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High-gain feedback and sliding modes in infinite dimensional systems

by

Laura Levaggi

Dipartimento di Matematica
Università di Genova
Via Dodecaneso, 35, 16146 Genova, Italy
e-mail: levaggi@dima.unige.it

Abstract: This paper focuses on the connection between sliding motions and low frequency modes of high-gain feedback systems in an infinite dimensional framework.
We study a particular class of abstract control systems in a Hilbert space setting and analyse their high-gain behaviour through singular perturbations. We show that the "slow" motion derived from the reduced model approximates the evolution of the closed loop after a fast transient. Moreover we prove a relation between this slow component of the high-gain feedback system and sliding motions, in the spirit of the analogous result in the finite dimensional setting by Young, Kokotovic and Utkin (Young et al., 1977).

Keywords: infinite dimensional systems, high-gain feedback, singular perturbations, sliding modes.

1. Introduction

Sliding mode techniques can be considered at present as a somewhat classical tool for the control of systems governed by ordinary differential equations. Their appealing features of order reduction and robustness have contributed to both their theoretical investigation and engineering application. The literature on the subject is wide and covers numerous aspects of these two fields (for an overview see for example Utkin, 1992).

The robustness properties with respect to unmodeled external disturbances these methods can guarantee, have increased the interest of researchers in their extension to the infinite dimensional setting, see Orlov and Utkin (1982, 1987, 1998), Orlov (1983, 2000), Utkin (1990). As for the early developments of the finite dimensional theory, the first mathematical obstacle to overcome is the approach means also choosing a state discontinuous feedback. This control is designed in order to make a chosen manifold globally attractive and so, after a reaching stage, to give rise to a constrained evolution. This choice implies the difficulty of giving a definition of solution for the closed loop dynamical system. The first answer to this problem was given in Orlov and Utkin (1987), where the validity of the sliding motion was related to a property of approximability through continuous controls. The study there, however, was limited to semilinear parabolic systems. Later the same approach has been shown in Orlov (2000) to be successful for a wider class of infinite dimensional systems. In Levaggi (2002a,b) the sliding mode validation was instead carried out using a generalized solution concept derived by the definition of a differential inclusion.

The study of the mathematical foundations of sliding mode control for infinite dimensional systems has thus received some attention, although it lacks the depth of investigation given to the finite dimensional one. This paper places itself in this framework, showing the extension of a particular result valid for finite dimensional linear time-invariant control systems to a Hilbert space setting. This appeared in the paper by Young, Kokotovic and Utkin (1977), showing a relationship between linear high-gain feedback systems and sliding motions. Using singular perturbation theory the authors showed that, under some stability assumption, as the gain tends to infinity the system acquires a mode separation property. The evolution is in fact the superposition of an asymptotically stable fast transient and a slow component. Moreover, the motion of the slow subsystem coincides with the evolution of the equivalent control system on a well chosen sliding surface.

Here we will present a class of infinite dimensional systems to which these results can be extended. We will analyse high-gain systems of the type

$$\begin{cases} \dot{x}_1 = A_{11}x_1 + A_{12}x_2, \\ \dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2 u, \end{cases} \qquad u = k(C_1x_1 + C_2x_2), \quad k > 0.$$

All operators will be supposed linear and bounded, apart from A_{11} and A_{22}, which are the generators of strongly continuous semigroups on Hilbert spaces H_1 and H_2. Just as in the finite dimensional case, when the gain k tends to infinity, the feedback system becomes singularly perturbed. For $\mu = 1/k$ we end up with a system in the form

$$\begin{cases} \dot{x}_1 = A_{11}x_1 + A_{12}x_2, \\ \mu\dot{x}_2 = A^{\mu}_{21}x_1 + A^{\mu}_{22}x_2, \end{cases} \tag{1}$$

where for any μ, A^{μ}_{22} still generates a C_0-semigroup, while A^{μ}_{21} is bounded.

We prove that, under some uniform exponential stability assumption, if the reduced order model (corresponding to $\mu = 0$) has a unique solution, then it approximates the evolution of (1) as $\mu \to 0$, away from $t = 0$. This new singular perturbation result is shown in Section 2.

As we said before, the sliding mode concept can be extended to the infinite enjoys the approximability property needed to make sense of the sliding motion. More precisely, we show that any feasible trajectory of the control system which in a specified sense evolves near the sliding constraint, uniformly approximates the sliding mode. The relation proved in Young, Kokotovic and Utkin (1977) with the high-gain slow motion is then generalized to this new setting.

In Section 4 we present an example of a controlled heat equation that fits into our scheme. We also show how to build the high-gain feedback (and thus the sliding surface) in order to satisfy the hypotheses needed to apply our perturbation theorem. The same goal is accomplished in Section 5, where we present a disturbance rejection problem for the output of a wave equation.

In both cases presented in Sections 4 and 5, the sliding mode can also be substantiated using the results from Orlov and Utkin (1998) and Orlov (2000), respectively. Here we show, in a different setting, under what conditions the constrained motion coincides with the slowly varying component of an high-gain feedback system.

2. A convergence result for singularly perturbed systems

Let us be given the following infinite dimensional control system

$$\begin{cases} \dot{x}_1 = A_{11}x_1 + A_{12}x_2, & x_1(0) = x_{1,0} \\ \dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2u, & x_2(0) = x_{2,0}, \end{cases} \quad u \in U \tag{2}$$

where for $i = 1, 2$ $A_{ii} : \mathcal{D}(A_{ii}) \subset H_i \to H_i$ is the generator of a C_0-semigroup $S_i(t)$, $t \geq 0$ on the Hilbert space H_i and $A_{ij} : H_j \to H_i$ is a continuous linear operator for $j \neq i$. The control space U is assumed to be an Hilbert space and the operator $B_2 : U \to H_2$ is linear and bounded. From now on we will always suppose that these structural hypotheses hold. When referring to the control system (2), the above properties will all have to be taken into account, even if not explicitly mentioned. Let $C_i : H_i \to U$ $i = 1, 2$ be two linear continuous operators, $k > 0$ and set

$$u(x_1, x_2) = k(C_1x_1 + C_2x_2).$$

Calling $\mu = \frac{1}{k}$ the closed loop system becomes

$$\begin{cases} \dot{x}_1 = A_{11}x_1 + A_{12}x_2, & x_1(0) = x_{1,0} \\ \mu\dot{x}_2 = (\mu A_{21} + B_2C_1)x_1 + (\mu A_{22} + B_2C_2)x_2, & x_2(0) = x_{2,0}. \end{cases} \tag{3}$$

Mild solutions of this differential system are well defined since A_{11} is the generator of a strongly continuous semigroup on H_1 by hypothesis, while the following classical perturbation result shows that $\mu A_{22} + B_2C_2$ generates a C_0-semigroup on H_2.

THEOREM 2.1 (Pazy, 1983, p. 76) *Let X be a Banach space and A the infinitesimal generator of a strongly continuous semigroup $T(t)$, $t \geq 0$ on X. If B is a* *bounded linear operator on X, then $A+B$ generates a C_0-semigroup $S(t)$, $t \geq 0$ on X satisfying*

$$S(t)x = T(t)x + \int_0^t T(t-s)BS(s)x\,ds, \quad \forall x \in X. \tag{4}$$

Moreover, if $T(\cdot)$ is a compact semigroup, so is $S(\cdot)$.

When the gain k tends to infinity, the parameter μ in (3) tends to zero, giving therefore rise to a singularly perturbed equation. If $\mu = 0$ the second equation in (3) is transformed into the following operator equation

$$B_2C_1x_1 + B_2C_2x_2 = 0.$$

If the operator $B_2C_2 : H_2 \to H_2$ is continuously invertible the unique solution is

$$\bar{x}_2(t) = -(B_2C_2)^{-1}B_2C_1\bar{x}_1(t), \tag{5}$$

with $\bar{x}_1$ solving the differential equation

$$\begin{cases} \dot{y} = [A_{11} - A_{12}(B_2C_2)^{-1}B_2C_1]y \\ y(0) = x_{1,0}\,. \end{cases} \tag{6}$$

Note that under the given hypotheses $P = -A_{12}(B_2C_2)^{-1}B_2C_1$ is a bounded linear operator from H_1 into itself, therefore, by Theorem 2.1, $A_{11} + P$ still generates a strongly continuous semigroup $\bar{S}_1(t)$, $t \geq 0$ on H_1. Thus, by substituting $\mu = 0$ into (3) we get the following evolution

$$\begin{cases} \bar{x}_1(t) = \bar{S}_1(t)x_{1,0} \\ \bar{x}_2(t) = -(B_2C_2)^{-1}B_2C_1\bar{x}_1(t). \end{cases} \tag{7}$$

We want to prove that, under some regularity assumptions, the motion of the singularly perturbed system (3) is the superposition of the slow motion (7) and a fast transient. To this end let us introduce the following definition.

DEFINITION 2.1 (Kreĭn, 1971, p. 284) *A family of functions $\{f_\mu(t) : 0 < \mu \leq \overline{\mu},\ t \in [0,T]\}$ tends to zero as $\mu \to 0$ almost uniformly on $(0,T]$ if for any $\varepsilon > 0$ there exist $\delta_\varepsilon > 0$, $t_\varepsilon > 0$ such that*

$$\|f_\mu(t)\| < \varepsilon, \quad \forall \mu < \delta_\varepsilon, \quad t \in [\mu t_\varepsilon, T].$$

A simple instance of such a family is given by $f_\mu(t) = \exp(-at/\mu)$, $a > 0$. Note also that the almost uniform convergence on $(0,T]$ implies the uniform convergence on $[t_0,T]$ for any $t_0 > 0$ but the converse is not always true (e.g. take $f_\mu(t) = \mu^{-1}\exp(-at/\mu)$).

THEOREM 2.2 *Let us suppose that B_2C_2 is continuously invertible, $\bar{x}_2(t) \in \mathcal{D}(A_{22})$ for almost all t and that there exists $p > 1$ such that the function $f(t) = A_{22}\bar{x}_2(t)$ belongs to $L^p(0,T;H_2)$ for all $T > 0$. Calling $K_\mu(t)$, $t \geq 0$ the semigroup generated by $\mu A_{22} + B_2C_2$, let us assume that, for all sufficiently small μ, the following holds:*

$$\|K_\mu(t)\| \leq Me^{\omega_\mu t}, \quad \text{with} \quad \omega^* = \sup_\mu \omega_\mu < 0. \tag{8}$$

Then, if (x_1^μ, x_2^μ) is the mild solution of the singularly perturbed system (3), as $\mu \to 0$ the following convergence result holds: for any $T > 0$ one has $\|x_1^\mu(t) - \bar{x}_1(t)\| \to 0$ uniformly on $[0,T]$, while $\|x_2^\mu(t) - \bar{x}_2(t)\| \to 0$ almost uniformly on $(0,T]$.

Proof. Let us call $x_\mu = x_1^\mu - \bar{x}_1$ and $z_\mu = x_2^\mu - \bar{x}_2$. From (3), (5) and (6) x_μ solves the following differential equation

$$\begin{cases} \dot{x} = A_{11}x + A_{12}z_\mu \\ x(0) = 0, \end{cases} \tag{9}$$

therefore the following holds

$$x_\mu(t) = \int_0^t S_1(t-s)A_{12}z_\mu(s)\,ds, \tag{10}$$

and the convergence properties of x_μ in the thesis depend on those of z_μ. Now, from the second equation in (3) we have

$$x_2^\mu(t) = K_\mu\left(\frac{t}{\mu}\right)x_{2,0} + \frac{1}{\mu}\int_0^t K_\mu\left(\frac{t-s}{\mu}\right)(\mu A_{21} + B_2C_1)x_1^\mu(s)\,ds,$$

since if $\mu A_{22} + B_2C_2$ generates the semigroup $K_\mu(\cdot)$, then $\frac{1}{\mu}(\mu A_{22} + B_2C_2)$ generates $K_\mu(\frac{t}{\mu})$, $t \geq 0$. We can then write

$$\begin{aligned} z_\mu(t) = {} & K_\mu\left(\frac{t}{\mu}\right)[x_{2,0} - \bar{x}_2(0)] + K_\mu\left(\frac{t}{\mu}\right)\bar{x}_2(0) - \bar{x}_2(t) \\ & + \frac{1}{\mu}\int_0^t K_\mu\left(\frac{t-s}{\mu}\right)(\mu A_{21} + B_2C_1)\,x_\mu(s)\,ds \\ & + \frac{1}{\mu}\int_0^t K_\mu\left(\frac{t-s}{\mu}\right)(\mu A_{21} + B_2C_1)\,\bar{x}_1(s)\,ds. \end{aligned}$$

Let us split the sum in the last integral. Recall that $B_2C_1\bar{x}_1 + B_2C_2\bar{x}_2 = 0$ and by hypothesis $\bar{x}_2$ takes values in $\mathcal{D}(A_{22})$, with $f(s) = A_{22}\bar{x}_2(s)$ p-integrable with $p > 1$. Therefore we have

$$\begin{aligned} \frac{1}{\mu}\int_0^t K_\mu\left(\frac{t-s}{\mu}\right)B_2C_1\bar{x}_1(s)\,ds = {} & \int_0^t K_\mu\left(\frac{t-s}{\mu}\right)A_{22}\bar{x}_2(s)\,ds \\ & - \frac{1}{\mu}\int_0^t K_\mu\left(\frac{t-s}{\mu}\right)(B_2C_2 + \mu A_{22})\bar{x}_2(s)\,ds. \end{aligned}$$

The second integral on the right-hand side can now be integrated and we get

$$\begin{aligned}\frac{1}{\mu}\int_0^t K_\mu\left(\frac{t-s}{\mu}\right) B_2C_1\bar{x}_1(s)\,ds &= \bar{x}_2(t) - K_\mu\left(\frac{t}{\mu}\right)\bar{x}_2(0)\\ &\quad + \int_0^t K_\mu\left(\frac{t-s}{\mu}\right) f(s)\,ds.\end{aligned}$$

Thus we obtain

$$\begin{aligned}z_\mu(t) &= K_\mu\left(\frac{t}{\mu}\right) z_\mu(0) + \frac{1}{\mu}\int_0^t K_\mu\left(\frac{t-s}{\mu}\right)(\mu A_{21} + B_2C_1)\,x_\mu(s)\,ds\\ &+ \int_0^t K_\mu\left(\frac{t-s}{\mu}\right) f(s)\,ds + \int_0^t K_\mu\left(\frac{t-s}{\mu}\right) A_{21}\,\bar{x}_1(s)\,ds.\end{aligned}$$

Plugging (10) into the above equation we get

$$z_\mu(t) = K_\mu\left(\frac{t}{\mu}\right) z_\mu(0) \tag{11}$$

$$+ \frac{1}{\mu}\int_0^t K_\mu\left(\frac{t-s}{\mu}\right)(\mu A_{21} + B_2C_1)\int_0^s S_1(s-\tau)A_{12}z_\mu(\tau)d\tau\,ds \tag{12}$$

$$+ \int_0^t K_\mu\left(\frac{t-s}{\mu}\right)[f(s) + A_{21}\,\bar{x}_1(s)]\,ds. \tag{13}$$

Let us first study the term (12). By the continuity properties of K_μ, A_{21}, B_2C_1 and Fubini's theorem this integral is equal to

$$\begin{aligned}&\frac{1}{\mu}\int_0^t d\tau \int_\tau^t K_\mu\left(\frac{t-s}{\mu}\right)(\mu A_{21} + B_2C_1)S_1(s-\tau)A_{12}z_\mu(\tau)\,ds\\ &= \frac{1}{\mu}\int_0^t W_\mu(t,\tau)z_\mu(\tau)\,d\tau,\end{aligned}$$

where for fixed μ, t and $\tau \in [0,t]$, for any $v \in H_2$ we define the linear operator

$$W_\mu(t,\tau)v = \int_\tau^t K_\mu\left(\frac{t-s}{\mu}\right)(\mu A_{21} + B_2C_1)S_1(s-\tau)A_{12}v\,ds.$$

We now prove that it is bounded and give an estimation of its norm. As $S_1(\cdot)$ is a C_0-semigroup on H_1, there exist a constant $N > 0$ and a real number α such that $\|S_1(t)\| \leq Ne^{\alpha t}$ for all t. Therefore, by (8) we get

$$\|W_\mu(t,\tau)v\| \leq M\,N(\mu\|A_{21}\| + \|B_2C_1\|)\,\|A_{12}\|\,\|v\|\int_\tau^t e^{\frac{\omega_\mu}{\mu}(t-s)}e^{\alpha(s-\tau)}ds.$$

Evaluating the integral and using (8), for all sufficiently small μ we thus obtain

$$\begin{aligned}\|W_\mu(t,\tau)v\| &\leq \text{const}\,\|v\|\frac{\mu}{\mu\alpha - \omega^*}(e^{\alpha(t-\tau)} - e^{\frac{\omega_\mu}{\mu}(t-\tau)})\\ &< \text{const}\,\|v\|\frac{\mu}{}g(t)\end{aligned} \tag{14}$$

with $g(t) = \max\{1, e^{\alpha t}\}$. From (8) we get the following bound for the right-hand side of (11)

$$\left\| K_\mu \left(\frac{t}{\mu} \right) z_\mu(0) \right\| \leq M \|z_\mu(0)\| e^{\frac{\omega^*}{\mu} t}. \tag{15}$$

Let us now study the integral in (13). As $\bar{x}_1$ is continuous, A_{21} is bounded and by hypothesis $f \in L^p$ with $p > 1$, we also have $h = f + A_{21}\bar{x}_1 \in L^p(0, T)$. Thus by (8) and Hölder inequality we get

$$\begin{aligned} \left\| \int_0^t K_\mu \left(\frac{t-s}{\mu} \right) h(s)\, ds \right\| &\leq M \|h\|_p \left(\int_0^t e^{\frac{\omega_\mu}{\mu}(t-s)q} ds \right)^{1/q} \\ &\leq M \|h\|_p \left(\frac{\mu}{-q\omega_\mu} \left(1 - e^{\frac{\omega_\mu}{\mu}} qt\right) \right)^{1/q} \\ &\leq \frac{M \|h\|_p}{|\omega^*|} \mu^{1/q} \end{aligned} \tag{16}$$

(here $1/q + 1/p = 1$ and $q < \infty$). Applying the bounds (14), (15) and (16), to relation (11)-(13) we obtain

$$\|z_\mu(t)\| \leq \text{const} \left[e^{\frac{\omega^*}{\mu} t} + \mu^{1/q} + \frac{g(t)}{\mu\alpha - \omega^*} \int_0^t \|z_\mu(\tau)\|\, d\tau \right].$$

Setting for simplicity $f_\mu(t) = \exp(\omega^* t/\mu) + \mu^{1/q}$, by Gronwall's Lemma (Lakshmikantham and Leela, 1969, Cor. 1.9.1) we have

$$\begin{aligned} \|z_\mu(t)\| &\leq \text{const}[f_\mu(t) + g(t) \int_0^t f_\mu(s) e^{(t-s)g(t)} ds] \\ &\leq \text{const}[f_\mu(t) + g(t) e^{t\, g(t)} \int_0^t f_\mu(s)\, ds]. \end{aligned}$$

Obviously $f_\mu(t) \to 0$ for $\mu \to 0$ whenever $t \neq 0$, while the integral converges to zero for all t. The uniformity stated in our result has only to be proved for the term $e^{\omega^* t/\mu}$ and this is straightforward. ■

REMARK 2.1 (FINITE DIMENSIONAL CONTROL) *As for the applicability of this result, an important remark about finite dimensional control should be made. Assume that our system is in the form (2) and* $\dim U = m$. *Then also* $B_2(U)$ *is a finite dimensional space. Without loss of generality we can assume that its dimension is still* m, *so that* B_2 *is surjective (otherwise we could reason on a subspace of* U *as the control space). In order to apply our singular perturbation result, we need the invertibility of the operator* $B_2C_2 : H_2 \to B_2(U)$, *thus we must have* $\dim H_2 = m$. *In this case the operator* A_{22} *is itself continuous and the hypotheses of Theorem 2.2 are satisfied whenever the matrix representing*

It also has to be noted that in this framework the overall dynamical system on $H_1 \oplus H_2$ has the property that the image of the input operator belongs to the domain of the the semigroup generator. The following Proposition shows a form of converse result.

PROPOSITION 2.1 *Given a general abstract control system in the form*

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu \\ x(0) = x_0 \end{cases}$$

on a Hilbert space H, with generator A, linear continuous input operator B and $\dim U = \dim B(U) = m$, if $B(U) \subset \mathcal{D}(A)$ then the system can be put into form (2).

Proof. Calling $H_2 = B(U)$ and H_1 any of its complements, the projections P_i on H_i along H_j, $i, j = 1, 2$ have the following properties: $P_1 + P_2 = I$, $P_2(H) \subset \mathcal{D}(A)$ and $P_1(\mathcal{D}(A)) \subset \mathcal{D}(A)$ because $\mathcal{D}(A)$ is a subspace. Thus the operators $A_{ij} = P_i A P_j$ are well defined on $\mathcal{D}(A)$, A_{11} is a generator on H_1 and so $P_1 B = 0$. As H_2 is finite dimensional we obviously have that A_{12} and A_{22} are continuous. As for A_{21}, its continuity on H_1 is equivalent to that of $P_2 A$ on H, since $A_{21} = P_2 A - A_{22}$. Using the Riesz representation theorem it is not difficult to prove that this condition is satisfied if and only if the space $H_1^{\perp}$ is contained in $\mathcal{D}(A^*)$. In fact if $P_2 A$ is continuous, it admits an extension to H and for any fixed $z \in H$ the operator $x \mapsto (P_2 Ax, z)$ is a functional. Thus using Riesz theorem one gets

$$\forall x \in \mathcal{D}(A), \quad \forall z \in H \quad \exists\, w \text{ s.t. } (x, w) = (P_2 Ax, z) = (Ax, P_2^* z),$$

i.e. $w \in \mathcal{D}(A^*)$ and $w = A^* P_2^* z$. This is possible only if $\operatorname{Im} P_2^* \subset \mathcal{D}(A^*)$, but $\operatorname{Im} P_2^* = (\ker P_2)^{\perp}$ and the previous statement is proved. Since $\dim H_2 = m$ it is always possible to choose a complement H_1 satisfying this property. ■

An example of application of these results is given in Section 5.

3. Sliding modes approximability

In this Section we introduce sliding modes and prove the approximation result pointed out in the introduction. Just as in the finite dimensional case (Young, Kokotovic and Utkin, 1977) for the class of systems under consideration, the interpretation of the sliding mode as the "slow" motion of a high-gain feedback system as the gain tends to infinity will easily come by.

Let us be given a control system of the form (2); for simplicity we suppose B_2 invertible (see, however, Remark 3.1). Now let $C_i : H_i \to U$ be two linear, continuous operators and set

Let us suppose that we can restrict the trajectory of system (2) to the manifold S, that is - we can induce a sliding motion on S. S will be then called a *sliding manifold.* How to do this in the case of a finite dimensional system through a discontinuous control law is by now well known (Utkin, 1992). The sliding mode control technique can also be extended to the infinite dimensional case, under some regularity assumptions. Here we will show how the equivalent control method extends to the particular case we are studying.

As in the finite dimensional case, the sliding mode existence requires the invertibility of operator C_2B_2. Thus, if B_2 is an isomorphism, the same will hold for C_2 and the manifold S will be given by

$$S = \{(x_1, -C_2^{-1}C_1x_1) \,:\, x_1 \in H_1\}. \tag{17}$$

If a sliding mode takes place on S the equation of motion of the sliding trajectory will therefore be given by

$$\begin{cases} \dot{\bar{x}}_1 = [A_{11} - A_{12}C_2^{-1}C_1]\bar{x}_1 \\ \bar{x}_2 = -C_2^{-1}C_1\bar{x}_1, \end{cases} \tag{18}$$

which is just the same as (6), (7). The pair $(\bar{x}_1(\cdot), \bar{x}_2(\cdot))$ will be referred to as the *sliding motion.*

One can easily show that this system can also be found through a formal application of the equivalent control method. In Levaggi (2002a,b) two different sets of regularity assumptions are shown to be sufficient to make this method rigorous, by generalizing the solution concept to infinite dimensional differential equations with discontinuous right-hand side. The sliding motion is then interpreted as a generalized solution which is viable on the constraint S.

Another way of validating the equivalent control method consists in showing that all trajectories obtained using continuous controls which are able to bound the state evolution in a boundary layer of the sliding manifold (real sliding modes), converge to the equation of motion produced by the equivalent control (ideal sliding mode). This approach can be found in both Orlov and Utkin (1998) and Orlov (2000): in the first one the result is proved under some compactness hypothesis on the semigroup governing the evolution. In Orlov (2000) the setting is rather general: no special regularity has to be assumed about the unbounded operators, although the definition of the boundary layer about S is given in the norm on the domain of the generator.

In the following proposition we show that the structure of our control system (2) allows us to simplify the proof of this result. The distance of the real sliding modes from the sliding surface is measured in the state space norm.

PROPOSITION 3.1 *Suppose that the operator B_2 in (2) is invertible and the manifold S in (17) is a sliding manifold for (2). For any $\delta > 0$ let $(x_{1,0}^{(\delta)}, x_{2,0}^{(\delta)})$ be a couple of initial values in $H_1 \oplus H_2$ and u_δ be any control law such that*

$$\begin{cases} \dot{y} = A_{11}y + A_{12}z, & y(0) = x_{1,0}^{(\delta)} \\ \dot{z} = A_{21}y + A_{22}z + B_2u_\delta, & z(0) = x_{2,0}^{(\delta)} \end{cases} \tag{19}$$

admits a unique strong solution $(x_1^{(\delta)}, x_2^{(\delta)})$. *Suppose that* u_δ *is chosen in such a way that*

$$s_\delta(t) = C_1 x_1^{(\delta)}(t) + C_2 x_2^{(\delta)}(t) \tag{20}$$

tends to zero as δ *tends to zero, uniformly on* $[0,T]$ *for any* $T > 0$. *If* $x_{1,0}^{(\delta)} \to \bar{x}_1(0)$ *for* $\delta \to 0$, *then* $(x_1^{(\delta)}, x_2^{(\delta)})$ *converges to the sliding motion (18) as* δ *tends to zero, uniformly on compact subsets of* $[0,+\infty)$.
Moreover, if s_δ *tends to zero uniformly on* $[0,+\infty)$ *and the semigroup* $\bar{S}_1(t)$, $t \geq 0$ *generated by* $A_{11} - A_{12}C_2^{-1}C_1$ *is exponentially stable, the convergence of* $(x_1^{(\delta)}, x_2^{(\delta)})$ *is uniform on* $[0,+\infty)$.

Proof. From (19) we have $x_2^{(\delta)} = C_2^{-1}[s_\delta - C_1 x_1^{(\delta)}]$ and by (17), (18) we can write

$$\begin{aligned} \frac{d}{dt}(x_1^{(\delta)} - \bar{x}_1)(t) &= A_{11}x_1^{(\delta)} + A_{12}x_2^{(\delta)} - [A_{11} - A_{12}C_2^{-1}C_1]\bar{x}_1(t) \\ &= [A_{11} - A_{12}C_2^{-1}C_1](x_1^{(\delta)} - \bar{x}_1)(t) + A_{12}C_2^{-1}s_\delta(t), \end{aligned}$$

so that

$$(x_1^{(\delta)} - \bar{x}_1)(t) = \bar{S}_1(t)[x_1^{(\delta)}(0) - \bar{x}_1(0)] + \int_0^t \bar{S}_1(t-s)A_{12}C_2^{-1}s_\delta(s)\,ds.$$

Thus, if $\|\bar{S}_1(t)\| \leq Me^{\omega_1 t}$ for all $t \geq 0$ we get

$$\begin{aligned} \|(x_1^{(\delta)} - \bar{x}_1)(t)\| &\leq Me^{\omega_1 t}\|x_1^{(\delta)}(0) - \bar{x}_1(0)\| \\ &\quad + M\|A_{12}\|\,\|C_2^{-1}\|\,(\sup_{s\in[0,t]}\|s_\delta(s)\|)\int_0^t e^{\omega_1(t-s)}ds. \end{aligned}$$

Therefore, if $\|x_1^{(\delta)}(0) - \bar{x}_1(0)\| \to 0$ for $\delta \to 0$ and s_δ tends to zero uniformly in $[0,T]$ for all $T > 0$, we have proved the uniform convergence of $x_1^{(\delta)}$ to $\bar{x}_1$ on compact subsets of $[0,+\infty)$. Moreover, if we have uniform convergence on $[0,+\infty)$ for s_δ and $\omega_1 < 0$, then $\|x_1^{(\delta)} - \bar{x}_1\|$ tends to zero uniformly on $[0,+\infty)$ as δ goes to zero. As

$$(x_2^{(\delta)} - \bar{x}_2)(t) = -C_2^{-1}C_1[(x_1^{(\delta)} - \bar{x}_1)(t)] + C_2^{-1}s_\delta(t)$$

the proof is completed. ■

REMARK 3.1 *The invertibility hypothesis imposed in this Section on* B_2 *has the advantage of simplifying further formulas, but is unnecessary to get the stated results. In fact it would be sufficient to require the invertibility of both* B_2C_2 *and* C_2B_2, *which are the nonsingularity conditions needed respectively to get (5) and to apply the equivalent control method. For example, let us show how to express the sliding motion in the general case. Any couple* $(x_1, x_2) \in S$ *obviously* *satisfies* $B_2C_1x_1 + B_2C_2x_2 = 0$, *which is equivalent to* $x_2 = -(B_2C_2)^{-1}B_2C_1x_1$. *The invertibility of* C_2B_2 *implies that* B_2 *has to be injective and thus* $S = \{(x_1, -(B_2C_2)^{-1}B_2C_1x_1) : x_1 \in H_1\}$ *as wanted.*

Proposition 3.1 has also the following physical interpretation: due to small imperfections e.g. in either the sensors or the actuators, ideal sliding modes cannot be realized by real-life control devices. However, if the real control law u_δ is able to constrain the motion in a boundary layer of the manifold S, the real trajectory will preserve the properties of the ideal one. Moreover, in the limit, as the amount δ of the imperfections tends to zero, the sliding motion will be realized.

COROLLARY 3.1 *Given a control system satisfying the hypotheses of (2), let* $C_i : H_i \to U$, $i = 1, 2$ *be two linear continuous operators. Suppose* B_2C_2 *and* C_2B_2 *are continuously invertible and a sliding mode exists on* $S = \{(x_1, x_2) \in H_1 \oplus H_2 : C_1x_1 + C_2x_2 = 0\}$. *Then if the hypotheses of Theorem 2.2 hold, the sliding motion can be interpreted as the limit slow mode of a high-gain feedback system as the gain tends to infinity.*

4. Application to a heat equation

In this section we are going to present a control example which fits into our framework. In Orlov and Utkin (1998) this system is stabilized by the application of a discontinuous control inducing a sliding motion on a suitably chosen surface S. Here we show that for a suitable choice of the sliding manifold the hypotheses of Theorem 2.2 are satisfied and thus the constrained motion can be interpreted as the the slowly varying component of a singularly perturbed system.

Let us be given the following boundary value problem

$$\begin{cases} \frac{\partial Q}{\partial t} = \frac{\partial^2 Q}{\partial y^2} + DQ + Fu(y) & y \in (0,1),\ t > 0 \\ \frac{\partial Q}{\partial y}(0,t) = \frac{\partial Q}{\partial y}(1,t) = 0 & t \geq 0 \\ Q(y,0) = Q_0(y) & y \in [0,1] \end{cases} \tag{21}$$

where $Q(y,t) \in \mathbb{R}^n$, $u \in L^2(0,1;\mathbb{R}^m)$, $D \in M_n(\mathbb{R})$ and $F \in M_{n\times m}(\mathbb{R})$ with $m < n$. The physical problem consists in the heating of n similar objects using m sources. The matrix D represents both the heat exchange between the objects and the environment.

Without loss of generality we can suppose that the constant matrix F has maximal rank m. In fact it is always possible to find a full rank matrix F' and a subspace U' of $\mathbb{R}^m$ such that for any $u \in \mathbb{R}^m$ there exists $u' \in U'$ with $Fu = F'u'$ and $\operatorname{rank} F' = \dim U'$.

We restrict the analysis to a family of candidate sliding manifolds, more precisely we will call

$$\mathcal{S} = \{Q \in \mathbb{R}^n : GQ = 0\},$$

with G a fixed but arbitrary $m \times n$ real matrix such that GF is invertible. The right choice for G will be carried out in the sequel. As $\operatorname{rank} F = m$ there exists a nonsingular matrix $M \in M_n(\mathbb{R})$ such that

$$MF = \begin{bmatrix} 0 \\ F_2 \end{bmatrix}, \qquad \text{with } F_2 \in M_m(\mathbb{R}), \quad \det F_2 \neq 0.$$

Setting $\tilde{Q} = MQ$, we get

$$\begin{aligned} \frac{\partial \tilde{Q}}{\partial t} &= M\frac{\partial Q}{\partial t} = M[\frac{\partial^2 Q}{\partial y^2} + DQ + Fu] \\ &= \frac{\partial^2 \tilde{Q}}{\partial y^2} + MDM^{-1}\tilde{Q} + \begin{bmatrix} 0 \\ F_2 \end{bmatrix} u. \end{aligned}$$

We now split vector $\tilde{Q} \in \mathbb{R}^n$ as $\tilde{Q} = (\tilde{Q}_1, \tilde{Q}_2)$ with $\tilde{Q}_1 \in \mathbb{R}^{n-m}$ and $\tilde{Q}_2 \in \mathbb{R}^m$. Using the identity above and substituting into (21) we obtain

$$\begin{cases} \frac{\partial \tilde{Q}_1}{\partial t} = \frac{\partial^2 \tilde{Q}_1}{\partial y^2} + D_{11}\tilde{Q}_1 + D_{12}\tilde{Q}_2 \\ \frac{\partial \tilde{Q}_2}{\partial t} = \frac{\partial^2 \tilde{Q}_2}{\partial y^2} + D_{21}\tilde{Q}_1 + D_{22}\tilde{Q}_2 + F_2 u \end{cases} \tag{22}$$

where the blocks D_{ij} correspond to the following decomposition

$$MDM^{-1} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix},$$

in accordance with the above splitting of $\mathbb{R}^n$. Moreover using the same block form we have

$$GF = GM^{-1}MF = [G_1\ G_2] \begin{bmatrix} 0 \\ F_2 \end{bmatrix} = G_2 F_2,$$

$G_1 \in M_{m \times n-m}(\mathbb{R})$, $G_2 \in M_{m \times m}(\mathbb{R})$, so that the invertibility of both GF and F_2 is equivalent to $\det G_2 \neq 0$. Therefore

$$\begin{aligned} \mathcal{S} &= \{(\tilde{Q}_1, \tilde{Q}_2) \in \mathbb{R}^n : G_1\tilde{Q}_1 + G_2\tilde{Q}_2 = 0\} \\ &= \{(\tilde{Q}_1, \tilde{Q}_2) : \tilde{Q}_2 = -G_2^{-1}G_1\tilde{Q}_1\}. \end{aligned} \tag{23}$$

We are now ready to restate our problem into the infinite dimensional setting we discussed before. In fact calling $H_1 = L^2(0,1;\mathbb{R}^{n-m})$, $H_2 = L^2(0,1;\mathbb{R}^m)$, $U = H_2$ we obtain a control system of type (2) with

$$\begin{array}{ll} A_{11}x_1 = x_1'' + D_{11}x_1, & \mathcal{D}(A_{11}) = \{x \in H^2(0,1;\mathbb{R}^{n-m}) : x'(0) = x'(1) = 0\} \\ A_{12}x_2 = D_{12}x_2, & \forall x_2 \in H_2 \\ A_{22}x_2 = x_2'' + D_{22}x_2, & \mathcal{D}(A_{22}) = \{z \in H^2(0,1;\mathbb{R}^m) : z'(0) = z'(1) = 0\} \\ A_{21}x_1 = D_{21}x_1, & \forall x_1 \in H_1 \\ B_2 u = F_2 u, & \forall u \in U. \end{array}$$

The initial values $(x_{1,0}, x_{2,0})$ will be given by $MQ_0(y) = (\tilde{Q}_{0,1}(y), \tilde{Q}_{0,2}(y))$.

The second derivative operator is the generator of a compact semigroup on either $\mathcal{D}(A_{11})$ or $\mathcal{D}(A_{22})$, therefore by the Hille-Phillips perturbation Theorem 2.1 both A_{11} and A_{22} generate a compact C_0-semigroup (on H_1 and H_2 respectively). The proofs of the continuity of A_{12}, A_{21} and B_2 are trivial. In accordance with the setting of Section 2, the operators C_1 and C_2 will be given by $C_1x_1 = G_1x_1$ and $C_2x_2 = G_2x_2$, respectively. The invertibility of operator B_2C_2 is of course an obvious consequence of the nonsingularity of both G_2 and F_2. From (6) and (7), the solution $(\bar{x}_1, \bar{x}_2)$ of the reduced order system will satisfy

$$\begin{cases} \dot{\bar{x}}_1 = [A_{11} - A_{12}G_2^{-1}G_1]\bar{x}_1 \\ \bar{x}_2 = -G_2^{-1}G_1\bar{x}_1 \end{cases} \tag{24}$$

which in terms of the variable $\tilde{Q}$ becomes

$$\begin{cases} \frac{\partial \tilde{Q}_1}{\partial t} = \frac{\partial^2 \tilde{Q}_1}{\partial y^2} + [D_{11} - D_{12}G_2^{-1}G_1]\tilde{Q}_1 \\ \tilde{Q}_2 = -G_2^{-1}G_1\tilde{Q}_1. \end{cases} \tag{25}$$

We are now going to show how the hypotheses required to apply Theorem 2.2 can be satisfied through a suitable choice of the matrix G, or equivalently the couple G_1, G_2. Let us start by studying the spectral properties of the singularly perturbed operator

$$A_\mu x = (\mu A_{22} + B_2C_2)x = \mu x'' + \mu D_{22}x + F_2G_2x. \tag{26}$$

From the Hille-Phillips theorem, A_μ generates a compact C_0-semigroup on H_2, therefore it satisfies the spectrum determined growth assumption (Curtain and Pritchard, 1978). This means that the semigroup type can be computed through the real part of its eigenvalues. Let us suppose that $\lambda_\mu \in \mathbb{C}$ is an eigenvalue of A_μ, i.e. that there exists a vector $v \in H^2(0,1;\mathbb{C}^m)$, $v'(0) = v'(1) = 0$, $v \neq 0$ such that

$$\mu v'' + \mu D_{22}v + F_2G_2v = \lambda_\mu v. \tag{27}$$

We are interested in the behaviour of the real part of the λ_μ-s as the parameter μ tends to zero. From (27) one expects that in the limit only the eigenvalues of matrix F_2G_2 do play a role. In fact it is not difficult to prove (see the Appendix) that if F_2G_2 is Hurwitz there exist constants $c_1, c_2 > 0$ such that

$$\mathcal{R}e\,\lambda_\mu \le c_1\left(\mu\, c_2 - \frac{1}{2}\right).$$

Therefore we have $\mathcal{R}e\,\lambda_\mu < 0$ for all $\mu < (2\,c_2)^{-1}$. Remember that G_2 is not given by the problem, but it belongs to our control tools. Thus, if we choose this matrix in order to place the eigenvalues of F_2G_2 in the left half plane, there exists a range of μ for which (8) is satisfied.

As for the other hypotheses, if we assume that $x_{1,0}$ belongs to $\mathcal{D}(A_{11})$, then $\bar{x}_1(t) \in \mathcal{D}(A_{11})$ for all t. Therefore $\bar{x}_2 \in \mathcal{D}(A_{22})$ because $\bar{x}_2$ is obtained from $\bar{x}_1$ by multiplying by a constant matrix and the regularity properties of $\bar{x}_1$ pass over to $\bar{x}_2$. This assumption is also sufficient to guarantee the p-integrability property required in the Theorem. In fact we have

$$\begin{aligned} A_{22}\bar{x}_2(t) &= [-G_2^{-1}G_1\bar{x}_1(t)]'' + D_{22}\bar{x}_2(t) \\ &= -G_2^{-1}G_1A_{11}\bar{x}_1(t) + [G_2^{-1}G_1D_{11} - D_{22}G_2^{-1}G_1]\bar{x}_1(t). \end{aligned}$$

It is not difficult to prove that $A_{22}\bar{x}_2$ belongs to $L^\infty(0,T;H_2)$. In fact $\bar{x}_1 \in L^\infty(0,T;H_1)$ because it is a continuous function and moreover as $x_{1,0} \in \mathcal{D}(A_{11})$, from (24) we have

$$A_{11}\bar{x}_1(t) = A_{11}\bar{S}_1(t)x_{1,0} = \bar{S}_1(t)[A_{11} - A_{12}G_2^{-1}G_1]x_{1,0} + A_{12}G_2^{-1}G_1\bar{x}_1(t),$$

so that also $A_{11}\bar{x}_1$ belongs to $L^\infty(0,T;H_1)$. This is obviously sufficient to prove the wanted boundedness property. Thus we get the following result.

PROPOSITION 4.1 *Given the control system (22), let $H \in M_{m\times m}(\mathbb{R})$ be a Hurwitz matrix, $G_1 \in M_{m\times n-m}(\mathbb{R})$ and set $G_2 = F_2^{-1}H$, $u = k(G_1\tilde{Q}_1 + G_2\tilde{Q}_2)$ with $k > 0$. When the gain k tends to infinity the closed loop system acquires a separation mode: the corresponding slow motion is the solution of (25). This evolution can also be interpreted as a sliding mode on the surface (23) obtained through the equivalent control method.*

If the couple (D_{11}, D_{12}) is controllable, the sliding motion can be stabilized by an appropriate choice of the matrix G_1.

Proof. The first part of the statement is just a collection of previous results. As for the second, just note that if (D_{11}, D_{12}) is controllable (observe that this is equivalent to (D, F) in (21) being controllable), then there exists $P \in M_{m\times n-m}(\mathbb{R})$ such that $D_{11} - D_{12}P$ is stable (Sontag, 1990). Then if we set $G_1 = H^{-1}F_2P$ we get the stability of $D_{11} - D_{12}F_2^{-1}HG_1 = D_{11} - D_{12}G_2G_1$. As before, this condition is sufficient for the exponential stability of the semigroup governing (24). ■

5. An application to the output control of a wave equation

Let us be given the following one-dimensional controlled wave equation

$$\begin{cases} \frac{\partial^2 Q}{\partial t^2} = \frac{\partial^2 Q}{\partial y^2} + \alpha\frac{\partial Q}{\partial t} + b(y)[u + h(t)] & y \in (0,1),\ t > 0 \\ \frac{\partial Q}{\partial t}(y,0) = Q_1(y), \quad Q(y,0) = Q_0(y) & y \in [0,1] \\ Q(0,t) = Q(1,t) = 0 & t \geq 0 \end{cases} \tag{28}$$

with scalar control u, initial values $Q_0 \in H_0^1(0,1)$, $Q_1 \in L^2(0,1)$ and $b \in L^2(0,1)$. The term h is in $L^\infty(0,+\infty)$ and represents the system's matched uncertainties. Suppose also that we are given the following output

$$y(t) = \int_0^1 Q(\omega, t)\, f(\omega)\, d\omega, \quad f \in L^2(0,1). \tag{29}$$

We show that we can choose an high-gain feedback control that ensures the independence of the closed loop slow motion with respect to the uncertain term h.

First we restate our control problem using a classical Hilbert form (see for instance Curtain and Pritchard, 1978). Let Δ be defined on $\mathcal{D}(\Delta) = H^2(0,1) \cap H_0^1(0,1)$ as $\Delta g = g''$. Then $-\Delta$ is positive definite and one can define $(-\Delta)^{1/2} g = g'$ on $H_0^1(0,1)$. Let $\mathcal{H}$ be the Hilbert space $\mathcal{H} = H_0^1(0,1) \times L^2(0,1)$ with the scalar product $\langle z, \tilde{z} \rangle_{\mathcal{H}} = (z_1, \tilde{z}_1)_{H_0^1} + (z_2, \tilde{z}_2)_{L^2}$. Then calling $z(t) = [Q(\cdot,t), \frac{\partial Q}{\partial t}(\cdot,t)]^T$, we can rewrite system (28) as follows:

$$\begin{cases} \dot{z}(t) = \mathcal{A} z(t) + \mathcal{B}[u+h], \quad t > 0 \\ z(0) = z_0 \end{cases}, \quad z(t) \in \mathcal{H} \tag{30}$$

with

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{H}, \quad \mathcal{A}z = \begin{bmatrix} 0 & I \\ \Delta & \alpha I \end{bmatrix} z, \quad \mathcal{D}(\mathcal{A}) = \mathcal{D}(\Delta) \times \mathcal{D}((-\Delta)^{1/2})$$

$$\mathcal{B} : I\!R \to \mathcal{H}, \quad \mathcal{B}v = \begin{bmatrix} 0 \\ b \end{bmatrix} v = Bv, \qquad z_0 = \begin{bmatrix} Q_0 \\ Q_1 \end{bmatrix}.$$

One can prove that $\mathcal{A}$ generates an analytic semigroup on $\mathcal{H}$ (Curtain and Pritchard, 1978). The control space U is finite dimensional, thus in order to be able to write our system in the form (2), we suppose that the image of $\mathcal{B}$ is contained in $\mathcal{D}(\mathcal{A})$, i.e. $b \in H_0^1(0,1)$. Proceeding as in Remark 2.1, we choose

$$\gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \in \mathcal{D}(\mathcal{A}^*) = \mathcal{D}(\mathcal{A}) \quad \text{such that} \quad \langle B, \gamma \rangle_{\mathcal{H}} = (\gamma_2, b)_{L^2} = 1$$

and set $H_2 = span\{B\}$, $H_1 = span\{\gamma\}^\perp$. The projection on H_2 will therefore be given by $P_1 x = \langle x, \gamma \rangle_{\mathcal{H}} B$. The vectors $z(t) \in \mathcal{H}$ will accordingly be split in the sum

$$z(t) = \lambda(t)\, B + w(t), \quad \lambda(t) \in I\!R, \quad w(t) \in \{\gamma\}^\perp \quad \forall t \geq 0$$

and (30) can be rewritten as

$$\begin{cases} \dot{\lambda}(t) = \lambda(t) c + \langle w(t), \mathcal{A}^* \gamma \rangle_{\mathcal{H}} + u + h, & \lambda(0) = \langle z_0, \gamma \rangle_{\mathcal{H}}, \\ \dot{w}(t) = \lambda(t)\, B_0 + \mathcal{A} w(t) - \langle w(t), \mathcal{A}^* \gamma \rangle_{\mathcal{H}}\, B, & \omega(0) = z_0 - \lambda(0) B, \end{cases} \tag{31}$$

where $c = (b, \gamma_1)_{H_0^1} + \alpha$ and $B_0 = [b, -(b, \gamma_1)_{H_0^1} b]^T$.

Using integration by parts it is straightforward to show that in our abstract setting, if $\varphi = -(\Delta)^{-1} f$, we can write this output as

$$y(t) = \langle z(t), \Phi \rangle_{\mathcal{H}} = (w_1(t), \varphi)_{H_0^1}, \quad \Phi = \begin{bmatrix} \varphi \\ 0 \end{bmatrix}$$

with $w(t) = [w_1(t), w_2(t)]^T \in H_2$. Also $\langle B, \Phi\rangle_{\mathcal{H}} = 0$, so that the output system has relative degree greater than one. Moreover, thanks to this condition, we have

$$\dot{y}(t) = \langle \mathcal{A}\, z(t) + \mathcal{B}\, u, \Phi\rangle_{\mathcal{H}} = \lambda(t)(b, f)_{L^2} + (w_2(t), f)_{L^2}.$$

Let us now suppose that the relative degree is two, i.e. $(b, f)_{L^2} \neq 0$ and to simplify formulas assume $(b, f)_{L^2} = 1$. Let $\beta > 0$ be a real number and define the following operator

$$C_1 : H_1 \to I\!R, \qquad C_1 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \beta(w_1, \varphi)_{H_0^1} + (w_2, f)_{L^2}.$$

Now choose the high-gain feedback control $u(t) = -k(\lambda(t) + C_1 w(t))$. Due to the hypothesis on the boundedness of h, it is sufficient to slightly modify the proof of Theorem 2.2, taking into account the presence of the disturbances, to show that the result is still true. Thus in the limit for k tending to infinity we obtain a slow motion satisfying the constraint $\lambda = -C_1 w$. Thus after the transient we have

$$\begin{aligned} \dot{y}(t) + \beta\, y(t) &= -\beta(w_1(t), \varphi)_{H_0^1} - (w_2(t), f)_L^2 + (w_2(t), f)_{L^2} \\ &+ \quad \beta(w_1(t), \varphi)_{H_0^1} = 0, \end{aligned}$$

which proves that the output $y(\cdot)$ is independent of the disturbances. By Corollary 3.1 the same result is obtained by imposing a sliding mode on the manifold $S = \{(\lambda, w) \in I\!R \times H_1 \, : \, \lambda = -C_1 w\}$.

REMARK 5.1 *A sliding surface for this control problem can also be chosen using the techniques shown in Orlov (2000). Indeed, that scheme is more general in that it does not need the image of the input operator to be contained in the domain of the generator. As we showed in Remark 2.1 this condition is necessary in our setting. Theorem 2.2 does not apply to the general case, but maybe a similar result can be proved by exploiting the regularity properties of the semigroup (in this case for example it is analytic). Here we have chosen to pay the price of imposing structural hypotheses on the systems to get a result which does not depend upon the operator's regularity.*

6. Conclusions

In this paper we presented a singular perturbation result for a class of high-gain infinite dimensional control systems and showed the relationship between reduced order model trajectories and sliding motions. However, not every distributed system to which sliding mode control can be extended is in this class. The question remains open whether for such control systems a similar result can be proved.

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Appendix

We study here the real part of scalars $\lambda \in \mathbb{C}$ that satisfy the following condition: there exists a vector $v \in H^2(0,1;\mathbb{C}^m) \cap H_0^1(0,1;\mathbb{C}^m)$, $v \neq 0$ such that

$$\mu v'' + \mu D v + K v = \lambda v, \tag{32}$$

for fixed $\mu > 0$ and $D, K \in \mathbb{R}^{m\times m}$ with K Hurwitz. The unique solution P of the Lyapunov equation $K^T P + PK = -I$ is symmetric and positive definite (see e.g. Sontag, 1990). Using such P we define on $L^2(0,1;\mathbb{C}^m)$ the following scalar product

$$(x|y) = \int_0^1 y^*(\xi) P x(\xi)\, d\xi, \quad y^* = \overline{y}^T.$$

Note that the induced norm is equivalent to the usual one: since P is symmetric and positive definite, there exists Q with $Q^*Q = I$ such that $P = Q^*\Delta Q$ with Δ the diagonal matrix of the eigenvalues of P (which are all real and positive). Since $\|Q\| = 1$ it is easy to show that

$$\lambda_{\min}\|y\|^2 \leq (y|y) \leq \lambda_{\max}\|y\|^2, \quad \forall y \in L^2(0,1;\mathbb{C}^m)$$

with $\lambda_{\max}$ and $\lambda_{\min}$ the largest and smallest eigenvalue of P respectively. From (32), through a scalar multiplication by v and applying integration by parts on $(v''|v)$ we get

$$\lambda(v|v) = -\mu(v'|v') + \mu(Dv|v) + (Kv|v).$$

Now, from the Lyapunov equation one has

$$2\mathcal{R}e\,(Kv|v) = -\int_0^1 v^*(\xi) v(\xi)\, d\xi,$$

therefore

$$\begin{aligned} \mathcal{R}e\,\lambda &\leq \frac{1}{(v|v)}[-\mu(v'|v') + \mu\lambda_{\max}\|D\|\,\|v\|^2 - \frac{1}{2}\|v\|^2] \\ &\leq \frac{1}{\lambda_{\min}}\left(\mu\,\|D\|\,\lambda_{\max} - \frac{1}{2}\right). \end{aligned}$$

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