Control and Cybernetics

vol. **33** (2004) No. 1

Robust performance of a class of control systems[1]

by

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Abstract: Some Kharitonov-like robust Hurwitz stability criteria are established for a class of complex polynomial families with nonlinearly correlated perturbations. These results are extended to the polynomial matrix case and non-interval D-stability case. Applications of these results in testing of robust strict positive realness of real and complex interval transfer function families are also presented.

Keywords: uncertain systems, robustness analysis, Kharitonov's theorem, complex interval polynomials, polynomial matrix family, Hurwitz stability, D-stability, transfer functions, strict positive realness.

1. Introduction

Motivated by the seminal theorem of Kharitonov on robust stability of interval polynomials (Kharitonov, 1978, 1979), a number of papers on robustness analysis of uncertain systems have been published in the past few years, see Hollot and Tempo (1994), Bartlett et al. (1988), Fu and Barmish (1989), Wang (1995, 2003), Wang and Huang (1994, 1994), Barmish et al. (1992), Chapellat et al. (1991), Ackermann (1991, 1992), Dasgupta (1988), Rantzer (1992), Wang et al. (2003), Yu and Wang (2001), Wang and Ackermann (2003). Kharitonov's theorem states that the Hurwitz stability of the real (or complex) interval polynomial family can be guaranteed by the Hurwitz stability of four (or eight)

[1]Supported by National Natural Science Foundation of China (10372002), National Key Basic Research and Development Program (2002CB312200) and National Hi-Tech R&D 863 Project (No. 2002AA755002).

prescribed critical vertex polynomials in this family. This result is significant since it reduces checking stability of infinitely many polynomials to checking stability of finitely many polynomials, and the number of critical vertex polynomials need to be checked is independent of the order of the polynomial family. An important extension of Kharitonov's theorem is the edge theorem discovered by Bartlett, Hollot and Huang (1998). The edge theorem states that the stability of a polytope of polynomials can be guaranteed by the stability of its one-dimensional exposed edge polynomials. The significance of the edge theorem is that it allows some (affine) dependency among polynomial coefficients, and applies to more general stability regions, e.g., unit circle, left sector, shifted half plane, hyperbola region, etc. When the dependency among polynomial coefficients is nonlinear, however, Ackermann shows that checking a subset of a polynomial family generally can not guarantee the stability of the entire family, see Ackermann (1991, 1992).

In this paper, we consider a class of complex polynomial families with nonlinear coefficient dependency. Based on our previous results, we will establish some Kharitonov-like robust stability criteria, i.e., the entire family is stable if and only if some critical vertices in this family are stable, and the number of critical vertices is independent of the order of the polynomial family. We will then extend our results to the polynomial matrix case and non-interval D-stability case. Applications of these results in testing strict positive realness of interval transfer function family are also presented.

2. Main results

A polynomial $p(s)$ is said to be Hurwitz stable, denoted by $p(s) \in H$, if all its roots lie within the open left half of the complex plane $\mathbb{C}$. A polynomial family P is said to be Hurwitz stable, denoted by $P \subset H$, if all polynomials in P are Hurwitz stable.

Consider the n-th order real interval polynomial family

$$\Gamma = \left\{ p(s) \mid p(s) = \sum_{i=0}^{n} q_i s^i \ , \ q_i \in [q_i^- \ , \ q_i^+] \ , \ i = 0, 1, \cdots, n \right\} \tag{1}$$

and define the four Kharitonov polynomials of Γ as

$$K_1(s) = q_0^- + q_1^- s + q_2^+ s^2 + q_3^+ s^3 + q_4^- s^4 + q_5^- s^5 + \cdots \tag{2}$$

$$K_2(s) = q_0^+ + q_1^+ s + q_2^- s^2 + q_3^- s^3 + q_4^+ s^4 + q_5^+ s^5 + \cdots \tag{3}$$

$$K_3(s) = q_0^+ + q_1^- s + q_2^- s^2 + q_3^+ s^3 + q_4^+ s^4 + q_5^- s^5 + \cdots \tag{4}$$

$$K_4(s) = q_0^- + q_1^+ s + q_2^+ s^2 + q_3^- s^3 + q_4^- s^4 + q_5^+ s^5 + \cdots \tag{5}$$

LEMMA 2.1 *(Kharitonov's Theorem for Real Polynomials, Kharitonov, 1978)*

$$\Gamma \subset H \iff K_1(s) \ , \ K_2(s) \ , \ K_3(s) \ , \ K_4(s) \in H. \tag{6}$$

Consider the n-th order complex interval polynomial family

$$\Delta = \Big\{\delta(s) \mid \delta(s) = \sum_{i=0}^{n} (\alpha_i + j\beta_i)s^i,\ \alpha_i \in [\alpha_i^-\ ,\ \alpha_i^+]\ ,\ \beta_i \in [\beta_i^-\ ,\beta_i^+]\ ,\ \ i = 0, 1, \cdots, n\Big\} \tag{7}$$

and define the eight Kharitonov polynomials of Δ as

$$K_1^+(s) = (\alpha_0^- + j\beta_0^-) + (\alpha_1^- + j\beta_1^+)s + (\alpha_2^+ + j\beta_2^+)s^2 + (\alpha_3^+ + j\beta_3^-)s^3 + (\alpha_4^- + j\beta_4^-)s^4 + (\alpha_5^- + j\beta_5^+)s^5 + \cdots \tag{8}$$

$$K_2^+(s) = (\alpha_0^- + j\beta_0^+) + (\alpha_1^+ + j\beta_1^+)s + (\alpha_2^+ + j\beta_2^-)s^2 + (\alpha_3^- + j\beta_3^-)s^3 + (\alpha_4^- + j\beta_4^+)s^4 + (\alpha_5^+ + j\beta_5^+)s^5 + \cdots \tag{9}$$

$$K_3^+(s) = (\alpha_0^+ + j\beta_0^-) + (\alpha_1^- + j\beta_1^-)s + (\alpha_2^- + j\beta_2^+)s^2 + (\alpha_3^+ + j\beta_3^+)s^3 + (\alpha_4^+ + j\beta_4^-)s^4 + (\alpha_5^- + j\beta_5^-)s^5 + \cdots \tag{10}$$

$$K_4^+(s) = (\alpha_0^+ + j\beta_0^+) + (\alpha_1^+ + j\beta_1^-)s + (\alpha_2^- + j\beta_2^-)s^2 + (\alpha_3^- + j\beta_3^+)s^3 + (\alpha_4^+ + j\beta_4^+)s^4 + (\alpha_5^+ + j\beta_5^-)s^5 + \cdots \tag{11}$$

$$K_1^-(s) = (\alpha_0^- + j\beta_0^-) + (\alpha_1^+ + j\beta_1^-)s + (\alpha_2^+ + j\beta_2^+)s^2 + (\alpha_3^- + j\beta_3^+)s^3 + (\alpha_4^- + j\beta_4^-)s^4 + (\alpha_5^+ + j\beta_5^-)s^5 + \cdots \tag{12}$$

$$K_2^-(s) = (\alpha_0^- + j\beta_0^+) + (\alpha_1^- + j\beta_1^-)s + (\alpha_2^+ + j\beta_2^-)s^2 + (\alpha_3^+ + j\beta_3^+)s^3 + (\alpha_4^- + j\beta_4^+)s^4 + (\alpha_5^- + j\beta_5^-)s^5 + \cdots \tag{13}$$

$$K_3^-(s) = (\alpha_0^+ + j\beta_0^-) + (\alpha_1^+ + j\beta_1^+)s + (\alpha_2^- + j\beta_2^+)s^2 + (\alpha_3^- + j\beta_3^-)s^3 + (\alpha_4^+ + j\beta_4^-)s^4 + (\alpha_5^+ + j\beta_5^+)s^5 + \cdots \tag{14}$$

$$K_4^-(s) = (\alpha_0^+ + j\beta_0^+) + (\alpha_1^- + j\beta_1^+)s + (\alpha_2^- + j\beta_2^-)s^2 + (\alpha_3^+ + j\beta_3^-)s^3 + (\alpha_4^+ + j\beta_4^+)s^4 + (\alpha_5^- + j\beta_5^+)s^5 + \cdots \tag{15}$$

LEMMA 2.2 *(Kharitonov's Theorem for Complex Polynomials, Kharitonov, 1979)*

$$\Delta \subset H \Longleftrightarrow K_1^+(s)\ ,\ K_2^+(s)\ ,\ K_3^+(s)\ ,\ K_4^+(s)\ ,\ K_1^-(s)\ ,\ K_2^-(s)\ ,\ K_3^-(s)\ , K_4^-(s) \in H. \tag{16}$$

Now consider the n_u-th, n_v-th order real interval polynomial families Γ_u and Γ_v. Denote their Kharitonov polynomials as $K_i^u(s)$, $i = 1, 2, 3, 4$ and $K_j^v(s)$, $j = 1, 2, 3, 4$ respectively.

Similarly, consider the n_u-th, n_v-th order complex interval polynomial families Δ_u and Δ_v. Denote their Kharitonov polynomials as $K_i^{+u}(s)$, $K_i^{-u}(s)$, $i = 1, 2, 3, 4$ and $K_j^{+v}(s)$, $K_j^{-v}(s)$, $j = 1, 2, 3, 4$ respectively.

For any function $f(x, y)$, define

$$f(\Gamma_u\,,\,\Gamma_v) = \{f(p_u(s)\,,\,p_v(s)) \mid p_u(s) \in \Gamma_u\,,\,p_v(s) \in \Gamma_v\} \tag{17}$$

$$f(\Delta_u\,,\,\Delta_v) = \{f(\delta_u(s)\,,\,\delta_v(s)) \mid \delta_u(s) \in \Delta_u\,,\,\delta_v(s) \in \Delta_v\}. \tag{18}$$

Specifically

$$\Gamma_u \times \Gamma_v = \{p_u(s) \times p_v(s) \mid p_u(s) \in \Gamma_u\,,\,p_v(s) \in \Gamma_v\} \tag{19}$$

$$\frac{\Gamma_u}{\Gamma_v} = \left\{\frac{p_u(s)}{p_v(s)} \mid p_u(s) \in \Gamma_u\,,\,p_v(s) \in \Gamma_v\right\}. \tag{20}$$

LEMMA 2.3 *(Hollot and Tempo, 1994) For any fixed complex number $z \in \mathbf{C}$, suppose the polynomial family $\Gamma_u - z\Gamma_v$ has a fixed order. Then*

$$\Gamma_u - z\Gamma_v \subset H \Longleftrightarrow K_i^u(s) - zK_j^v(s) \in H, \quad i, j = 1, 2, 3, 4. \tag{21}$$

If the location of z is known, then the number of critical vertices needed to be checked can further be reduced. For example, if z is on the negative (or positive) real axis, then only four out of the 16 critical vertices need to be checked, namely, if z is negative

$$\Gamma_u - z\Gamma_v \subset H \Longleftrightarrow K_i^u(s) - zK_i^v(s) \in H, \quad i = 1, 2, 3, 4; \tag{22}$$

if z is on the imaginary (or real) axis, then only eight critical vertices need to be checked; if z is in the left (or right) half of the complex plane, then only twelve critical vertices need to be checked, see Hollot and Tempo (1994), Wang (2003), Wang and Huang (1994), Barmish et al. (1992).

For complex polynomials, we have the following similar result:

LEMMA 2.4 *For any fixed complex number $z \in \mathbf{C}$, suppose the polynomial family $\Delta_u - z\Delta_v$ has a fixed order. Then*

$$\Delta_u - z\Delta_v \subset H \Longleftrightarrow K_i^{+u}(s) - zK_j^{+v}(s),\ \ K_i^{-u}(s) - zK_j^{-v}(s) \in H, \quad i, j = 1, 2, 3, 4. \tag{23}$$

THEOREM 2.1 *Consider the polynomial family*

$$a_m\Gamma_u^m + a_{m-1}\Gamma_u^{m-1}\Gamma_v + a_{m-2}\Gamma_u^{m-2}\Gamma_v^2 + \cdots\cdots + a_2\Gamma_u^2\Gamma_v^{m-2} + a_1\Gamma_u\Gamma_v^{m-1} + a_0\Gamma_v^m \tag{24}$$

where $a_k \in \mathbf{R}$, $k = 0, 1, \cdots, m$. Suppose it has a fixed order. Then

$$\begin{aligned} &a_m\Gamma_u^m + a_{m-1}\Gamma_u^{m-1}\Gamma_v + \cdots\cdots + a_1\Gamma_u\Gamma_v^{m-1} + a_0\Gamma_v^m \subset H \Longleftrightarrow \\ &a_m[K_i^u(s)]^m + a_{m-1}[K_i^u(s)]^{m-1}K_j^v(s) + \cdots\cdots \\ &\quad + a_1K_i^u(s)[K_j^v(s)]^{m-1} + a_0[K_j^v(s)]^m \in H, \quad i, j = 1, 2, 3, 4. \end{aligned} \tag{25}$$

Proof. Consider the polynomial

$$q(z) = a_m z^m + a_{m-1} z^{m-1} + a_{m-2} z^{m-2} + \cdots\cdots + a_2 z^2 + a_1 z + a_0. \quad (26)$$

Let $r = \max\{k \mid a_k \neq 0\}$. Then $q(z)$ can be expressed as

$$q(z) = a_r (z - z_1)(z - z_2) \cdots\cdots (z - z_{r-1})(z - z_r) \quad (27)$$

where $z_1, z_2, \cdots\cdots, z_{r-1}, z_r \in \mathbf{C}$. Hence, we have

$$a_m \Gamma_u^m + a_{m-1} \Gamma_u^{m-1} \Gamma_v + \cdots\cdots + a_1 \Gamma_u \Gamma_v^{m-1} + a_0 \Gamma_v^m \subset H$$

$$\Longleftrightarrow \quad \Gamma_v^m \left[a_r \left(\frac{\Gamma_u}{\Gamma_v}\right)^r + a_{r-1} \left(\frac{\Gamma_u}{\Gamma_v}\right)^{r-1} + \cdots\cdots + a_1 \left(\frac{\Gamma_u}{\Gamma_v}\right) + a_0 \right] \subset H$$

$$\Longleftrightarrow \quad \Gamma_v^m \left[a_r \left(\frac{\Gamma_u}{\Gamma_v} - z_1\right) \left(\frac{\Gamma_u}{\Gamma_v} - z_2\right) \cdots\cdots \left(\frac{\Gamma_u}{\Gamma_v} - z_{r-1}\right) \left(\frac{\Gamma_u}{\Gamma_v} - z_r\right) \right] \subset H$$

$$\Longleftrightarrow \quad a_r \Gamma_v^{m-r} (\Gamma_u - z_1 \Gamma_v)(\Gamma_u - z_2 \Gamma_v) \cdots\cdots (\Gamma_u - z_{r-1} \Gamma_v)(\Gamma_u - z_r \Gamma_v) \subset H$$

$$\Longleftrightarrow \quad \begin{cases} \Gamma_u - z_k \Gamma_v \subset H, \; k = 1, 2, \cdots\cdots, r-1, r, & r = m \\ \Gamma_u - z_k \Gamma_v \subset H, \; k = 1, 2, \cdots\cdots, r-1, r \text{ and } \Gamma_v \subset H, & r < m \end{cases}$$

$$\overset{\text{Lemmas 2.1\&2.3}}{\Longleftrightarrow} \quad \begin{cases} K_i^u(s) - z_k K_j^v(s) \in H, & \\ i, j = 1, 2, 3, 4, \; k = 1, 2, \cdots, r-1, r & r = m \\ & \\ K_i^u(s) - z_k K_j^v(s) \in H, & \\ i, j = 1, 2, 3, 4, \; k = 1, 2, \cdots, r-1, r & r < m \\ \text{and } \; K_j^v(s) \in H, \; \; j = 1, 2, 3, 4 & \end{cases} \quad (28)$$

$$\Longleftrightarrow \quad a_r [K_j^v(s)]^{m-r} [K_i^u(s) - z_1 K_j^v(s)][K_i^u(s) - z_2 K_j^v(s)] \cdots$$

$$\cdots [K_i^u(s) - z_{r-1} K_j^v(s)][K_i^u(s) - z_r K_j^v(s)] \in H, \quad i, j = 1, 2, 3, 4$$

$$\Longleftrightarrow \quad [K_j^v(s)]^m \left[a_r \left(\frac{K_i^u(s)}{K_j^v(s)} - z_1\right) \left(\frac{K_i^u(s)}{K_j^v(s)} - z_2\right) \cdots \right.$$

$$\left. \cdots \left(\frac{K_i^u(s)}{K_j^v(s)} - z_{r-1}\right) \left(\frac{K_i^u(s)}{K_j^v(s)} - z_r\right) \right] \in H, \quad i, j = 1, 2, 3, 4$$

$$\Longleftrightarrow \quad [K_j^v(s)]^m \left[a_r \left(\frac{K_i^u(s)}{K_j^v(s)}\right)^r + a_{r-1} \left(\frac{K_i^u(s)}{K_j^v(s)}\right)^{r-1} + \cdots \right.$$

$$\left. \cdots + a_1 \left(\frac{K_i^u(s)}{K_j^v(s)}\right) + a_0 \right] \in H, \quad i, j = 1, 2, 3, 4$$

$$\Longleftrightarrow \quad a_m [K_i^u(s)]^m + a_{m-1} [K_i^u(s)]^{m-1} K_j^v(s) + \cdots$$

$$\cdots + a_1 K_i^u(s) [K_j^v(s)]^{m-1} + a_0 [K_j^v(s)]^m \in H, \quad i, j = 1, 2, 3, 4. \quad (29)$$

This completes the proof. ■

From the proof of Theorem 2.1 and by Lemmas 2.2 and 2.4, we have

THEOREM 2.2 *Consider the polynomial family*

$$\begin{aligned} c_m\Delta_u^m &+ c_{m-1}\Delta_u^{m-1}\Delta_v + c_{m-2}\Delta_u^{m-2}\Delta_v^2 + \cdots\cdots + c_2\Delta_u^2\Delta_v^{m-2} \\ &+ c_1\Delta_u\Delta_v^{m-1} + c_0\Delta_v^m \end{aligned} \tag{30}$$

where $c_k \in \mathbf{C}$, $k = 0, 1, \cdots, m$. *Suppose it has a fixed order. Then*

$$\begin{aligned} & c_m\Delta_u^m + c_{m-1}\Delta_u^{m-1}\Delta_v + \cdots\cdots + c_1\Delta_u\Delta_v^{m-1} + c_0\Delta_v^m \subset H \Longleftrightarrow \\ & c_m[K_i^{+u}(s)]^m + c_{m-1}[K_i^{+u}(s)]^{m-1}K_j^{+v}(s) + \cdots\cdots \\ & \quad +c_1K_i^{+u}(s)[K_j^{+v}(s)]^{m-1} + c_0[K_j^{+v}(s)]^m \in H, \qquad (31) \\ & c_m[K_i^{-u}(s)]^m + c_{m-1}[K_i^{-u}(s)]^{m-1}K_j^{-v}(s) + \cdots\cdots \\ & \quad +c_1K_i^{-u}(s)[K_j^{-v}(s)]^{m-1} + c_0[K_j^{-v}(s)]^m \in H\,, \quad i,j = 1,2,3,4. \end{aligned}$$

REMARK 2.1 *We have established strong Kharitonov-like criteria for the stability of a class of polynomial families with nonlinearly correlated perturbations. The number of critical polynomials that need to be checked is independent of the order of the polynomial family.*

EXAMPLE 2.1 Consider a negative unity feedback system with the forward path composed of three identical blocks in tandem. Each block consists of an interval plant $\frac{N(s)}{D(s)}$ with negative unity feedback. Then, the characteristic polynomial of the closed-loop system is

$$[N(s)]^3 + [N(s) + D(s)]^3. \tag{32}$$

By Theorem 2.1, we only need to check 16 vertex systems for the stability of the entire uncertain system family. Furthermore, since all the roots of

$$q(z) = 2z^3 + 3z^2 + 3z + 1 \tag{33}$$

lie within the left half of the complex plane, only twelve out of the 16 vertex systems need to be checked to verify robust stability of the entire system family.

EXAMPLE 2.2 Consider a negative unity feedback system with the forward path composed of a controller and an interval plant $\frac{N(s)}{D(s)}$ in tandem. The controller is simply a gain k, but can be switched among $\{k_1\,, k_2\,, \cdots\cdots, k_m\}$ under different working conditions. Thus, robust stability of the entire system family is tantamount to

$$[k_1N(s) + D(s)][k_2N(s) + D(s)]\cdots\cdots[k_mN(s) + D(s)] \subset H. \tag{34}$$

By Theorem 2.1, we only need to check 16 vertex systems for the stability of the entire uncertain system family. Furthermore, since all the roots of

$$q(z) = (k_1z + 1)(k_2z + 1)\cdots\cdots(k_mz + 1) \tag{35}$$

lie on the real axis, only eight out of the 16 vertex systems need to be checked. Moreover, if $k_1, k_2, \cdots\cdots, k_m$ have the same sign, then only four out of the 16 vertex systems need to be checked, see Hollot and Tempo (1994), Wang (2003), Wang and Huang (1004), Barmish et al. (1992).

3. Some extensions

3.1. Extension to non-interval D-stability case

Given any stability region D in the complex plane $\mathbf{C}$, a polynomial $p(s)$ is said to be D-stable, denoted by $p(s) \in D$, if all its roots lie within D. A polynomial family P is said to be D-stable, denoted by $P \subset D$, if all polynomials in P are D-stable.

Let the uncertainty bounding set (hyperbox) be

$$Q = \{q = (q_1\,,\, q_2,\; \cdots\,,\, q_l)^T \mid q_i \in [q_i^-\,,\; q_i^+],\;\; i = 1, 2, \cdots, l\} \tag{36}$$

and define its one-dimensional edge set as

$$\begin{aligned} Q_E = \{q = (q_1\,,\, q_2\,,\, \cdots\,,\, q_l)^T \mid q_k \in [q_k^-\,,\, q_k^+] \;\text{ for some } k \in \{1, 2, \cdots, l\} \\ \text{and } q_i \in \{q_i^-\,,\, q_i^+\} \text{ for all } i \neq k\}. \end{aligned} \tag{37}$$

Consider the n_1-th, n_2-th order complex polynomials

$$n(s,q) = \sum_{i=0}^{n_1} c_i(q) s^i \tag{38}$$

$$d(s,q) = \sum_{j=0}^{n_2} b_j(q) s^j \tag{39}$$

where the complex coefficients $c_i(q)\,,\, b_j(q)$ are affine functions of the uncertain parameters $q = (q_1\,,\, q_2\,,\, \cdots\cdots\,,\, q_l)^T$, respectively.

In the sequel, we will suppose that D^c is a connected set. Note that Hurwitz stability and Schur stability are special cases of D-stability.

LEMMA 3.1 *For any fixed complex numbers $z_{01}\,,\, z_{02} \in \mathbf{C}$, suppose the polynomial family $\{z_{01}n(s,q) + z_{02}d(s,q) \mid q \in Q\}$ has a fixed order. Then*

$$\begin{aligned} &\{z_{01}n(s,q) + z_{02}d(s,q) \mid q \in Q\} \subset D \Longleftrightarrow \\ &\{z_{01}n(s,q) + z_{02}d(s,q) \mid q \in Q_E\} \subset D. \end{aligned} \tag{40}$$

Proof. Since the coefficients of $z_{01}n(s,q) + z_{02}d(s,q)$ are also affine functions of $q = (q_1\,,\, q_2\,,\, \cdots\cdots\,,\, q_l)^T$, the result follows directly from the Edge Theorem, see Bartlett, Hollot and Huang (1988), Fu and Barmish (1989). ∎

For notational simplicity, define

$$\begin{aligned} g(s,q) \;&=\; a_m[n(s,q)]^m + a_{m-1}[n(s,q)]^{m-1}d(s,q) \\ &+\; a_{m-2}[n(s,q)]^{m-2}[d(s,q)]^2 + \cdots\cdots \\ &+\; a_2[n(s,q)]^2[d(s,q)]^{m-2} + a_1 n(s,q)[d(s,q)]^{m-1} \\ &+\; a_0[d(s,q)]^m \end{aligned} \tag{41}$$

where $a_k \in \mathbf{C}\,,\; k = 0, 1, \cdots\cdots, m\,.$

THEOREM 3.1 *Consider the polynomial family*

$$\{g(s,q) \mid q \in Q\}. \tag{42}$$

Suppose it has a fixed order. Then

$$\{g(s,q) \mid q \in Q\} \subset D \Longleftrightarrow \{g(s,q) \mid q \in Q_E\} \subset D. \tag{43}$$

Proof. Consider the polynomial

$$q(z) = a_m z^m + a_{m-1} z^{m-1} + a_{m-2} z^{m-2} + \cdots\cdots + a_2 z^2 + a_1 z + a_0. \tag{44}$$

Let $r = \max\{k \mid a_k \neq 0\}$. Then $q(z)$ can be expressed as

$$q(z) = a_r (z - z_1)(z - z_2) \cdots\cdots (z - z_{r-1})(z - z_r) \tag{45}$$

where z_1 , z_2 , $\cdots\cdots$, z_{r-1} , $z_r \in \mathbf{C}$. Hence, we have

$$\begin{aligned}
\{g(s,q) \mid q \in Q\} \subset D \quad &\Longleftrightarrow \quad g(s,q) \in D\,,\ \forall q \in Q \\
\Longleftrightarrow \quad & [d(s,q)]^m \left\{ a_m \left[\tfrac{n(s,q)}{d(s,q)}\right]^m + a_{m-1} \left[\tfrac{n(s,q)}{d(s,q)}\right]^{m-1} + \cdots \right. \\
& \left. \cdots + a_1 \left[\tfrac{n(s,q)}{d(s,q)}\right]^1 + a_0 \right\} \in D\,,\ \forall q \in Q \\
\Longleftrightarrow \quad & [d(s,q)]^m \left\{ a_r \left[\tfrac{n(s,q)}{d(s,q)} - z_1\right] \left[\tfrac{n(s,q)}{d(s,q)} - z_2\right] \cdots \right. \\
& \left. \cdots \left[\tfrac{n(s,q)}{d(s,q)} - z_{r-1}\right] \left[\tfrac{n(s,q)}{d(s,q)} - z_r\right] \right\} \in D\,,\ \forall q \in Q \\
\Longleftrightarrow \quad & a_r [d(s,q)]^{m-r} [n(s,q) - z_1 d(s,q)][n(s,q) - z_2 d(s,q)] \cdots \\
& \cdots [n(s,q) - z_{r-1} d(s,q)][n(s,q) - z_r d(s,q)] \in D\,,\ \forall q \in Q \\
\Longleftrightarrow \quad & \begin{cases} n(s,q) - z_k d(s,q) \in D\,, & \\ k = 1, 2, \cdots\cdots, r-1, r\,,\ \forall q \in Q & r = m \\[1em] n(s,q) - z_k d(s,q) \in D\,, & \\ k = 1, 2, \cdots\cdots, r-1, r\,,\ \forall q \in Q & r < m \\ \text{and } \ d(s,q) \in D\,,\ \forall q \in Q & \end{cases}
\end{aligned}$$

$$\overset{\text{Lemma 3.1}}{\Longleftrightarrow} \quad \begin{cases} n(s,q) - z_k d(s,q) \in D\,, & \\ k = 1, 2, \cdots\cdots, r-1, r\,,\ \forall q \in Q_E & r = m \\[1em] n(s,q) - z_k d(s,q) \in D\,, & \\ k = 1, 2, \cdots\cdots, r-1, r\,,\ \forall q \in Q_E & r < m \\ \text{and } \ d(s,q) \in D\,,\ \forall q \in Q_E & \end{cases}$$

$$\begin{aligned}
&\iff a_r[d(s,q)]^{m-r}[n(s,q)-z_1d(s,q)][n(s,q)-z_2d(s,q)]\cdots\\
&\qquad \cdots[n(s,q)-z_{r-1}d(s,q)][n(s,q)-z_rd(s,q)]\in D\,,\ \forall q\in Q_E\\
&\iff [d(s,q)]^m\left\{a_r\left[\tfrac{n(s,q)}{d(s,q)}-z_1\right]\left[\tfrac{n(s,q)}{d(s,q)}-z_2\right]\cdots\right.\\
&\qquad\quad \left.\cdots\left[\tfrac{n(s,q)}{d(s,q)}-z_{r-1}\right]\left[\tfrac{n(s,q)}{d(s,q)}-z_r\right]\right\}\in D\,,\ \forall q\in Q_E\\
&\iff [d(s,q)]^m\left\{a_m\left[\tfrac{n(s,q)}{d(s,q)}\right]^m+a_{m-1}\left[\tfrac{n(s,q)}{d(s,q)}\right]^{m-1}+\cdots\right.\\
&\qquad\quad \left.\cdots+a_1\left[\tfrac{n(s,q)}{d(s,q)}\right]^1+a_0\right\}\in D\,,\ \forall q\in Q_E\\
&\iff g(s,q)\in D\,,\ \forall q\in Q_E \iff \{g(s,q)\mid q\in Q_E\}\subset D. \qquad (46)
\end{aligned}$$

This completes the proof. ■

REMARK 3.1 *Theorem 3.1 reveals that, for a class of polynomial family with nonlinearly correlated perturbations, D-stability of the entire family can be ascertained by only checking one-dimensional edge polynomials in this family.*

3.2. Extension to polynomial matrix families

Consider the uncertain polynomial matrix

$$M(s,q)=\begin{bmatrix} 2n(s,q)+3d(s,q) & 3n(s,q)+4d(s,q) & 0\\ 0 & 4n(s,q)+5d(s,q) & 2n(s,q)\\ 9d(s,q) & 6n(s,q) & 5n(s,q)+6d(s,q)\end{bmatrix} \qquad (47)$$

It is easy to see that

$$\begin{aligned}\det[M(s,q)] &= 16[n(s,q)]^3+176[n(s,q)]^2d(s,q)\\ &\quad +279n(s,q)[d(s,q)]^2+90[d(s,q)]^3.\end{aligned} \qquad (48)$$

By Theorem 3.1, we have

$$\{\det[M(s,q)]\mid q\in Q\}\subset D \iff \{\det[M(s,q)]\mid q\in Q_E\}\subset D. \qquad (49)$$

Namely, robust D-stability of the entire polynomial matrix family can be ascertained by only checking one-dimensional edges. More generally, for any uncertain polynomial matrix of the form

$$M(s,q)=[\alpha_{ij}n(s,q)+\beta_{ij}d(s,q)]_{n\times n} \qquad (50)$$

it is easy to see that the above edge result also holds. Moreover, if $n(s,q)$, $d(s,q)$ are replaced by interval polynomial families $\Gamma_u\,,\Gamma_v$ or $\Delta_u\,,\Delta_v$ as defined in the last section, then Kharitonov-like results can be established for robust Hurwitz stability of the corresponding polynomial matrix families.

THEOREM 3.2 *Consider the polynomial matrix family*

$$M(\delta_u(s), \delta_v(s)) = [\gamma_{ij}\delta_u(s) + \eta_{ij}\delta_v(s)]_{n\times n} \tag{51}$$

where $\delta_u(s) \in \Delta_u$, $\delta_v(s) \in \Delta_v$, *and* $\gamma_{ij}, \eta_{ij}, i, j = 1, 2, ..., n$ *are complex numbers. Then*

$$\begin{aligned} &\{\det[M(\delta_u(s), \delta_v(s))] \mid \delta_u(s) \in \Delta_u \, , \, \delta_v(s) \in \Delta_v\} \subset H \iff \\ &\{\det[M(K_i^{+u}(s) \, , \, K_j^{+v}(s))] \mid i, j = 1, 2, 3, 4\} \bigcup \\ &\qquad \{\det[M(K_i^{-u}(s) \, , \, K_j^{-v}(s))] \mid i, j = 1, 2, 3, 4\} \subset H. \end{aligned} \tag{52}$$

4. Some applications

A proper transfer function $\frac{p(s)}{q(s)}$ is said to be strictly positive real, denoted by $\frac{p(s)}{q(s)} \in SPR$, if

$$\begin{aligned} &1) \; q(s) \in H \\ &2) \; \Re\frac{p(j\omega)}{q(j\omega)} > 0 \, , \; \forall \omega \in R. \end{aligned} \tag{53}$$

Suppose $p(s)$, $q(s)$ have positive leading coefficients. Then, it is easy to see that

$$\frac{p(s)}{q(s)} \in SPR \iff \lambda p^2(s) + (1-\lambda)q^2(s) \in H, \lambda \in [0, 1]. \tag{54}$$

Now consider the proper interval transfer function family

$$T = \left\{ \frac{p_u(s)}{p_v(s)} \mid p_u(s) \in \Gamma_u \, , \, p_v(s) \in \Gamma_v \right\}. \tag{55}$$

In order to have

$$\frac{p_u(s)}{p_v(s)} \in SPR \, , \, p_u(s) \in \Gamma_u \, , \, p_v(s) \in \Gamma_v \tag{56}$$

we must have

$$\lambda\Gamma_u^2 + (1-\lambda)\Gamma_v^2 \subset H, \lambda \in [0, 1] \tag{57}$$

since $\lambda z^2 + (1-\lambda)$ has purely imaginary roots. By Theorem 2.1, we only need to have, see Hollot and Tempo (1994), Wang and Huang (1994), Barmish et al. (1992)

$$\lambda[K_1^u(s)]^2 + (1-\lambda)[K_4^v(s)]^2 \in H, \lambda \in [0, 1] \tag{58}$$

$$\lambda[K_2^u(s)]^2 + (1-\lambda)[K_3^v(s)]^2 \in H, \lambda \in [0, 1] \tag{59}$$

$$\lambda[K_3^u(s)]^2 + (1-\lambda)[K_1^v(s)]^2 \in H, \lambda \in [0, 1] \tag{60}$$

$$\lambda[K_4^u(s)]^2 + (1-\lambda)[K_2^v(s)]^2 \in H, \lambda \in [0,1] \tag{61}$$

$$\lambda[K_1^u(s)]^2 + (1-\lambda)[K_3^v(s)]^2 \in H, \lambda \in [0,1] \tag{62}$$

$$\lambda[K_2^u(s)]^2 + (1-\lambda)[K_4^v(s)]^2 \in H, \lambda \in [0,1] \tag{63}$$

$$\lambda[K_3^u(s)]^2 + (1-\lambda)[K_2^v(s)]^2 \in H, \lambda \in [0,1] \tag{64}$$

$$\lambda[K_4^u(s)]^2 + (1-\lambda)[K_1^v(s)]^2 \in H, \lambda \in [0,1]. \tag{65}$$

Equivalently

$$\begin{aligned}&\frac{K_1^u(s)}{K_4^v(s)}, \frac{K_2^u(s)}{K_3^v(s)}, \frac{K_3^u(s)}{K_1^v(s)}, \frac{K_4^u(s)}{K_2^v(s)},\\ &\quad \frac{K_1^u(s)}{K_3^v(s)}, \frac{K_2^u(s)}{K_4^v(s)}, \frac{K_3^u(s)}{K_2^v(s)}, \frac{K_4^u(s)}{K_1^v(s)} \in SPR.\end{aligned} \tag{66}$$

Namely, in order to guarantee that every member of the interval transfer function family T is strictly positive real, we only need to check eight specially selected vertex transfer functions, that is

$$\begin{aligned}&\frac{p_u(s)}{p_v(s)} \in SPR\ ,\ \forall p_u(s) \in \Gamma_u\ ,\ \forall p_v(s) \in \Gamma_v\\ &\Longleftrightarrow \frac{K_1^u(s)}{K_4^v(s)}, \frac{K_2^u(s)}{K_3^v(s)}, \frac{K_3^u(s)}{K_1^v(s)}, \frac{K_4^u(s)}{K_2^v(s)},\\ &\qquad \frac{K_1^u(s)}{K_3^v(s)}, \frac{K_2^u(s)}{K_4^v(s)}, \frac{K_3^u(s)}{K_2^v(s)}, \frac{K_4^u(s)}{K_1^v(s)} \in SPR.\end{aligned} \tag{67}$$

which is consistent with the result of Chapellat, Dahleh and Bhattacharyya (1991), and Wang and Huang (1991).

Moreover, for any $\gamma \in R$, in order to have

$$\gamma + \frac{p_u(s)}{p_v(s)} \in SPR\ ,\ p_u(s) \in \Gamma_u\ ,\ p_v(s) \in \Gamma_v, \tag{68}$$

we must have

$$\lambda[\gamma\Gamma_v + \Gamma_u]^2 + (1-\lambda)\Gamma_v^2 \subset H, \lambda \in [0,1]. \tag{69}$$

since $\lambda(\gamma+z)^2 + (1-\lambda)$ has roots either at first and fourth quadrants (when $\gamma < 0$) or at second and third quadrants (when $\gamma > 0$). By Theorem 2.1, we only need to check twelve vertices to guarantee robust stability, see Hollot and Tempo (1994), Wang (2003), Wang and Huang (1994), Barmish et al. (1992). Namely, in order to guarantee that

$$\gamma + \frac{p_u(s)}{p_v(s)} \in SPR\ ,\ p_u(s) \in \Gamma_u\ ,\ p_v(s) \in \Gamma_v \tag{70}$$

we only need to check the same property for twelve specially selected vertex transfer functions.

REMARK 4.1 *The above result can be easily extended to the case of complex interval transfer function family. Namely, every member in the complex interval transfer function family is strictly positive real, if and only if, sixteen specially selected vertex transfer functions in this family are strictly positive real, see Wang and Huang (1991).*

5. Conclusions

Some Kharitonov-like robust Hurwitz stability criteria have been established for a class of complex polynomial families with nonlinearly correlated perturbations. These results have been extended to the polynomial matrix case and non-interval D-stability case. Applications of these results in testing of robust strict positive realness of real and complex interval transfer function families have also been presented.

Acknowledgements

The author would like to express his sincere gratitude to Dr Jan W. Owsinski and the two anonymous reviewers for their constructive suggestions.

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