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Maximal stability bounds of discrete-time singularly perturbed systems

by

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Abstract: The exact stability bound of the parasitic parameter for a discrete-time singularly perturbed system is determined by the linear fractional transformation (LFT) framework. Two systematic approaches including time-domain and frequency-domain methods are proposed to solve this problem based on the unified LFT framework. One employs the Kronecker product of LFTs and the guardian map theory. The other is to plot the eigenvalue loci of a real rational function matrix. Two examples are given to show the feasibility of the approaches.

Keywords: discrete-time singularly perturbed system, stability bound, eigenvalue loci, Kronecker product of LFTs.

1. Introduction

Singular perturbations arise naturally in many control systems as a result of the presence of small physical parameters, such as small time constant, mass, capacitance, etc. Singularly perturbed systems or two-time-scale systems, consist of a slow subsystem and a fast subsystem. An overview of discrete-time or continuous-time singularly perturbed systems was given in Naidu and Pao (1985), Naidu, Price and Hibey (1987), Kokotovic, Khalil and O'Reilly (1986).

The stability bound problem is of great practical significance for analysis and synthesis in singularly perturbed systems. For continuous-time cases, this problem has been investigated by various methods and some solutions have been proposed in Fang (1988), Chen and Lin (1990), Sen and Datta (1993), Mustafa

(1995), Li et al. (1997), Chen and Lin (1999). For discrete-time cases, an approach based on Nyquist plot to determine the stability bounds was developed in Li and Li (1992). In Kafri and Abed (1996), the authors used guardian map theory to address this problem and gave less stringent assumption than Li and Li (1992). Recently, Li, Chiou and Kung (1999) and Ghosh, Sen and Datta (1999) used the critical stability criterion to solve this problem by the bialternate product.

In this paper two systematic approaches in time domain and frequency domain are proposed to determine the exact stability bounds of a discrete-time singularly perturbed system. The Schur stability is considered. It means that all eigenvalues of the system lie in the open unit circle. The approaches are essentially based on the LFT framework which was successfully applied to continuous cases, see Chen and Li (1999). For the time domain approach, the stability bounds are determined by using the Kronecker product of LFTs and the guardian map theory. For the frequency domain approach, they can be also determined by plotting the eigenvalue loci of a rational function matrix. It is seen that the results in Li and Li (1992) can be easily obtained by the frequency domain approach. An appealing advantage of the system in LFT description is to provide a unified framework for evaluating the exact stability bounds.

In Li and Li (1992), Li, Chiou and Kung (1999), a nonsingularity assumption for a subsystem matrix was made, but it is not necessary in our approaches. Compared with Kafri and Abed (1996), the LFT framework proposed in this paper provides a matrix with smaller dimension, which is used to find out the stability bounds in constructing the guardian map. Although the method from Ghosh, Sen and Datta (1999) could be used to reduce the dimension of the computational matrix, however, it involved a one-dimensional search over the positive real domain.

This paper is organized as follows. In the following section some preliminaries are briefly reviewed and the LFT description systems are proposed. Main results for determining the exact bounds by time-domain and frequency-domain approaches are given in Sections 3 and 4, respectively. Two examples are given in Section 5 to show the feasibility of the approaches. Finally, a brief conclusion will be given in the last section.

2. Preliminaries and LFT description systems

Suppose that M is a matrix partitioned as $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ and $\triangle$ is a matrix of appropriate dimension. The upper and lower linear fractional transformations (LFTs) are defined as, Doyle, Packard and Zhon (1991)

$$F_u(M, \triangle) := M_{22} + M_{21}\triangle(I - M_{11}\triangle)^{-1}M_{12}, \tag{1}$$

$$F_l(M, \triangle) := M_{11} + M_{12}\triangle(I - M_{22}\triangle)^{-1}M_{21}, \tag{2}$$

respectively. Clearly, the existence of the LFTs depends, respectively, on the invertibility of matrices $I - M_{11}\Delta$ and $I - M_{22}\Delta$.

The Kronecker product of two square real matrices A and B is denoted by $A \otimes B$, Lancester and Tismenetsky (1985). If A is an $m_1 \times m_1$ matrix and B is an $m_2 \times m_2$ matrix, then $A \otimes B$ is an $m_1 m_2 \times m_1 m_2$ matrix with (i,j)-th block $a_{ij}B$. The Kronecker product of two LFTs was established in Lin and Chen (1999).

LEMMA 2.1 *(Lin and Chen, 1999) Let $\Delta_M \in \mathsf{C}^{q_2 \times p_2}$ and $\Delta_N \in \mathsf{C}^{s_2 \times r_2}$. Suppose that M and N are complex matrices partitioned as*

$$\begin{aligned} M &= \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \mathsf{C}^{(p_1+p_2)(q_1+q_2)}, \\ N &= \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \in \mathsf{C}^{(r_1+r_2)\times(s_1+s_2)} \end{aligned}$$

then

$$F_l(M, \Delta_M) \otimes F_l(N, \Delta_N) = F_l(G, \Delta_G)$$

where

$$G = \left[\begin{array}{c:cc} M_{11} \otimes N_{11} & I_{p_1} \otimes N_{12} & M_{12} \otimes N_{11} \\ \hdashline M_{11} \otimes N_{21} & I_{p_1} \otimes N_{22} & M_{12} \otimes N_{21} \\ M_{21} \otimes I_{s_1} & 0 & M_{22} \otimes I_{s_1} \end{array}\right] = \left[\begin{array}{c:c} G_{11} & G_{12} \\ \hdashline G_{21} & G_{22} \end{array}\right],$$

$$\Delta_G = \begin{bmatrix} I_{p_1} \otimes \Delta_N & 0 \\ 0 & \Delta_M \otimes I_{s_1} \end{bmatrix}.$$

The discrete-time models of sampling singularly perturbed continuous systems can be classified into two categories: the slow sampling model $\sum_1$ and the fast sampling model $\sum_2$, described in Naidu, Price and Hibey (1987), Li and Li (1992), Kafri and Abed (1996), Li, Chiou and Kung (1999), and Ghosh, Sen and Datta (1999),

$$\sum\nolimits_1 : \begin{aligned} x(k+1) &= A_{11}x(k) + \varepsilon A_{12}y(k) \\ y(k+1) &= A_{21}x(k) + \varepsilon A_{22}y(k) \end{aligned} \tag{3}$$

and

$$\sum\nolimits_2 : \begin{aligned} x(k+1) &= (I_{n_1} + \varepsilon A_{11})x(k) + \varepsilon A_{12}y(k) \\ y(k+1) &= A_{21}x(k) + A_{22}y(k). \end{aligned} \tag{4}$$

In both of them $x(k) \in \mathsf{R}^{n_1}$ and $y(k) \in \mathsf{R}^{n_2}$ are the state vectors at the k-th instant. The constant matrices A_{ij}, i, $j = 1, 2$, are of consistent dimensions. The singular perturbation parameter ε is a small positive scalar.

The stability bound problem of a discrete-time singularly perturbed system is to determine the upper stability bound ε^* such that the overall system remains stable for all $\varepsilon \in (0, \varepsilon^*)$.

Let $X(k) = [x^T(k)\ y^T(k)]^T$ and ε be considered as a parameter uncertainty, then the system $\sum_1$ can be rewritten as an LFT description of

$$\sum\nolimits_1 : X(k+1) = F_l(M, \varepsilon I_{n_2})X(k) = F_l\left(\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \varepsilon I_{n_2}\right) X(k) \quad (5)$$

where

$$M_{11} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & 0 \end{bmatrix}_{n \times n}, \quad M_{12} = \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}_{n \times n_2},$$

$$M_{21} = [\ 0 \quad I_{n_2}\ ]_{n_2 \times n}, \quad M_{22} = 0_{n_2 \times n_2}.$$

Similarly, the system $\sum_2$ can be rewritten as an LFT description of

$$\sum\nolimits_2 : X(k+1) = F_l(N, \varepsilon I_{n_1})X(k) = F_l\left(\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}, \varepsilon I_{n_1}\right) X(k) \quad (6)$$

where

$$N_{11} = \begin{bmatrix} I_{n_1} & 0 \\ A_{21} & A_{22} \end{bmatrix}_{n \times n}, \quad N_{12} = \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}_{n \times n_1},$$

$$N_{21} = [\ A_{11} \quad A_{12}\ ]_{n_1 \times n}, \quad N_{22} = 0_{n_1 \times n_1}.$$

REMARK 2.1 *Although the slow sampling model can be represented in different forms, Li and Li (1992), Kafri and Abed (1996), Li, Chiou and Kung (1999), Ghosh, Sen and Datta (1999), these forms can be equivalently transformed to the canonical model* $\sum_1$ *by a state-variable transformation, Li, Chiou and Kung (1999), Ghosh, Sen and Datta (1999). Thus, the stability analysis can be carried out only for the system* $\sum_1$ *since stability is invariant under any state-variable transformation. On the contrary, the representation of the fast sampling model* $\sum_2$ *is unique.*

In the sequel, two systematic approaches including time-domain and frequency-domain methods are developed to determine the exact stability bounds for both models $\sum_1$ and $\sum_2$.

3. Time-domain approach

The guardian map was introduced in Saydy, Tits and Abed (1990) as a useful tool for studying generalized stability of parametrized families of matrices. With the system matrix $F_l(M, \varepsilon I_{n_2})$, the LFT description of the system $\sum_1$, a guardian map for Schur stability is given by

$$\det(F_l(M, \varepsilon I_{n_2}) \otimes F_l(M, \varepsilon I_{n_2}) - I_n \otimes I_n) := \det(F_l(\hat{M}, \varepsilon I_{2nn_2})) \quad (7)$$

where

$$\hat{M} = \left[\begin{array}{c:cc} M_{11} \otimes M_{11} - I_{n^2} & I_r \otimes M_{12} & M_{12} \otimes M_{11} \\ \hdashline M_{11} \otimes M_{21} & 0 & M_{12} \otimes M_{21} \\ M_{21} \otimes I_r & 0 & 0 \end{array}\right] := \left[\begin{array}{c:c} \hat{M}_{11} & \hat{M}_{12} \\ \hdashline \hat{M}_{21} & \hat{M}_{22} \end{array}\right]. \quad (8)$$

The matrix $F_l(\hat{M}, \varepsilon I_{2nn_2})$ can be calculated directly from Lemma 2.1, and it is required to be nonsingular to guarantee Schur stability of the system $\sum_1$ for all $\varepsilon \in (0, \varepsilon^*)$. The only assumption in the following theorem is that A_{11} is Schur stable. The invertibility of A_{11}, assumed in Li and Li (1992), Li, Chiou and Kung (1999), is not required. Thus, this paper gives less stringent conditions for stability analysis as shown in Kafri and Abed (1996).

THEOREM 3.1 *Let A_{11} be Schur stable. The discrete-time singularly perturbed system $\sum_1$ is Schur stable for all $\varepsilon \in (0, \varepsilon^*)$. Then the exact stability bound ε^* is given by*

$$\varepsilon^* = \frac{1}{\lambda^+_{\max}}(\tilde{M})$$

where $\tilde{M} = \hat{M}_{22} - \hat{M}_{21}\hat{M}_{11}^{-1}\hat{M}_{12}$ and $\lambda^+_{\max}(\tilde{M})$ denotes the largest positive real eigenvalue of the matrix $\tilde{M}$.

Proof. Since A_{11} is Schur stable, and thus $M_{11} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & 0 \end{bmatrix}$, the nominal system matrix of $F_l(M, \varepsilon I_{n_2})$ with $\varepsilon = 0$ in $\sum_1$, is Schur stable. This implies that matrix $\hat{M}_{11} = M_{11} \otimes M_{11} - I_{n^2}$ is nonsingular. Hence, from the guardian map theory we have

$$\begin{aligned}
&\text{System } \textstyle\sum_1 \text{ is Schur stable} \\
&\Leftrightarrow F_l(\hat{M}, \varepsilon I_{2nn_2}) \text{ is nonsingular} \\
&\Leftrightarrow \det(\hat{M}_{11} + \varepsilon \hat{M}_{12}(I_{2nn_2} - \varepsilon \hat{M}_{22})^{-1}\hat{M}_{21}) \neq 0 \\
&\Leftrightarrow \det(I_{n^2} + \varepsilon \hat{M}_{11}^{-1}\hat{M}_{12}(I_{2nn_2} - \varepsilon \hat{M}_{22})^{-1}\hat{M}_{21}) \neq 0 \\
&\Leftrightarrow \det(I_{2nn_2} + \varepsilon (I_{2nn_2} - \varepsilon \hat{M}_{22})^{-1}\hat{M}_{21}\hat{M}_{11}^{-1}\hat{M}_{12}) \neq 0 \\
&\Leftrightarrow \det(I_{2nn_2} - \varepsilon \hat{M}_{22} + \varepsilon \hat{M}_{21}\hat{M}_{11}^{-1}\hat{M}_{12}) \neq 0 \\
&\Leftrightarrow \det(I_{2nn_2} - \varepsilon \tilde{M}) \neq 0.
\end{aligned}$$

As a result, ε^* is the smallest positive real value such that $\det(I_{2nn_2} - \varepsilon \tilde{M}) = 0$. Therefore, the system $\sum_1$ preserves the Schur stability for all $\varepsilon \in (0, \varepsilon^*)$. ■

Similarly, the guardian map for Schur stability of the system $\sum_2$ is given by

$$\det(F_l(N, \varepsilon I_{n_1}) \otimes F_l(N, \varepsilon I_{n_1}) - I_n \otimes I_n) := \det(F_l(\hat{N}, \varepsilon I_{2nn_1})) \quad (9)$$

where

$$\hat{N} = \left[\begin{array}{c|cc} N_{11} \otimes N_{11} - I_{n^2} & I_n \otimes N_{12} & N_{12} \otimes N_{11} \\ \hline N_{11} \otimes N_{21} & 0 & N_{12} \otimes N_{21} \\ N_{21} \otimes I_n & 0 & 0 \end{array}\right] := \left[\begin{array}{c|c} \hat{N}_{11} & \hat{N}_{12} \\ \hline \hat{N}_{21} & \hat{N}_{22} \end{array}\right]. \qquad (10)$$

The matrix $F_l(\hat{N}, \varepsilon I_{2nn_1})$ must be nonsingular to guarantee Schur stability of the system $\sum_2$ for all $\varepsilon \in (0, \varepsilon^*)$. It is worth noting that there is a significant difference between $\sum_1$ and $\sum_2$. The nominal system matrix, $N_{11} = \begin{bmatrix} I_{n_1} & 0 \\ A_{21} & A_{22} \end{bmatrix}$ in $\sum_2$ with fixed n_1 eigenvalues of one, is not Schur stable. Thus, Theorem 3.1 can not be directly applied to the system $\sum_2$ because the invertibility of $\hat{N}_{11} = N_{11} \otimes N_{11} - I_{n^2}$ does not exist. However, the guardian map is also applicable, and thus the following theorem is established.

THEOREM 3.2 *Assume that A_{22} is Schur stable. The discrete-time singularly perturbed system $\sum_2$ is Schur stable for all $\varepsilon \in (0, \varepsilon^*)$. Then the exact stability bound ε^* is given by*

$$\varepsilon^* = \lambda^+_{\min}(U, V)$$

where $U = \begin{bmatrix} I_{2nn_1} & 0 \\ 0 & 0 \end{bmatrix}$, $V = \begin{bmatrix} \hat{N}_{22} & -\hat{N}_{21} \\ \hat{N}_{12} & -\hat{N}_{11} \end{bmatrix}$ *and* $\lambda^+_{\min}(U, V)$ *denotes the smallest positive real generalized eigenvalues of the matrix pair* (U, V), *i.e.,* $\det(U - \lambda V) = 0$.

Proof. Analogous to the Theorem 3.1,

$$\begin{aligned} &\text{System } \textstyle\sum_2 \text{ is Schur stable} \\ &\Leftrightarrow F_l(\hat{N}, \varepsilon I_{2nn_1}) \text{ is nonsingular} \\ &\Leftrightarrow \det(\hat{N}_{11} + \varepsilon \hat{N}_{12}(I_{2nn_1} - \varepsilon \hat{N}_{22})^{-1} \hat{N}_{21}) \neq 0 \\ &\Leftrightarrow \det(\hat{N}_{11} + \hat{N}_{12}(\tfrac{1}{\varepsilon} I_{2nn_1} - \hat{N}_{22})^{-1} \hat{N}_{21}) \neq 0. \end{aligned} \qquad (11)$$

It is worth noting that the matrix $(\frac{1}{\varepsilon} I_{2nn_1} - \hat{N}_{22})$ is not singular for any $\varepsilon > 0$. The following determinant identity holds

$$\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \det(A) \cdot \det(D - CA^{-1}B)$$

if the matrix A is nonsingular. Eqn. 11 is thereby equivalent to

$$\det\left(\begin{bmatrix} \frac{1}{\varepsilon} I_{2nn_1} - \hat{N}_{22} & \hat{N}_{21} \\ -\hat{N}_{12} & \hat{N}_{11} \end{bmatrix}\right) \neq 0 \Leftrightarrow \det\left(\frac{1}{\varepsilon} U - V\right) \neq 0.$$

Then the maximal stability bound ε^* is the smallest positive number ε such that

$$\det\left(\frac{1}{\varepsilon}U - V\right) = 0$$

holds. Mathematically, the above equation can be rewritten as $\det(U - \varepsilon V) = 0$ for $\varepsilon \neq 0$. Hence $\varepsilon^* = \lambda^+_{\min}(U, V)$. This completes the proof. ■

REMARK 3.1 *The generalized eigenvalues of the matrix pair (U, V) can be easily be obtained by using the existing software packages, for example, the Matlab (Matlab..., 1991) command "eig(U, V)". If there does not exist any positive real generalized eigenvalue for the matrix pair (U, V), then* ***either*** *the system $\sum_2$ is Schur stable for all $\varepsilon > 0$* ***or*** *it is not Schur stable for any $\varepsilon > 0$. The precise situation can be checked by an arbitrarily chosen positive number ε.*

4. Frequency-domain approach

In this section, an LFT block diagram is proposed to evaluate the exact stability bound for a discrete-time singularly perturbed system. Taking the z-transform for system $\sum_1$ yields

$$zX(z) = F_l(M, \varepsilon I_{n_2})X(z) \tag{12}$$

with zero initial conditions. Clearly, (12) can be represented by an LFT block diagram illustrated in Fig. 1, where $z^{-1}I_n$ denotes the time-shift operator.

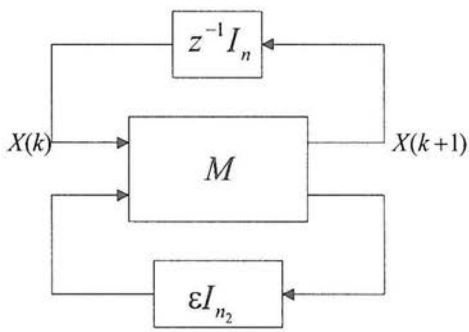

Figure 1. An LFT block diagram of system $\sum_1$

Absorbing the time-shift operator $z^{-1}I_n$ into the matrix M by upper LFT, leads to an equivalent block diagram shown in Fig. 2, where

$$\begin{aligned} P(z) &= F_u(M, z^{-1}I_n) \\ &= M_{22} + z^{-1}M_{21}(I_n - z^{-1}M_{11})^{-1}M_{12} \\ &= 0 + [0\ I_{n_2}]\left(zI_n - \begin{bmatrix} A_{11} & 0 \\ A_{21} & 0 \end{bmatrix}\right)^{-1}\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} \\ &= [0\ I_{n_2}]\left(\begin{bmatrix} (zI_{n_1} - A_{11})^{-1} & 0 \\ z^{-1}A_{21}(zI_{n_1} - A_{11})^{-1} & z^{-1}I_{n_2} \end{bmatrix}\right)^{-1}\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} \\ &= z^{-1}(A_{21}(zI_{n_1} - A_{11})^{-1}A_{12} + A_{22}). \end{aligned} \quad (13)$$

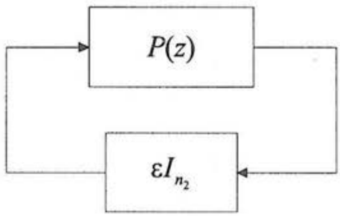

Figure 2. An equivalent block diagram of Figure 1

Assume that matrix A_{11} is Schur stable, then $P(z)$ is a stable rational function matrix of dimension $n_2 \times n_2$ with poles in the unit circle. Clearly, the characteristic equation of the system $\sum_1$ is given by

$$\det(I_{n_2} - \varepsilon P(z)) = 0 \quad (14)$$

Since the nominal system matrix of the system $\sum_1$ is Schur stable, of all eigenvalues lie inside the unit circle. For the special case of $\varepsilon = \varepsilon^*$, there exist at least an eigenvalue just on the unit circle. Consequently, the problem of finding the exact stability bound ε^* is equivalent to determining the smallest positive number ε such that

$$\det(I_{n_2} - \varepsilon P(e^{j\theta})) = 0 \ \text{ for } \ 0 \le \theta < 2\pi. \quad (15)$$

Let $\lambda_i(e^{j\theta})$, $i = 1, 2, \ldots, n_2$, be the i-th eigenvalue of the rational function matrix $P(e^{j\theta})$ for a given θ, then the graphs of $\lambda_i(e^{j\theta})$ for $0 \le \theta < 2\pi$ are the eigenvalue loci of the matrix $P(e^{j\theta})$. Based on the fact that eigenvalues are continuous functions of the entries of a matrix, $\lambda_i(e^{j\theta})$ is continuous on θ for $i = 1, 2, \ldots, n_2$. In view of this, an explicit solution for ε^* is obtained by plotting the eigenvalue loci of $P(e^{j\theta})$ for $0 \le \theta < 2\pi$.

THEOREM 4.1 *Assume that A_{11} is Schur stable. The discrete-time singularly perturbed system $\sum_1$ is Schur stable for all $\varepsilon \in (0, \varepsilon^*)$. Then the exact stability bound ε^* is given by*

$$\varepsilon^* = \frac{1}{\lambda_{\max}^{+}(P(e^{j\theta}))}$$

where $\lambda_{\max}^{+}(P(e^{j\theta}))$ denotes the largest positive real eigenvalue of matrix $P(e^{j\theta})$ for $0 \le \theta < 2\pi$.

Proof. If the eigenvalue loci of matrix $P(e^{j\theta})$ for $0 \le \theta < 2\pi$ intersect the real axis at $(\delta, 0)$, then there exists a real number $\varepsilon = \frac{1}{\delta}$ satisfying $\det(I_{n_2} - \varepsilon P(e^{j\theta})) = 0$. Mathematically,

$$\begin{aligned}\varepsilon^* &= \min\left\{\varepsilon > 0 : \det\left(\frac{1}{\varepsilon}I_{n_2} - P(e^{j\theta})\right) = 0 \ \text{ for } \ 0 \le \theta < 2\pi\right\}\\ &= \frac{1}{\lambda_{\max}^{+}(P(e^{j\theta}))}.\end{aligned}$$

As a result, the exact stability bound ε^* is the reciprocal of the largest positive real value where the eigenvalue loci of matrix $P(e^{j\theta})$ for $0 \le \theta < 2\pi$ intersect the real axis. This completes the proof. ■

Analogous to the development of Theorem 4.1, the exact stability bound of system $\sum_2$ can be obtained in the following theorem.

THEOREM 4.2 *Assume that A_{22} is Schur stable. The discrete-time singularly perturbed system $\sum_2$ is Schur stable for all $\varepsilon \in (0, \varepsilon^*)$. Then the exact stability bound ε^* is given by*

$$\varepsilon^* = \frac{1}{\lambda_{\max}^{+}(G(e^{j\theta}))}$$

where $G(e^{j\theta}) = (e^{j\theta} - 1)^{-1}[A_{11} + A_{12}(e^{j\theta}I_{n_2} - A_{22})^{-1}A_{21}]$ and $\lambda_{\max}^{+}(G(e^{j\theta}))$ denotes the largest positive real eigenvalue of $G(e^{j\theta})$ for $0 \le \theta < 2\pi$.

Proof. An LFT block diagram can be depicted as shown in Fig. 1 with matrix M replaced by matrix N and εI_{n_2} replaced by εI_{n_1}. Then the upper LFT of the time-shift operator $z^{-1}I_n$ and matrix N is given by

$$\begin{aligned}&G(z) = F_u(N, z^{-1}I_n)\\ &= N_{22} + z^{-1}N_{21}(I_n - z^{-1}N_{11})^{-1}N_{12}\\ &= 0 + [\, A_{11} \ \ A_{12} \,]\left(zI_n - \begin{bmatrix} I_{n_1} & 0 \\ A_{21} & A_{22}\end{bmatrix}\right)^{-1}\begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}\\ &= [\, A_{11} \ \ A_{12} \,]\begin{bmatrix} (zI_{n_1} - I_{n_1})^{-1} & 0 \\ (zI_{n_2} - A_{22})^{-1}A_{21}(zI_{n_1} - I_{n_1})^{-1} & (zI_{n_2} - A_{22})^{-1}\end{bmatrix}\begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}\\ &= (z-1)^{-1}(A_{11} + A_{12}(zI_{n_2} - A_{22})^{-1}A_{21}).\end{aligned}$$

$G(e^{j\theta})$ is a stable rational function matrix of dimension $n_1 \times n_1$ with assumption of A_{22} being Schur stable. The remainder is similar to the development of Theorem 4.1. ■

REMARK 4.1 *In the complex plane of eigenvalues of matrices $P(e^{j\theta})$ or $G(e^{j\theta})$, if the eigenvalue loci intersect the positive real axis at points $(\lambda_1^+, 0)$, $(\lambda_2^+, 0)$, $\ldots$, $(\lambda_m^+, 0)$, with $\lambda_m^+ > \cdots > \lambda_2^+ > \lambda_1^+ > 0$, then $\varepsilon^* = 1/\lambda_m^+$. In the case where the eigenvalue loci do not intersect the positive real axis,* **either** *the system $\sum_2$ is Schur stable for all $\varepsilon > 0$* or *it is not Schur stable for any $\varepsilon > 0$. The precise situation can be checked by an arbitrarily chosen positive number ε.*

REMARK 4.2 *The conditions determining the maximal stability bound ε^* in Theorem 4.1 and Theorem 4.2 coincide with Li's results in Li and Li (1992). It is clear that the LFT framework provides a more systematic and straightforward development.*

REMARK 4.3 *Comparing time-domain approach with frequency-domain approach, it is seen that the dimension of matrix to be dealt with in Theorems 4.1 and 4.2 is lower but a sweeping parameter θ is needed. On the other hand, the dimension of matrix to be dealt with in Theorems 3.1 and 3.2 is larger but no sweeping parameter is needed.*

5. Illustrative examples

In this section, two examples are presented to illustrate the proposed approaches.

EXAMPLE 5.1 Let us consider the same example adopted from Li, Chiou and Kung (1999). The slow sampling model $\sum_1$ is given by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ y_1(k+1) \\ y_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.9 & 1.1 & -1.7357\varepsilon & 0.5357\varepsilon \\ 0 & 0.8 & 0 & -2.7882\varepsilon \\ -0.05 & 1.65 & 2.5963\varepsilon & 0.8036\varepsilon \\ 0 & 0.0453 & 0 & 1.3423\varepsilon \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ y_1(k) \\ y_2(k) \end{bmatrix}.$$

For the time-domain approach, the set of the non-repeated positive real eigenvalues of the matrix $\tilde{M} = \hat{M}_{22} - \hat{M}_{21}\hat{M}_{11}\hat{M}_{12}$ is {0.7108, 0.8556, 1.2001, 2.2499, 3.4642}. The dimension of the matrix $\tilde{M}$ is 16×16. Based on Theorem 3.1, we obtain $\varepsilon^* = 1/3.4642 = 0.2887$. On the other hand, for the frequency-domain approach, the eigenvalue loci plot of $P(e^{j\theta})$ for $0 \leq \theta < 2\pi$ is shown in Fig. 3.

$P(e^{j\theta})$ is a matrix of dimensions 2×2. It can be seen that the largest value of the eigenvalue loci intersecting the positive real axis is 3.4642. Hence, Theorem 3.2 leads to the stability bound $\varepsilon^* = 1/3.4642 = 0.2887$. Both results derived from the two approaches are clearly the same. They also coincide with that of Li, Chiou and Kung (1999), which was determined by calculating four real eigenvalues. Although the same result can be obtained by the method of Kafri and Abed (1996), the matrix size for calculation is 32×32.

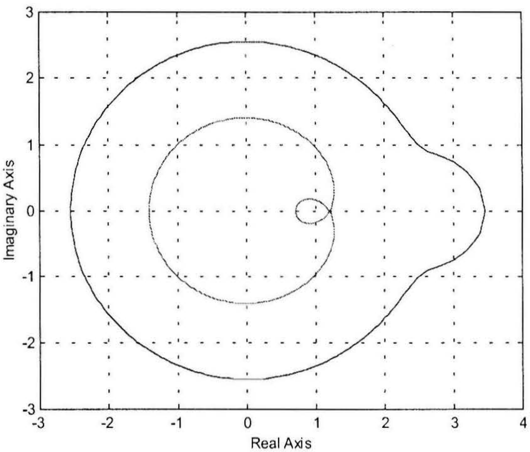

Figure 3. The eigenvalue plot of matrix $P(^{e\theta})$ for $0 \leq \theta < 2\pi$

EXAMPLE 5.2 Let us consider the same example as the one analysed in Li and Li (1992), Kafri and Abed (1996), Ghosh, Sen and Datta (1999). The fast sampling model $\sum_2$ is described by

$$\begin{bmatrix} x(k+1) \\ y_1(k+1) \\ y_2(k+1) \\ y_3(k+1) \end{bmatrix} = \begin{bmatrix} 1-6.71\varepsilon & \varepsilon & -\varepsilon & \varepsilon \\ -1 & -0.65 & 0 & 0 \\ -0.05 & 0 & 0.45 & 0 \\ 0.98 & 0 & 0 & -0.54 \end{bmatrix} \begin{bmatrix} x(k) \\ y_1(k) \\ y_2(k) \\ y_3(k) \end{bmatrix}.$$

For the time-domain approach, the set of the non-repeated positive real generalized eigenvalues of the matrix pair (U, V) is given by $\{0.347, 0.3778, 0.5528\}$, and thus we have the stability bound $\varepsilon^* = 0.347$ by Theorem 3.2.

On the other hand, for the frequency-domain approach, $G(e^{j\theta})$ is a scalar function since the dimension of the fast mode is one. The eigenvalue loci plot of $G(e^{j\theta})$ for $0 < \theta < 2\pi$ is shown in Fig 4. The local plot of interest is shown in Fig. 5.

It is seen that the largest value of the eigenvalue loci intersecting the positive real axis is 2.8815. Hence, Theorem 4.1 leads to the stability bound $\varepsilon^* = 1/2.8815 = 0.347$. The results derived from both approaches are clearly the same. The results also coincide with those obtained in Li and Li (1992), Kafri and Abed (1996), Ghosh, Sen and Datta (1999).

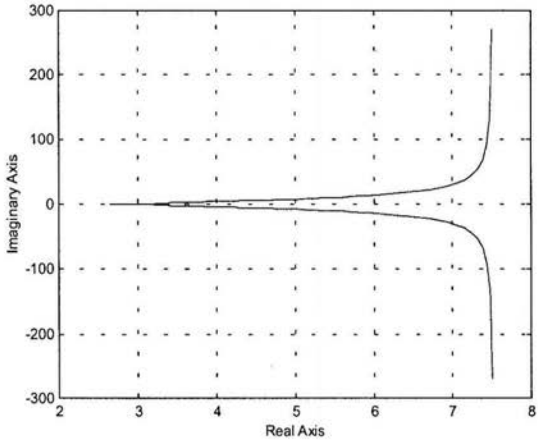

Figure 4. The eigenvalue loci of matrix $G(e^{j\theta})$ for $0 < \theta < 2\pi$

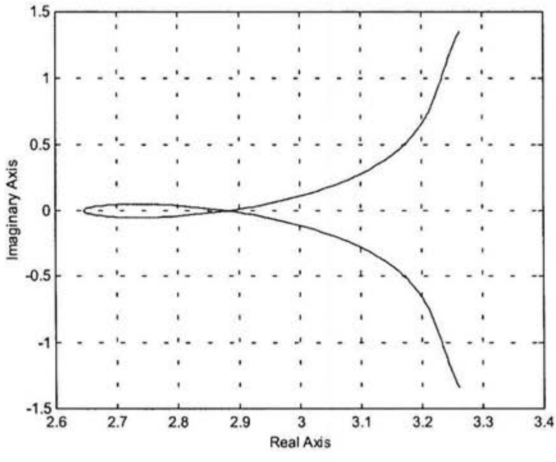

Figure 5. The local eigenvalue loci of matrix for $G(e^{j\theta})$

6. Conclusions

The exact stability bound is derived in an explicit and closed form to guarantee Schur stability of a discrete-time singularly perturbed system by time-domain and frequency-domain approaches. An LFT description for the system is proposed to serve as a unified framework for evaluating the exact stability bound in both approaches. In fact, the approaches can be extended to address more general stability region with a suitable guardian map for time-domain approach or a sweeping parameter on the regional boundary for frequency-domain approach. Illustrative examples show that the results obtained by both approaches are the same, and also the same as those of the existing criteria proposed in the literature.

Acknowledgments

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