and Applics.* **27**, 103–141.

VAN DOOREN, P. (1981A) A generalized eigenvalue approach for solving Riccati equations. *SIAM J. Sci. Stat. Comput.* **2** (2), 121–135.

VAN DOOREN, P. (1981B) The generalized eigenstructure problem in linear system theory. *IEEE Trans. Autom. Control* **26** (1), 111–129.

VARGA, A. (1996) Computation of Kronecker-like forms of a system pencil: applications, algorithms and software. *Proc. of the IEEE Int. Symp. on Computer Aided Control System Design, CACSD'96,* Dearborn, MI, 77-82.

WEINMANN, A. (1991) *Uncertain Models and Robust Control.* Wien, New York, Springer Verlag.

WILLEMS, J. S. (1981) Almost invariant subspaces: an approach to high gain feedback deign - Part I: almost controlled invariant subspaces. *IEEE Trans. Autom. Control* **26** (1), 235–252.

WILLEMS, J.S., KITAPÇI, A. and SILVERMAN, L.M. (1986) Singular optimal control: a geometric approach. *SIAM J. Control Optim.* **24** (2), 323–337.

ZHOU, K., DOYLE, J. C. and GLOVER, K. (1996) *Robust and Optimal Control.* Upper Saddle River, N.J., Prentice Hall, Inc.