Control and Cybernetics

vol. 33 (2004) No. 1

Reachability and controllability of time-variant discrete-time positive linear systems

by

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Abstract: In this paper necessary and sufficient conditions (and criteria) for null-controllability, reachability and controllability of time-variant discrete-time positive linear systems are established. These properties appear to be entirely structural properties, that is, they do depend on the zero-nonzero pattern of the pair $(A(k),\ B(k)) \geq 0$ and do not depend on the values of its entries. An interesting phenomenon has been revealed namely the time needed to reach the origin for a null-controllable system as well as the time to reach a (non-negative) state from the origin for a reachable system can be less, equal or greater than the dimension of the system. This phenomenon has no equivalent in the case of time-invariant discrete-time positive linear systems where this time is always less or equal to the system dimension. Examples are provided.

Keywords: positive linear systems, discrete-time systems, time-variant systems, reachability and controllability, non-negative matrices.

1. Introduction

Positive systems are defined as systems in which the state trajectory lies entirely in the non-negative orthant whenever the initial state and the inputs are non-negative. A number of models having positive linear systems behaviour can be found in engineering, economics, pharmacology and medicine, biology and other fields, see Farina and Rinaldi (2000), Kaczorek (2002), Luenberger (1979). Reachability and controllability are fundamental properties of the system with direct implications in a number of control problems. While the reachability and controllability properties of time-invariant discrete-time positive linear systems have been well studied during the last decade and a variety of criteria in both algebraic and digraph forms (Bru et al., 2003; Caccetta and Rumchev, 2000; Rumchev and James, 1998; Rumchev, 2003) for recognising these important properties have been developed, there is not much work done for time-variant positive linear systems except, possibly, Kaczorek (2001), where the author considers reachability of time-variant continuous-time positive linear systems and obtains sufficient type conditions for reachability. In this paper we study null-controllability, reachability and controllability properties of time-variant discrete-time positive linear systems. Our motivation to consider this problem is twofold. We are interested in examining the problem in the course of developing a system theory for positive systems. At the same time such models naturally arise in some inventory and production systems.

2. Preliminaries

Consider the positive time-varying linear discrete time system, Kaczorek (2002),

$$x(k+1) = A(k)x(k) + B(k)u(k),\ k = k_0, k_0+1, k_0+2, \dots \tag{1}$$

$$A(k) \in R_+^{n\times n},\ B(k) \in R_+^{n\times m},\ u(k) \in R_+^m, \tag{2}$$

where $x(k)$ is state of the system at time k, $u(k)$ is the control vector sequence and for each pair of positive integers (r,s) the symbol $R_+^{r\times s}$ denotes the set of all $r\times s$ real matrices (vectors) with non-negative entries $a_{ij}(k) \geq 0$ for all k.

The reachability matrix of the pair $(A(k), B(k)) \geq 0$ (and the system (1)-(2) is defined as

$$\begin{aligned}&\Re(k,k_0) = \\ &\quad \left[\ B(k-1)\ \ A(k-1)B(k-2)\ \ \dots\ \ A(k-1)A(k-2)\cdots A(k_0+1)B(k_0)\ \right], \\ &\quad k > k_0,\end{aligned}$$

$$\Re(k_0,k_0) = I. \tag{3}$$

The reachability matrix $\Re(k,k_0)$ is clearly a non-negative matrix for $k \geq k_0$, i.e. $\Re(k,k_0) \geq 0$. By introducing the reachability matrix the state $x(k)$ can be represented as

$$x(k) = \Phi(k,k_0)x(k_0) + \Re(k,k_0)u^{(k)}, \tag{4}$$

where the transition matrix $\Phi(k,k_0)$ is given by

$$\Phi(k,k_0) = A(k-1)A(k-2)\dots A(k_0+1)A(k_0) \text{ for } k > k_0,\ \Phi(k_0,k_0) = I \tag{5}$$

and $u^{(k)} = [u^T(k-1), \cdots, u^T(k_0+1), u(k_0)]^T$ is the expanded control vector.

The positive system (1)-(2) (and the nonnegative pair $(A(k), B(k)) \geq 0$) is said to be, Rumchev and James (1989),

a) *reachable* (or *controllable-from-the-origin*) if for any state $x \in R^n_+, x \neq 0$, and some finite k there exists a non-negative control sequence $\{u(s),\ s = k_0, k_0+1, k_0+2, \ldots, k-1\}$ that transfers the system from the origin to the state $x = x(k)$;
b) *null-controllable* (or *controllable-to-the-origin*) if for any state $x \in R^n_+$ and some finite k there exists a non-negative control sequence $\{u(s),\ s = k_0, k_0+1, k_0+2, \ldots, k-1\}$ that transfers the system from the state $x = x(k_0)$ to the origin $x(k) = 0$;
c) *controllable* if for any non-negative pair $(x_0, x) \in R^n_+$ and some finite k there exists a non-negative control sequence $\{u(s),\ s = k_0, k_0+1, k_0+2, \ldots, k-1\}$ that transfers the system from the state $x_o = x(k_0)$ to the state $x = x(k)$.

A column vector with exactly one non-zero entry is called *monomial*. The product of a non-singular diagonal matrix and a permutation matrix is called *monomial matrix*. A monomial matrix consists of linearly independent monomial columns. A monomial vector is called i-monomial if the non-zero entry is in the i-th position. All monomial matrices (vectors) in this paper are non-negative. An $n \times n$ matrix A is called *nil-potent* if $A^s = 0$ for some $s \leq n$ but $A^k \neq 0$ for $k < s$. The integer s is called the index of nil-potency of the matrix A.

The following result is proved in Coxson, Larson and Schneider (1987), see also Caccetta and Rumcher (2000).

LEMMA 2.1 *Let A be an $n\times n$ non-negative matrix and b be an $n\times 1$ non-negative column vector. If there is an i-monomial in the sequence $\{A^k b,\ k = n, n+1, \ldots\}$, then there is an i-monomial in the sequence $\{A^k b,\ k = 0, 1, \ldots, n-1\}$.*

Lemma 2.1 simply tells us (Caccetta and Rumchev, 2000) that all monomial columns in the reachability matrix $\Re_k$ for $k > n$ of a time-invariant discrete-time positive linear system (with scalar as well as with vector control sequences) are in the reachability matrix $\Re_n$.

Let $D(A)$ be the *digraph* of an $n \times n$ non-negative matrix A constructed as follows: the set of vertices of $D(A)$ is denoted as $N = \{1, 2, \ldots, n\}$ and there is an arc (i, j) in $D(A)$ if and only if $a_{ij} > 0$; the set of arcs is denoted by U. A *walk* in $D(A)$ is an alternating sequence of vertices and arcs of D, i.e. $\{i_1, (i_1, i_2), i_2, \ldots, (i_k, i_k+1), i_k+1\}$. A walk is called *closed* if $i_1 = i_{k+1}$ and *spanning* if it passes through all the vertices of $D(A)$. A walk is said to be a *path* if all of its vertices are distinct and a *cycle* if it is a closed path. The path length is defined to be equal to the number of arcs in the path. A cycle of length one is called a *loop*.

One way to associate a matrix with digraph is by the use of *adjacency matrix* $\tilde{A}$. The entries of the adjacency matrix $\tilde{A}$ are defined as

$$\tilde{a}_{ij} = \begin{cases} 1 & \text{if } (i,j) \in D(A), \\ 0 & \text{if } (i,j) \notin D(A). \end{cases}$$

The matrix $\tilde{A}$ is a binary matrix. It is clear that $\tilde{A} \geq 0$ and $D(\tilde{A}) = D(A)$, where A is any non-negative matrix having the same zero-nonzero pattern as $\tilde{A}$.

Let $D_1 = \{N_1, U_1\}$ and $D_2 = \{N_2, U_2\}$ be two digraphs. The operation union $D_1 \cup D_2$ produces the digraph $\{N_1 \cup N_2, U_1 \cup U_2\}$. Thus, if the vertex sets are the same, then the union of two digraphs is just the superposition of their arcs. Given m $n \times n$ nonnegative matrices A_1, A_2, ..., A_m, we define a joint digraph $D(A_1, A_2, \ldots, A_m)$ as $D(A_1) \cup D(A_2) \cup \ldots \cup D(A_m)$ in which each arc is labelled (coloured) with a subset of $\{1, 2, \ldots, m\}$ depending upon which of the digraphs $D(A_1)$, $D(A_2)$, ..., $D(A_m)$ includes that arc. The digraph of the product of m $n \times n$ nonnegative matrices A_1, A_2, ..., A_m is denoted as $D(A_1 A_2 \ldots A_m)$.

LEMMA 2.2 *(Bru and Johnson, 1993)*
There is an arc from i to j in $D(A_1 A_2 \ldots A_m)$ if and only if there is a path of length m from i to j in the joint graph $D(A_1, A_2, \ldots, A_m)$ coloured $1, 2, \ldots, m$, in that order.

Proof. The matrices $A_s \geq 0$, $s = 1, 2, \ldots, m$. Thus, there is an arc $(i, j) \in D(A_1 A_2 \ldots A_m)$ if and only if there are arcs $(i, s_1) \in D(A_1)$, $(s_1, s_2) \in D(A_2)$, ..., $(s_{m-1}, s_m) \in D(A_m)$. These arcs are coloured in the order $1, 2, \ldots, m$ in the joint graph $D(A_1, A_2, \ldots, A_m)$. ■

The inverse of Lemma 2.2 also holds true:

LEMMA 2.3 *There is no arc from i to j in $D(A_1 A_2 \ldots A_m)$ if and only if there is no path of length m from i to j in the joint graph $D(A_1, A_2, \ldots, A_m)$ coloured $1, 2, \ldots, m$, in that order.*

3. Main results

3.1. Null-controllability

THEOREM 3.1 *The positive system (1)-(2) is null-controllable if and only if there exists a finite time $s \geq k_o$ such that the transition matrix $\Phi(s, k_0) = 0$.*

Proof. Since the system is positive the control sequence does not contribute to reaching the origin so the *if part* of the proof readily follows from (4). To prove the *only if part* assume that null-controllability of the system (1)-(2) does not imply $\Phi(s, k_0) = [\phi_{ij}(s, k_o)] = 0$, that is, there exists a finite time $s \geq k_o$ such that $\Phi(s, k_0)x = 0$ for every $x \geq 0$ but $\Phi(s, k_0) \neq 0$. Without loss of generality take $\phi_{rp}(s, k_o) > 0$, $\phi_{ij}(s, k_o) = 0$ for $i \neq r$ and $j \neq p$, and $x = e_p$. Then $[\Phi(s, k_0)x]_r > 0$. A contradiction. The theorem is proved. ■

REMARK 3.1 *The transitivity property of the transition matrix implies that the system (1)-(2) is null-controllable for any j and s, $k_o \leq j < s$ such that $\Phi(s, j) = 0$.*

EXAMPLE 3.1 *Consider the system (1)-(2) with a system matrix given by*

$$A(k) = \begin{bmatrix} 0 & \left(\frac{1}{4}\right)^k & \left(\frac{1}{2}\right)^k \\ \delta_{1k} + \delta_{2k} & 0 & k \\ 0 & 0 & 0 \end{bmatrix}, \ k = 0, 1, 2, \ldots,$$

where $\delta_{ij} = \begin{cases} 1, & i = j \\ 0 & i \neq j \end{cases}$ *i.e. the Kronecker* δ.
It is easy to see that

$$A(1)A(0) \neq [0] \text{ and } A(2)A(1)A(0) \neq [0]$$

but

$$A(3)A(2)A(1)A(0) = \\ \begin{bmatrix} 0 & \frac{1}{64} & \frac{1}{8} \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{16} & \frac{1}{4} \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{2} \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so the system is null-controllable in four steps while the dimension of the system is three.

THEOREM 3.2 *The system (1)-(2) is null-controllable if and only if for some finite* $s \geq k_o$ *there exists no path of length* $(s - k_o)$ *in the joint diagraph* $D(A(s-1), \ldots, A(k_o))$ *coloured with* $1, 2, \ldots, (s - k_o)$ *in the order of the matrices in the product from the left to the right.*

Proof. Let the system (1)-(2) be null-controllable. Then, from Theorem 3.1, with necessity $\Phi(s, k_0) = 0$ for some finite $s \geq k_o$ and, therefore, there is no arc between any two vertices in the digraph $D(\Phi(s, k_0))$. This implies, according to Lemma 2.3, that there is no path of length $(s - k_o)$ in the joint graph $D(A(s-1), \ldots, A(k_o))$ coloured $1, 2, \ldots, (s-k_o)$ in order. The *only if* part is thus proved.

Assume now that there is no path of length $(s-k_o)$ in the joint graph $D(A(s-1), \ldots, A(k_o))$ coloured $1, 2, \ldots, (s - k_o)$ in order. Then, by Lemma 2.3, there is no arc between any two vertices in $D(A(s-1) \ldots A(k_o))$, which corresponds to all the entries of the transition matrix $\Phi(s, k_0) = A(s-1) \ldots A(k_o)$ being zero. Since $\Phi(s, k_0) = 0$, according to Theorem 3.1, the system (1)-(2) is null-controllable. ■

EXAMPLE 3.2 *Consider the positive system with*

$$A(k) = \begin{bmatrix} 0 & \left(\frac{1}{2}\right)^k & k^2 \\ 0 & 0 & k+2 \\ 0 & k^2 - k & 0 \end{bmatrix} \geq 0, \quad k = 1, 2, \ldots .$$

The transition matrix $\Phi(3, 0) = A(2)A(1)A(0) = 0$, *so by Theorem 3.1 the positive system is null-controllable in three steps.*

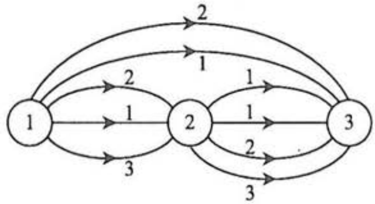

Figure 1.

The joint digraph $D(A(2), A(1), A(0))$ coloured in order 1, 2, 3 that is $A(2)$, $A(1)$, $A(0)$, respectively, is given in Fig. 1. There does not exist a path of length three from any vertex coloured with 1, 2, 3 in order in the above joint digraph. Therefore, by Theorem 3.2 the system is null-controllable in three steps.

Since $A(k) \geq 0$ and $\Phi(k, k_0) = A(k-1)A(k-2)\ldots A(k_0)$, $k_0 \leq k$, Theorem 3.1 and Theorem 3.2 tell us that *null-controllability* of the system (1)-(2) is an entirely structural *property*. It depends on the zero-nonzero patterns of the system matrix $A(k)$, $k \geq k_0$, and does not depend on the values of its entries. The *time* needed to reach the origin can be less, equal or *greater than the dimension* of the system. This *phenomenon* has no equivalent in the case of time-invariant positive linear systems where the time of reaching the origin is always less or equal to the dimension of the system, Caccetta and Rumchev (2000). It is important for many reasons to determine the classes of time-variant discrete-time positive linear systems that are null-controllable and the time of reaching the origin is less or at most equal to the dimension of the system. Theorems 3.3 and 3.4 below define such classes of systems.

THEOREM 3.3 *Let the adjacency matrix $\tilde{A}(k) = \tilde{A} \geq 0$ be a constant matrix for $k_o + j \leq k < k_o + j + n$ and let $\tilde{A}$ be a nil-potent matrix. Then the positive system (1)-(2) is null-controllable and the time k of reaching the origin is $k < k_o + j + n$. In particular, if $j = 0$ the origin is reachable in $k - k_0 \leq n$ steps.*

Proof. The zero-nonzero pattern of the transition matrix $\Phi(k, k_0)$ depends on the zero-nonzero patterns of $A(k)$, $k \geq k_0$, only so that the adjacency matrix of $\Phi(k_0 + j + n, k_0)$ is given by

$$\tilde{\Phi}(k_0 + j + n, k_0) = \tilde{\Phi}(k_0 + j + n, k_0 + j)\tilde{\Phi}(k_0 + j, k_0).$$

Let now $\tilde{A}(k) = \tilde{A}$ for $k_o + j \leq k < k_o + j + n$ and $\tilde{A}$ be a nil-potent matrix, i.e. $\tilde{A}^n = 0$. Then

$$\tilde{\Phi}(k_0 + j + n, k_0) = \tilde{A}^n \tilde{\Phi}(k_0 + j, k_0) = 0,$$

so the system (1)-(2) is null-controllable by Theorem 3.1. For $j = 0$

$$\tilde{\Phi}(k_0 + n, k_0) = \tilde{A}^n \tilde{\Phi}(k_0, k_0) = \tilde{A}^n I = 0,$$

and the origin is reachable in $s \leq n$ steps, where $s = k - k_0$ is the index of nil-potency of the adjacency matrix $\tilde{A}$ and k is the time of reaching the origin. This completes the proof. ■

THEOREM 3.4 *Let $\tilde{A}(k) \geq 0$ be a nil-potent matrix for $k = 0, 1, 2, \ldots, n-1$ and let $\tilde{A}(k) \leq \tilde{A}(k+1)$ for at least $k = 0, 1, 2, \ldots, n-2$. Then the positive system (1)-(2) is null-controllable and the origin can be reached in $s \leq n$ steps.*

Proof. Since $\tilde{A}(n-1)$ is nil-potent, $\tilde{A}^n(n-1) = 0$ and therefore there are no arcs between any two vertices in the digraph $D(\tilde{A}^n(n-1))$. Consequently, there is no path of length n in the joint digraph $D(\tilde{A}(n-1), \ldots, \tilde{A}(n-1))$ coloured in order $1, 2, \ldots, n$. By hypothesis $\tilde{a}_{ij}(k) \leq \tilde{a}_{ij}(k+1)$ for $i,\ j = 1, \ldots, n$ and $k = 0, 1, 2, \ldots, n-2$, so that the joint digraph $D(\tilde{A}(n-1), \ldots, \tilde{A}(1), \tilde{A}(0))$ contains only the arcs $(i,j) \in D(\tilde{A}(n-1), \ldots, \tilde{A}(n-1))$, but coloured in order $1, 2, \ldots, s(i,j)$ with $s(i,j) \leq n$. This means that there is no path of length n in $D(A(n-1), \ldots, A(1), A(0))$ coloured in order $1, 2, \ldots, n$, since there is no path of length n in $D(\tilde{A}(n-1), \ldots, \tilde{A}(n-1))$ coloured in the same order. If there is no path of length n in the joint digraph $D(\tilde{A}(n-1), \ldots, \tilde{A}(1), \tilde{A}(0))$ coloured in order $1, 2, \ldots, n$, then by Lemma 2.3 there are no edges between any two vertices in $D(\tilde{A}(n-1), \ldots, \tilde{A}(1), \tilde{A}(0))$ and hence $\Phi(n, 0) = 0$. The system (1)-(2) is null-controllable. ■

3.2. Reachability

THEOREM 3.5 *The non-negative pair $(A(k), b(k)) \geq 0$ (and the system (1)-(2)) is reachable if and only if, for some finite s, the reachability matrix $\Re(s, k_0)$ contains an $n \times n$ monomial submatrix M, i.e. $M \subseteq \Re(s, k_0)$.*

Proof. Considering the *if part* of the theorem assume that $M \subseteq \Re(s, k_0) \geq 0$ for some finite s. Set all the controls that do not correspond to the columns of M equal to zero. Then, from (4) for $x(k_0) = 0$,

$$x(s) = \sum_{j=1}^{n} \alpha_j e_j \text{ with } \alpha_j \geq 0 \text{ for all } j = 1, 2, \ldots, n, \tag{6}$$

where e_j are the basis unit vectors and α_j are scalar multiples of the controls corresponding to the respective monomial column of M. The non-negative linear combination (6) represents the non-negative orthant (a polyhedral cone). Therefore, for every $x = x(s) \geq 0$ it is always possible to find a non-negative control sequence that transfers the system from the origin into the state x. The system (1)-(2) is reachable.

To prove the *only if part* assume that the system (1)-(2) is reachable for some finite s but the reachability matrix $\Re(s, k_0)$ does not contain a monomial submatrix. We may assume without loss of generality that $\Re(s, k_0)$ contains n linearly independent columns, $(n-1)$ of which are monomial, say scalar multiples of the unit basis vectors $e_1, \dots, e_{n-1}$, respectively, and the nth column has two positive entries, that $\alpha_n(e_n + e_j)$ for $j \in \{1, \dots, n\}$. Then,

$$x(s) = \sum_{j=1}^{n} \alpha_j e_j + \alpha_n(e_n + e_j) \text{ with } \alpha_j \geq 0 \text{ for all } j = 1, 2, \dots, n. \tag{7}$$

It is not difficult to see from (7) that the states on the ray $\alpha_n e_n$, $\alpha_n \geq 0$ cannot be reached by non-negative controls. Thus, the system is not reachable, which contradicts the assumption. The theorem is proved. ■

REMARK 3.2 *Note that in Theorem 3.5 the time of reaching a non-negative state from the origin for a reachable time-variant discrete-time positive linear system can be less, equal or greater than the dimension of the system. Example 3.3 illustrates this.*

EXAMPLE 3.3 *The non-negative pair $(A(k), b(k)) \geq 0$ with*

$$A(k) = \begin{bmatrix} 0 & 0 & k^2 - k \\ k^2 - k & 0 & 0 \\ 0 & \left(\frac{1}{2}\right)^k & k^2 - k \end{bmatrix}, \quad b(k) = \begin{bmatrix} \left(\frac{1}{2}\right)^k \\ 0 \\ 0 \end{bmatrix}, \quad k = 0, 1, 2, \dots$$

is not reachable in three steps but it is reachable in four steps since

$$\Re(3,0) = \begin{bmatrix} b(2) & A(2)b(1) & A(2)A(1)b(0) \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is not monomial but $\Re(4,0)$ does contain a monomial submatrix,

$$\begin{aligned} \Re(4,0) &= \begin{bmatrix} b(3) & A(3)b(2) & A(3)A(2)b(1) & A(3)A(2)A(1)b(0) \end{bmatrix} = \\ &= \begin{bmatrix} \frac{1}{8} & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 \end{bmatrix}. \end{aligned}$$

It is interesting to notice that a similar phenomenon has been observed in the study of singular linear systems Kaczorek, 2002, p.164, Example 3.14, where the system with $n = 2$ is not reachable in two steps but it is reachable in three steps.

At the same time the system considered above is null-controllable in two steps since the transition matrix $\Phi(2,0) = 0$.

THEOREM 3.6 *Let $(\tilde{A}(k), \tilde{B}(k)) = (\tilde{A}, \tilde{B})$ for $k \geq k_o$, where the adjacency matrices $\tilde{A}$ and $\tilde{B}$ are constant matrices. Then if the system (1)-(2) is reachable it is reachable in $s \leq n$ steps.*

Proof. Since $(\tilde{A}(k), \tilde{B}(k)) = (\tilde{A}, \tilde{B})$ for $k \geq k_o$ the pair $(A(k), B(k)) \geq 0$ and $(\tilde{A}, \tilde{B}) \geq 0$ have the same zero-nonzero pattern. Thus, $(A(k), B(k))$ is reachable if and only if the pair $(\tilde{A}, \tilde{B})$ is reachable. By Lemma 2.1, all monomial columns in the reachability matrix $\Re_k$ for $k > n$ of $(\tilde{A}, \tilde{B}) \geq 0$ are in the reachability matrix $\Re_n$ so that if the pair $(\tilde{A}, \tilde{B}) \geq 0$ is reachable it is reachable in $s \leq n$ steps. The theorem is proved. ∎

3.3. Controllability

THEOREM 3.7 *The system (1)-(2) is controllable if it is null-controllable and reachable.*

Proof. Let $(x_0, x) \in \Re_+^n$ and assume the system (1)-(2) is null-controllable and reachable. Since the system (1)-(2) is null-controllable, according to Theorem 3.1, $\Phi(s, k_0) = 0$ for (some) finite s, and the system can reach the origin from $x_o = x(k_o)$ in s steps. On the other hand, by hypothesis, the system (1)-(2) is reachable, so it follows that for some finite k and any state $x \in R_+^n$ there exists a nonnegative control sequence such that the system (1)-(2) can move from $x(k_0) = 0$ into $x = x(k)$. Then it readily follows from linearity of the system (as well as from (4)) that there exists a non-negative control sequence such that the system can be moved from the state x_o into the state x in $\max\{s, k\}$ and thus, according to Definition 2.1, the system (1)-(2) is controllable. ∎

EXAMPLE 3.4 *The non-negative pair $(A(k), b(k)) \geq 0$ in Example 3.3 is null-controllable in two steps but reachable in four, and so the system can be transferred from any $x_0 \geq 0$ into any $x \geq 0$ in four steps.*

4. Conclusions

In this paper necessary and sufficient conditions (and criteria) for null-controllability, reachability and controllability of time-variant discrete-time positive linear systems are established. These properties appear to be entirely structural properties. They do depend on the zero-nonzero pattern of the pair $(A(k), B(k)) \geq 0$ and do not depend on the values of its entries. An interesting phenomenon has been revealed namely the time needed to reach the origin for a null-controllable system as well as the time to reach a (non-negative) state from the origin for a reachable system can be less, equal or greater than the dimension of the system. This phenomenon has no equivalent in the case of time-invariant discrete-time positive linear systems where this time is always less or equal to the system dimension. Classes of null-controllable and reachable time-variant discrete-time positive linear systems with transition time less or equal to the dimension of the system are identified. Examples are provided.

The authors wish to thank the anonymous referees for their comments.

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