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A sufficient condition for Hurwitz stability of the convex combination of two matrices

by

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Abstract: The paper provides a simple sufficient condition for Hurwitz stability of the convex combination of two real matrices.

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1. Introduction

Stability analysis of many control systems is concerned with locations of the eigenvalues of the matrix. If the elements of the matrix are known exactly, then the locations of its eigenvalues can be determined readily by the well–known methods. However, in practice, an approximation of the real process is used for the given system model. Thus, one of the real problems of stability analysis is to determine the stability of a family of matrices. One of such families of matrices is constituted by the convex combinations of the given matrices.

Białas (1985) has given the necessary and sufficient conditions for the Hurwitz stability of the convex combinations of two real matrices, but these conditions are not practical.

In this paper, we give another, but simpler in terms of calculation, sufficient condition for the Hurwitz stability of the convex combination of two matrices.

Let $A = [a_{ij}] \in R^{n \times n}, \quad B = [b_{ij}] \in R^{n \times n}$.

We apply the following notations:

$$V_2 = \{(\alpha_1, \alpha_2) \in R^2 : \quad \alpha_1 \geq 0,\ \alpha_2 \geq 0,\ \alpha_1 + \alpha_2 = 1\},$$

$$C(A, B) = \{\alpha_1 A + \alpha_2 B : \quad (\alpha_1, \alpha_2) \in V_2\}.$$

Let A^T denote the transpose of the matrix A, and $\lambda_i(A)$ be eigenvalues of the matrix $A \in R^{n\times n}$. The matrix A is called Hurwitz stable if $Re(\lambda_i(A)) < 0$ for $i = 1, 2, \ldots, n$. Let

$$H_n = \{A \in R^{n\times n} : Re(\lambda_i(A)) < 0 \quad (i = 1, 2, \ldots, n)\}.$$

If $C(A, B) \subset H_n$, then we say that the set $C(A, B)$ is Hurwitz stable.

2. Main result

We give a sufficient condition for the Hurwitz stability of the set $C(A, B)$, which is simple in calculation.

We denote by $A \bullet B$ the bialternate product of the matrices A and B, see Jury (1974).

Let $L(A) = (2A) \bullet I$ denote the bialternate product of the matrix $2A$ and I, where I is the identity matrix $n \times n$. It is quite easy to see that $L(A) \in R^{(n(n-1)/2)\times(n(n-1)/2)}$.

The following theorems are true:

THEOREM 2.1 (JURY, 1974) *If $A \in R^{n\times n}$, then*

$$\lambda_i(A) + \lambda_j(A) \quad (i = 2, 3, \ldots, n;\ j = 1, 2, \ldots, i-1)$$

are the eigenvalues of the matrix $L(A) = (2A) \bullet I$.

THEOREM 2.2 (BIAŁAS, 1985) *If the matrices $A, B \in H_n$, then the convex combination*

$$C(A, B) = \{\alpha_1 A + \alpha_2 B :\ (\alpha_1, \alpha_2) \in V_2\} \subset H_n$$

if and only if

$$\lambda_i(L(A)L^{-1}(B)) \notin (-\infty, 0>$$

for $i = 1, 2, \ldots, n(n-1)/2$.

If the matrix $Q^T = Q \in R^{n\times n}$ and $\lambda_i(Q) > 0$ $(i = 1, 2, \ldots, n)$, then we write $Q > 0$.

We apply

LEMMA 2.1 (NEUMANN, 1979) *If the matrix $A \in R^{n\times n}$ and there exists a matrix $P \in R^{n\times n}$ such that*

$$P^T + P > 0,$$
$$A^T P^T + PA > 0,$$

then $\lambda_i(A) \notin (-\infty, 0> \quad (i = 1, 2, \ldots, n)$.

Now, we will prove the simple sufficient condition for Hurwitz stability of the convex combination of two matrices.

THEOREM 2.3 *If the matrices $A, B \in H_n$ and there exists a matrix $P \in R^{(n(n-1)/2)\times(n(n-1)/2)}$ such that*

$$\begin{aligned} &L^T(A)P^T + PL(A) > 0, \\ &L^T(B)P^T + PL(B) > 0 \end{aligned} \tag{1}$$

then the convex combination

$$C(A,B) = \{\alpha_1 A + \alpha_2 B : \alpha_1 \geq 0,\ \alpha_2 \geq 0,\ \alpha_1 + \alpha_2 = 1\}$$

is Hurwitz stable.

Proof. From the assumption $A, B \in H_n$ and from Theorem 2.1, it follows that there exist inverse matrices $L^{-1}(A), L^{-1}(B)$. It is easy to see that for the matrix $L^{-1}(A)L(B)$ we have

$$\begin{aligned} &[L^{-1}(A)L(B)]^T[PL(A)]^T + [PL(A)][L^{-1}(A)L(B)] = \\ &= L^T(B)P^T + PL(B) > 0. \end{aligned}$$

Hence, and from the inequality (1), by applying Lemma 2.1, we get

$$\lambda_i(L(A)L^{-1}(B)) \notin (-\infty, 0> \quad (i = 1, 2, \ldots, n(n-1)/2).$$

From this and from Theorem 2.2, it follows that $C(A,B) \subset H_n$. This completes the proof of Theorem 2.3. ■

From Theorem 2.3 we have the following corollaries:

COROLLARY 2.1 *If the matrices $A, B \in H_n$ and there exists a matrix $P = P^T \in R^{(n(n-1)/2)\times(n(n-1)/2)}$ such that*

$$\begin{aligned} &L^T(A)P + PL(A) > 0, \\ &L^T(B)P + PL(B) > 0, \end{aligned}$$

then $C(A,B) \subset H_n$.

COROLLARY 2.2 *If the matrices $A, B \in R^{n\times n}$ and*

$$L^T(A) + L(A) > \quad \text{and} \quad L^T(B) + L(B) > 0,$$

then $C(A,B) \subset H_n$.

EXAMPLE 2.1 *For the matrices*

$$A = \begin{bmatrix} 19 & 0 & -28 \\ 0 & 18 & -27 \\ 1 & 36 & 18 \end{bmatrix}, \quad B = \begin{bmatrix} 9 & 0 & -28 \\ 0 & 8 & -27 \\ 2 & 36 & 8 \end{bmatrix}$$

we have

$$L(A) = \begin{bmatrix} 37 & -27 & 28 \\ 26 & 37 & 0 \\ -1 & 0 & 36 \end{bmatrix}, \quad L(B) = \begin{bmatrix} 17 & -27 & 28 \\ 36 & 17 & 0 \\ -2 & 0 & 16 \end{bmatrix},$$

$$L^T(A) + L(A) = \begin{bmatrix} 74 & 9 & 27 \\ 9 & 74 & 0 \\ 27 & 0 & 72 \end{bmatrix} > 0,$$

$$L^T(B) + L(B) = \begin{bmatrix} 34 & 9 & 26 \\ 9 & 34 & 0 \\ 26 & 0 & 32 \end{bmatrix} > 0.$$

Hence, given that $L^T(A) + L(A) > 0$ *and* $L^T(B) + L(B) > 0$, *by applying Corollary 2.2 it follows that the convex combination* $C(A, B)$ *is Hurwitz stable.*

Hurwitz stability of the convex combination of more than two matrices is an open problem.

References

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