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Mean-variance optimal local reinsurance contracts

by

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Abstract: Reinsurance reduces the risk but it also reduces the potential profit. The aim of the paper is to derive optimal local reinsurance contracts balancing the risk measured by variance and expected profits under various mean-variance premium principles of the reinsurer. The reinsurer's premium is calculated per claim. It is found that the optimal rules are combinations of excess of loss and quota share contracts.

Keywords: reinsurance, optimal reinsurance contracts, quota share, excess of loss.

1. Introduction

Let X be a nonnegative random variable, called risk, defined on a given probability space $(\Omega, S, \Pr)$. Reinsurance is a transfer of risk from a direct insurer, the *cedent*, to a second insurance carrier, the *reinsurer*. The reinsurer's share of the total claim amount is given by $R(X)$, where $R : [0, \infty) \to [0, \infty)$ is the *compensation function*. The best known examples of reinsurance are stop loss and quota share. In stop loss reinsurance, $R(X) = (X - b)_+$, with b being a parameter. (From now on, a_+ means $\{a, 0\}$.) In quota share reinsurance, $R(X) = aX$, where $a \in (0, 1)$ is a parameter. Both stop loss and quota share rules are optimal, i.e. stop loss (resp. quota share) minimizes the variance of cedent's payment under the constraint $P_{Re} = (1 + \beta)ER$ (resp. $P_{Re} = \mathrm{Var}(R)$) with a safety loading parameter $\beta > 0$ (see, e.g., Gerber, 1979; Pesonen, 1984; Daykin et al., 1994; and Bühlmann, 1996). Here and subsequently, P_{Re} stands for the premium of reinsurer, $\mathrm{Var}(X)$ denotes the variance of random variable X, and $D(X)$ denotes the standard deviation of X. Throughout the paper we

The results cited above suggest that the following rule

$$R(X) = a(X - b)_+ , \tag{1}$$

with $0 < a \leq 1,\ b > 0$, may also be optimal in a suitable setting. In fact, by employing a straightforward method Kałuszka (2001) showed that the rule (1) is a solution of the following problem

$$\begin{aligned} &\text{minimize } \operatorname{Var}(X - R(X)) \\ &\text{subject to } \ E[R(X)] = f(P_{Re}, D[R(X)]), \ \ 0 \leq R(X) \leq X, \end{aligned} \tag{2}$$

with f being a function defining the reinsurer's premium principle. Examples include the following premium principles:

$$\text{expected value principle } P_{Re} = (1 + \beta)E(R), \tag{3}$$

$$\text{standard deviation principle } P_{Re} = E(R) + \beta D(R), \tag{4}$$

$$\text{variance principle } P_{Re} = E(R) + \beta \operatorname{Var}(R), \tag{5}$$

$$\text{modified variance principle } P_{Re} = E(R) + \beta \operatorname{Var}(R)/E(R), \tag{6}$$

$$\text{mixed principle } P_{Re} = E(R) + \alpha D(R) + \beta \operatorname{Var}(R), \tag{7}$$

$$\text{mean value principle } P_{Re} = [\,E(R^2)\,]^{\frac{1}{2}} = [\,(ER)^2 + \operatorname{Var}(R)\,]^{\frac{1}{2}}, \tag{8}$$

$$\text{quadratic utility principle } P_{Re} = E(R) + c - (c^2 - \operatorname{Var}(R))^{\frac{1}{2}}, \tag{9}$$

where $R = R(X)$ and $\alpha, \beta, c > 0$. For an overview of the premium calculation principles, we refer the reader to Gerber (1979), Goovaerts et al. (1984), Straub (1988), Embrechts et al. (1997), and Rolski et al. (1998). The result of Kałuszka (2001) is a generalization of the work of Gajek and Zagrodny (2000), which concerns optimality under standard deviation principle. By applying the involved convex programming method of Gajek and Zagrodny, Mazur (2000) found an optimal reinsurance rule under the variance principle. Kałuszka (2001) also showed that a combination of excess of loss and quota share contracts is optimal for local reinsurance with reinsurer's premium per claim. Other related optimality results can be found in Centeno (1985), Centeno and Simões (1991), Hesselager (1990), and Kałuszka(2002). We follow the most common terminology for excess of loss and stop loss rules, i.e. stop loss is an aggregate type of cover while excess of loss stands for individual claim amounts.

The rule (1) is well known as a combination of stop loss and quota share arrangements (see Centeno, 1986, and Hesselager, 1990). To the best of my knowledge, this kind of reinsurance was first considered in actuarial literature by Kahn (1961), p. 270. Unfortunately, there is no official name for it. Samson it is called the *change loss rule*. The change loss reinsurance has been used in practice (see Gerathewohl, **2**, 1980, 371; Lehrke, 1997; Schmitter, 2001).

The purchase of reinsurance is a compromise between expected gain and security. Reinsurance reduces the cedent's risk; on the other hand, it will reduce the expected gain of the cedent. The aim of this paper is to derive optimal rules of the reinsurance provided the cedent trades off between the variance and the expected value of his gains. Our approach is related to the Markowitz methodology of optimal portfolio. The paper is a continuation of the work of Kałuszka (2001), which does not take into account the regulation of the expected gain of the cedent. In Section 2 we assume that the cedent is interested in minimizing the variance of his retained risk under a fixed expected value of the gain. Section 3 contains results on minimizing some meaningful functions of the expected gain and variance like the coefficient of variation.

2. Regulation of expected gains of cedent

Let $X, X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables defined on a common probability space $(\Omega, S, \Pr)$. Let N be an integer valued random variable such that $0 \leq \operatorname{Var} N < \infty$. Assume that N is independent of $X, X_1, X_2, \ldots$. We use $X_1, X_2, \ldots$ as the sequence of successive claims occurring in a time interval. N models the number of claims over that period.

Consider local reinsurance with a common compensation function R, i.e. each claim is divided between the cedent and the reinsurer as follows: for the i-th claim of size X_i the part $R = R(X_i)$ is carried by the reinsurer. We assume that the reinsurer's premium, say P_{Re}, is defined by $g(P_{Re}, ER) = DR$ with a nonnegative function g on $\{ (x, y) \mid x \geq y,\, y \geq 0 \}$ such that $g(x, y)$ is increasing in x for each y. Observe that the principles (4)-(9) are included in this class.

Suppose that the cedent wants to have an agreement which minimizes his risk measured by the variance of his global payment, but also wants to keep control of his expected gain. Moreover, the cedent is willing to pay not more than P for reinsurance of each claim. Hence the following problem arises

$$\begin{aligned} &\text{minimize } \operatorname{Var} \sum_{i=1}^{N} [\, X_i - R(X_i) \,] \\ &\text{subject to } DR \leq g(P, ER),\ E(X - R) = m,\ 0 \leq R(X) \leq X, \qquad (10) \end{aligned}$$

where $P > 0$ and $0 < m < EX$. Here and subsequently, the symbol $X \leq Y$ means that the random variables X, Y are defined on $(\Omega, S, \Pr)$ and $\Pr\{X \leq Y\} = 1$. In order that the problem (10) have nontrivial solution we assume that $D(X) > g(P, EX)$. Observe that (10) is a constrained optimization problem on

THEOREM 2.1 *a) Suppose that there exists a nonnegative real b such that*

$$\min\left\{1, \frac{g(P, EX-m)}{D(X-b)_+}\right\} = \frac{EX-m}{E(X-b)_+}. \tag{11}$$

Then a solution of the problem (10) is given by

$$R^*(x) = \frac{EX-m}{E(X-b)_+}(x-b)_+. \tag{12}$$

b) There exists a nonnegative real b satisfying (11) if

$$(EX-m)\,DX \le g(P, EX-m)EX. \tag{13}$$

REMARK 2.1 *If equality in (13) holds, then the quota share coverage $R^*(x) = (1-\frac{m}{EX})x$ is a solution of the problem (10) since $b=0$ is a solution of (11). Moreover, if there exists a strictly positive solution, say b^*, of the following equation $EX-m = E(X-s)_+$ in $s \ge 0$, such that $D(X-b^*)_+ \le g(p, EX-m)$, then the excess of loss contract $R^*(x) = (x-b^*)_+$ solves the problem (10).*

Proof of Theorem 2.1

a) Let b be a nonnegative solution of (11). Recall that

$$E\sum_{i=1}^{N}[X_i - R(X_i)] = E(N)E(X-R), \tag{14}$$

$$\mathrm{Var}\sum_{i=1}^{N}[X_i - R(X_i)] = E(N)\mathrm{Var}(X-R) + (EX-ER)^2\mathrm{Var}(N) \tag{15}$$

(see e.g. Rolski et al., 1998, Corollary 4.2.1). Here and subsequently, $R = R(X)$. Put

$$\Re_m(g) = \{R \mid DR \le g(P, ER),\ E(X-R) = m,\ 0 \le R(X) \le X\} \tag{16}$$

Since $X - b = (X-b)_+ - (b-X)_+$, we have

$$\mathrm{Cov}(X,R) = \mathrm{Cov}(X-b,\, R) = \mathrm{Cov}((X-b)_+,\, R) - \mathrm{Cov}((b-X)_+,\, R). \tag{17}$$

From (17) we get

$$\mathrm{Cov}(X,R) \le \mathrm{Cov}((X-b)_+,\, R) + E(R)E(b-X)_+ \tag{18}$$

with equality if $R(x) = 0$ for $0 \le x \le b$. From (18) and the Cauchy-Schwarz inequality it follows that for every $R \in \Re_m(g)$

and (19) is attained if $R(x) = c(x-b)_+$ for $x \geq 0$ with a real c. Combining (15) with (19) yields

$$\operatorname{Var}\sum_{i=1}^{N}[X_i - R(X_i)] \geq [\operatorname{Var} X - 2D(R)D(X-b)_+ + \operatorname{Var} R + \\ -2(EX-m)E(b-X)_+]EN + m^2 \operatorname{Var} N \tag{20}$$

for every $R \in \Re_m(g)$. Taking $t = D(R)/D(X-b)_+$ we get

$$\operatorname{Var}\sum_{i=1}^{N}[X_i - R(X_i)] \geq [\operatorname{Var} X - \operatorname{Var}(X-b)_+ + (t-1)^2 \operatorname{Var}(X-b)_+ + \\ -2(EX-m)E(b-X)_+]EN + m^2 \operatorname{Var} N \tag{21}$$

for every $R \in \Re_m(g)$, and equality in (21) holds if $R(x) = t(x-b)_+$ for $x \geq 0$ with a real t such that $R \in \Re_m(g)$. Since

$$0 \leq t = \frac{D(R)}{D(X-b)_+} \leq \frac{g(P,\, ER)}{D(X-b)_+} = \frac{g(P,\, EX-m)}{D(X-b)_+} \tag{22}$$

the minimum of the left-hand side of (21) under constraint (22) is attained at

$$t_0 = \min\left\{1, \frac{g(P,\, EX-m)}{D(X-b)_+}\right\}. \tag{23}$$

By (11), $t_0 = (EX-m)/E(X-b)_+ \leq 1$. Clearly, $0 \leq R^*(X) \leq X$ and $ER^* = t_0 E(X-b)_+ = EX - m$ so $E(X - R^*) = m$. By (23) $D(R^*) = t_0 D(X-b)_+ \leq g(P,\, EX-m) = g(P,\, E(R^*))$. Hence, $R^* \in \Re_m(g)$ and from (21) it follows that for every $R \in \Re_m(g)$ $\operatorname{Var}\sum_{i=1}^{N}[X_i - R(X_i)] \geq \operatorname{Var}\sum_{i=1}^{N}[X_i - R^*(X_i)]$, which completes the proof of part (a) of Theorem 2.1.

b) Define

$$u(s) = \frac{EX-m}{E(X-s)_+},\ 0 \leq s < \sup X. \tag{24}$$

From now on, $\sup X = \sup\{b : \Pr\{X > b\} > 0\}$. Denote by $d^+/ds^+ f(s)$ the right-hand derivative of f at s and observe that

$$\frac{d^+}{ds^+}E(X-s)_+ = \frac{d^+}{ds^+}\int_s^\infty \Pr\{X > t\}dt = -\Pr\{X > s\} \leq 0$$

which implies that $u(\cdot)$ is a continuous increasing function on $[\,0,\, \sup X\,)$. Moreover, $u(s) \to \infty$ as $s \to \sup X$, and $0 < u(0) < 1$ because of $0 < m < EX$. Define

$$\tau(s) = \min\left\{1, \frac{g(P,\, EX-m)}{\ldots}\right\} \quad \text{for } 0 \leq s < \sup X \tag{25}$$

Since

$$\mathrm{Var}(X-s)_+ = 2\int_s^\infty\int_t^\infty \Pr\{X\geq u\}dudt - \left(\int_s^\infty \Pr\{X\geq u\}du\right)^2,$$

$\tau(\cdot)$ is a continuous function on $[0, \sup X)$. By (13), $u(0) \leq \tau(0)$, so there exists a nonnegative real b such that $u(b) = \tau(b)$, completing the proof of Theorem 2.1. ∎

EXAMPLE 2.1 Suppose X is a Bernoulli claim with parameters M, p, i.e. X takes the value M with probability p and the value of 0 with probability $q = 1-p$, where M is a positive real. Assume the reinsurer uses standard deviation principle, i.e. $ER + \beta DR = P_{Re} \leq P$, where $\beta > 0$ is a safety loading coefficient. Recall that P is an amount of money which the cedent wants to spend on the reinsurance per claim. A standard algebra gives

$$E(X-b)_+ = (M-b)p,\ D(X-b)_+ = (M-b)\sqrt{pq}\ \text{ for } 0 \leq b < M.$$

From part (a) of Theorem 2.1 it follows that the excess of loss rule defined by

$$R^*(X) = (X - \frac{m}{p})_+ \tag{26}$$

is a solution of the problem (10) with $g(P, x) = (P - x)/\beta$ if and only if

$$\frac{EX - m}{EX} \leq \min\left\{1, \frac{P - EX + m}{\beta DX}\right\} \text{ and } P < EX + \beta DX. \tag{27}$$

Since $0 < m < EX$, (27) holds provided

$$P_{Re} = (Mp - m)(1 + \beta\sqrt{q/p}) \leq P < Mp(1 + \beta\sqrt{q/p}).$$

Observe that the premium of reinsurer is a linear function of m.

With a choice of $M = \$10^6$, $p = 10^{-4}$, $\beta = 2.3$, and $m = 0.4\,EX = \$40$, we have $P_{Re} = \$13859.3$ for $P \geq \$13859.3$ and the reinsurer has to pay $\$6 \cdot 10^5$ provided $X = \$10^6$.

EXAMPLE 2.2 Assume each claim has the exponential distribution with expectation μ^{-1}. It follows after a little algebra that

$$E(X-s)_+ = \mu^{-1}\exp(-\mu s),$$
$$\mathrm{Var}(X-s)_+ = \mu^{-2}[2 - \exp(-\mu s)]\exp(-\mu s).$$

Assume that the reinsurer uses the variance principle (3). By Theorem 2.1 and Remark 2.1, the quota share coverage given by $R^*(X) = (1 - m\mu)X$ is a solution

P_{Re}. Moreover, if $(\mu^{-1} - m)[1 + \beta(\mu^{-1} + m)] \le P < \mu^{-1} + \beta\mu^{-2}$ then the following excess of loss coverage $R^*(X) = (x + \mu^{-1}\ln(1 - \mu m))_+$ solves (10) and $P_{Re} = (\mu^{-1} - m)[1 + \beta(\mu^{-1} + m)]$. If $(\mu^{-1} - m)[1 + \beta(\mu^{-1} - m)] < P < (\mu^{-1} - m)[1 + \beta(\mu^{-1} + m)]$ then a solution of (10) is the change loss coverage $R^*(X) = a(X - b)_+$ with $0 < a < 1$, $b > 0$, and $P_{Re} = P$, where

$$a = \frac{EX - m}{E(X - b)_+} = \frac{(1 - m\mu)}{2}\left[\frac{P - \mu^{-1} + m}{\beta(\mu^{-1} - m)^2} + 1\right]$$

and b is the solution of the following equation

$$g(P, EX - m)E(X - b)_+ = (EX - m)D(X - b)_+,$$

namely

$$b = \frac{1}{\mu}\ln\left(\frac{1}{2}\left[\frac{P - \mu^{-1} + m}{\beta(\mu^{-1} - m)^2} + 1\right]\right).$$

EXAMPLE 2.3 Let X have the Pareto distribution with parameters $\alpha > 2$, $k > 0$, i.e. $\Pr\{X < x\} = 1 - (\frac{k}{k+x})^\alpha$, $x > 0$. A standard algebra leads to

$$E(X - b)_+ = \frac{k^\alpha}{\alpha - 1}\frac{1}{(k + b)^{\alpha-1}},$$

$$\mathrm{Var}(X - b)_+ = \frac{k^\alpha}{(\alpha - 1)^2(k + b)^{\alpha-2}}\left[\frac{2(\alpha - 1)}{\alpha - 2} - (\frac{k}{k + b})^\alpha\right]$$

Remark 2.1 implies that the excess of loss rule defined by

$$R^*(X) = \left(X - k\left(\frac{k}{k - (\alpha - 1)m}\right)^{1/(\alpha-1)} + k\right)_+$$

is an optimal arrangement provided the reinsurer uses the variance principle (5) and the following condition holds

$$\frac{2k^{\alpha/(\alpha-1)}}{(\alpha - 2)(\alpha - 1)^{1/(\alpha-1)}}\left(\frac{k}{\alpha - 1} - m\right)^{1-1/(\alpha-1)} +$$
$$+\frac{1}{\beta}\left(\frac{k}{\alpha - 1} - m\right) - \left(\frac{k}{\alpha - 1} - m\right)^2 \le \frac{P}{\beta}.$$

Let $k = \$2000$, $\alpha = 3.5$, $\beta = 1/\$100$, $m = \$770$, and let $P = \$3000 < EX + \beta \operatorname{Var} X = \$15733$. In this case $EX = \$800$ and the payment $R(X) = (X - 5437.33)_+$ is received from the reinsurer per each claim.

Following Koller and Dettwyler (1997) we now present a result for a larger class of admissible reinsurance arrangements, i.e. we assume that $0 \le ER \le EX$ instead of $0 \le R(X) \le X$. Set with $0 < m \leq EX$ (see (16)). We look for a solution of the following problem

$$\operatorname{Var} \sum_{i=1}^{N} [X_i - R(X_i)] = \min! \tag{29}$$

where the minimum is taken over all R in $\Re_m^E(g)$.

THEOREM 2.2 *Suppose the minimum of the function $t \to (t-1)^2$ under constraint $0 \leq t \leq g(P, EX - m)/D(X)$ is attained at $t = a$, where a is a real. Then the solution of (29) is given by*

$$R^{**}(x) = a(x - EX) + EX - m. \tag{30}$$

Proof. In view of (15) and the Cauchy-Schwarz inequality we get

$$\begin{aligned}\operatorname{Var} \sum_{i=1}^{N} [X - R(X_i)] &= E(N)\,[\operatorname{Var}(X) - 2\operatorname{Cov}(X, R) + \operatorname{Var}(R)] + \\ &+ (EX - ER)^2 \operatorname{Var} N \geq E(N)\,[D(X) - D(R)]^2 + \\ &+ (EX - ER)^2 \operatorname{Var} N \end{aligned} \tag{31}$$

with equality if $R(x) = \alpha\, x + \beta$ with reals α, β. Hence for every $R \in \Re_m^E(g)$

$$\operatorname{Var} \sum_{i=1}^{N} [X_i - R(X_i)] \geq (t-1)^2 E(N) \operatorname{Var}(X) + m^2 \operatorname{Var}(N), \tag{32}$$

where $t = D(R)/D(X)$. By the assumption, the minimum of the right-hand side of (32) under the constraint $0 \leq t \leq g(P, EX - m)/D(X)$ is attained at $t = a$. Observe that $E(R^{**}) = EX - m$ and $\operatorname{Var}(R^{**}) = a^2 \operatorname{Var}(X) \leq g^2(P, EX - m) = g^2(P, E(R^{**}))$. Therefore $R^{**} \in \Re_m^E(g)$. By (32) with $t = a$, we get

$$\begin{aligned}\operatorname{Var} \sum_{i=1}^{N} [X_i - R(X_i)] &\geq (a-1)^2 E(N) \operatorname{Var}(X) + m^2 \operatorname{Var}(N) = \\ = E(X) \operatorname{Var}(X - R^{**}) &+ (EX - ER^{**})^2 \operatorname{Var}(N) = \operatorname{Var} \sum_{i=1}^{N} [X_i - R^{**}(X_i)],\end{aligned}$$

for every $R \in \Re_m^E(g)$. The proof is complete. ∎

REMARK 2.2 *If $m = EX$, then the rule R^{**} is the APS reinsurance arrange-*

3. Trade off between gain and security of the cedent

In Section 3 we assume that the cedent is interested in minimization of a function which depends not only on the variance of cedent's payment but also on his expected gain. Adopt the notation of Section 2 and recall that the aggregate of the ceded claims equals $\sum_{i=1}^{N}[X_i - R(X_i)]$.

EXAMPLE 3.1 The following criterion for balancing the risk and profits of the cedent can be found in the financial literature

$$\alpha \left(E \sum_{i=1}^{N} [X_i - R(X_i)] \right)^2 + \mathrm{Var} \sum_{i=1}^{N} [X_i - R(X_i)], \tag{33}$$

where $\alpha > 0$. Given $\alpha = 1$, the cedent wants to minimize the second moment of his payment after reinsurance:

$$E \left(\sum_{i=1}^{N} X_i - \sum_{i=1}^{N} R(X_i) \right)^2 \tag{34}$$

EXAMPLE 3.2 From the actuarial point of view it may be of interest to solve the following problem

$$\begin{aligned} &\text{minimize } CV \left(\sum_{i=1}^{N} [X_i - R(X_i)] \right) \\ &\text{subject to } ER = f(P_{Re},\, DR),\ 0 \leq R \leq X, \end{aligned} \tag{35}$$

where $CV(X) = DX/EX$ is the *coefficient of variation.* The reciprocal of the coefficient of variation is known in the financial literature as the *Sharp ratio.* For instance, suppose that the cedent uses the standard deviation premium principle and wants to keep control of the *claim ratio* of a given period defined by

$$r = \frac{\sum_{i=1}^{N} [X_i - R(X_i)]}{N(E(X-R) + \beta D(X-R))}$$

with $\beta \geq 0$ (see De Vylder and Goovaerts, 1999). After a standard algebra we get

$$\mathrm{Var}(r) = \frac{\mathrm{Var}(X-R)}{[E(X-R) + \beta D(X-R)]^2} E(1/N)$$

$$[CV(X-R)]^2$$

So, if the cedent is interested in minimization of the variance of the claim ratio then the coefficient of variation should be minimized. Other problems leading to minimization of the coefficient of variation of risk can be found in Hart et al. (1996) and Schnieper (2000).

EXAMPLE 3.3 As observed in Dickson and Waters (1997) an approximate way to minimize the cedent's probability of ruin would be to maximize the following approximation of the adjustment coefficient, say R_A,

$$R_A \approx 2\frac{Expected\ profit}{Variance\ of\ profit}$$

(see also Gerber, 1979, and Straub, 1988, Chapter 5). Hence the following problem arises

$$\begin{aligned}&\text{minimize Var} \sum_{i=1}^{N}[X_i - R(X_i)] \left(P_{Ce}EN - E\sum_{i=1}^{N}[X_i - R(X_i)]\right)^{-1}\\ &\text{subject to } P_{Re} = ER + \beta\,\text{Var}R \quad 0 \leq R(X) \leq X,\end{aligned}$$

provided the reinsurer uses the principle (5). Here P_{Ce} denotes the cedent's premium.

In a general setting, we minimize the target function defined by

$$\begin{aligned}&h\left(E\sum_{i=1}^{N}[X_i - R(X_i)],\ \text{Var}\sum_{i=1}^{N}[X_i - R(X_i)]\right)\\ &\text{subject to } E(R(X)) = f(P_{Re},\, D(R(X)),\ 0 \leq R(X) \leq X, \qquad (36)\end{aligned}$$

where $f, h : [0,\infty) \times [0,\infty) \to (-\infty,\infty)$ and $h(x,y)$ is strictly increasing in y for each x. Put

$$\begin{aligned}\varphi(t,b) = h(&EN[EX - f(P_{Re}, tD(X-b)_+)],\, EN[\text{Var}\,X - \text{Var}(X-b)_+ +\\ &+(t-1)^2\,\text{Var}(X-b)_+ - 2f(P_{Re},\, tD(X-b)_+)E(b-X)_+] +\\ &+\text{Var}\,N[EX - f(P_{Re},\, tD(X-b)_+)]^2) \qquad (37)\end{aligned}$$

with $0 \leq t \leq 1$ and $b \geq 0$.

THEOREM 3.1 *a) Assume there exist reals a, b such that*

i) $0 \leq a \leq 1,\ b \geq 0,$

ii) $\varphi(a,b) = \min_{0\leq t\leq 1} \varphi(t,b),$

iii) $f(P_{Re},\, aD(X-b)_+) = aE(X-b)_+.$

Then $R^*(X) = (X-b)_+$ *is a solution of the problem (36).*

b) Suppose h is continuous on $[0,\infty)^2$. *Let* $f(P_{Re}, 0) = P_{Re}$, *let* $f(P_{Re}, D(X)) < EX$, *and let* $f(P_{Re}, t)$ *be decreasing, concave, and differentiable in t. Then there exist reals a, b such that conditions (i)-(iii) are satisfied.*

Proof.

a) Arguments like those in the proof of Theorem 2.1 show that

$$h\left(E\sum_{i=1}^{N}[X_i - R(X_i)],\ \mathrm{Var}\sum_{i=1}^{N}[X_i - R(X_i)]\right) \geq \varphi(t, b) \geq \varphi(a, b), \tag{38}$$

for each rule $R = R(X)$ such that $R \in \Re(f)$, where $\Re(f) = \{R \mid ER = \hat{f}(P_{Re}, DR),\ 0 \leq R(X) \leq X\}$. The first inequality in (38) becomes equality if $R(X) = t(X - b)_+$ with a real t such that $R \in \Re(f)$ By assumptions *(i)-(iii)*, $R^* \in \Re(f)$, which completes the proof of part (a) of Theorem 3.1.

b) It is clear that for every $0 \leq a \leq 1$ there is only one point, say $b(a)$, such that $f(P_{Re}, aD(X - b(a))_+) = aE(X - b(a))_+$ and the function $a \to b(a)$ is continuous (see Kałuszka, 2001, the proof of Theorem 1). By the Weierstrass theorem, there exists a real a such that $\varphi(a, b(a)) = \min_{0\leq t\leq 1} \varphi(t, b(t))$, as desired.

REMARK 3.1 *Let us discuss under which conditions the quota share contract is a solution of (36). Adopt the assumptions of Theorem 3.1(b). Assume additionally that h is convex and differentiable and assume $P_{Re} < EX$. Then for every $0 \leq t \leq 1$ and $b > 0$ we have*

$$f(P_{Re}, tD(X - b)_+) \leq f(P_{Re}, 0) = P_{Re} < EX.$$

In consequence, $t \to \varphi(t, b)$ is convex for each b. Furthermore, $\varphi(t, 0) = h(EN[EX - f(P_{Re}, tDX)], (t-1)^2 EN\,\mathrm{Var}\,X + \mathrm{Var}\,N[EX - f(P_{Re}, tDX)]^2)$.

Denote by a_0 a solution of the equation $f(P_{Re}, aDX) = aEX$ Clearly, $R(X) = a_0 X$ is a solution of (36) if (A) $d\varphi(t, 0)/dt|_{t=a_0} = 0$. For instance, if the reinsurer uses the rule (4), then condition (A) is equivalent to

$$\begin{aligned} &h_1[EN\,EX\pi_\beta, (EN\,\mathrm{Var}\,X + \mathrm{Var}\,N(EX)^2)\pi_\beta^2]EN\,\beta + \\ &+2h_2[EN\,EX\pi_\beta, (EN\,\mathrm{Var}\,X + \mathrm{Var}\,N(EX)^2)\pi_\beta^2]\,\pi_\beta(\beta EX\,\mathrm{Var}\,N + \\ &-EN\,DX) = 0 \end{aligned}$$

in which $h_1(t, s) = \partial h(t, s)/\partial t$, $h_2(t, s) = \partial h(t, s)/\partial s$ and $\pi_\beta = 1 - P_{Re}/(EX + \beta DX)$.

EXAMPLE 3.4 The following problem is to be solved

$$\begin{aligned} &\text{minimize } \alpha E\sum_{i=1}^{N}[X_i - R(X_i)] + \mathrm{Var}\sum_{i=1}^{N}[X_i - R(X_i)] \\ &\text{subject to } ER(X) + \beta DR(X) = P_{Re},\ 0 \leq R(X) \leq X. \end{aligned} \tag{39}$$

From Remark 3.1 it follows that the rule $R(X) = XP_{Re}/(EX + \beta DX)$ solves (39) provided

$$\beta\alpha EN + 2\left(1 - \frac{P_{Re}}{\quad}\right)(\ldots$$

Since $P_{Re} < EX$, (40) has a positive solution, say β_0, but the explicit formula for β_0 is rather complicated so it will be omitted.

EXAMPLE 3.5 Given $\alpha > 0$,

$$\text{minimize } \alpha \left(E \sum_{i=1}^{N} [X_i - R(X_i)] \right)^2 + \text{Var} \sum_{i=1}^{N} [X_i - R(X_i)]$$
$$\text{subject to } \; ER + \beta DR = P_{Re}, \quad 0 \le R(X) \le X. \tag{41}$$

An easy computation shows that $\varphi(\cdot, b)$, defined by (37), attains its minimum over $(-\infty, \infty)$ at

$$t(b) = \frac{\beta(P_{Re} - EX)[\text{Var}N + \alpha(EN)^2] + E(N)D(X-b)_+ - \beta E(b-X)_+}{[\beta^2(\text{Var}N + \alpha(EN)^2) + EN]D(X-b)_+}.$$

Moreover, $t(b) \le 1$ because of

$$P_{Re} = a[E(X-b)_+ + \beta D(X-b)_+] \le EX + \beta D(X-b)_+.$$

Consequently, if there exists a nonnegative real b^* such that $P_{Re} = t(b^*)[E(X - b^*)_+ + \beta D(X - b^*)_+]$, then $R^*(X) = t(b^*)(X - b^*)_+$ is a solution of (41). For instance, if each X_i is a Bernoulli claim (see Example 2.1), and if the number of claims N has the Poisson distribution with mean λ, then

$$b^* = \lambda \frac{\sqrt{pq} - \beta p(1+\alpha\lambda)}{\beta q + \lambda\sqrt{pq}} [Mp + \beta M \sqrt{pq} - P_{Re}) > 0,$$

and

$$t(b^*) = \frac{\beta(P_{Re} - Mp)(\lambda + \alpha\lambda^2) + \lambda(M - b^*)\sqrt{pq} - \beta q b^*}{[\beta^2(\lambda + \alpha\lambda^2) + \lambda](M - b^*)\sqrt{pq}}$$

provided $0 < \beta < (1 + \alpha\lambda)^{-1}\sqrt{p/q}$.

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