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Sufficient conditions and duality for multiobjective variational problems with generalized B-invexity

by

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Abstract: In this paper, we consider the multiobjective variational problem. We propose a class of generalized B-type I vector-valued functions and use this concept to establish sufficient optimality conditions and mixed type duality results.

Keywords: multiobjective variational programming, efficient solution, B-type I vector-valued functions, generalized B-type I vector-valued functions, sufficient conditions, mixed type duality.

1. Introduction

The relationship between mathematical programming and classical calculus of variation was explored and extended by Hanson (1964). Thereafter variational programming problems have attracted some attention in literature. Optimality conditions and duality results were obtained for scalar valued variational problems by Mond and Hanson (1967) under convexity and by Bhatia and Kumar (1996) under B-vexity assumptions. Motivated by the approach of Bector and Husain (1992), Nahak and Nanda (1996) extended the results of Mond, Chandra, and Husain (1988) to multiobjective variational problems involving invex functions. Type I functions were first introduced by Hanson and Mond (1987). Rueda and Hanson (1988) have defined the classes of pseudo-type I and quasi-type I functions as generalizations of type I functions. Kaul, Sunja and Srivastava (1994), Aghezzaf and Hachimi (2000) obtained optimality conditions and duality results for multiobjective programming problems involving type I and generalized type I functions.

Recently, Bhatia and Mehra (1999) introduced a class of B-type I functions, an extension of invex functions defined by Mond, Chandra, and Husain (1988) and B-vex functions defined by Bhatia and Kumar (1996), and used the concept to obtain various sufficient optimality conditions and duality results for multiobjective variational problems.

In this paper, we propose a class of vector-valued functions called generalized B-type I and derive various sufficient optimality conditions and mixed type duality results for multiobjective variational problems. To establish our results with our new classes of vector-valued functions, we do not require the assumption of scalarization of vector objective function made in Bhatia and Mehra (1999).

2. Notations and preliminaries

Throughout this paper the following conventions for vectors in $\mathbb{R}^n$ $(n > 1)$ will be followed:

$$\begin{aligned}
&x = y \Longleftrightarrow x_i = y_i \quad \forall\, i = 1, \ldots, n;\\
&x \leqq y \Longleftrightarrow x_i \leq y_i \quad \forall\, i = 1, \ldots, n;\\
&x \leq y \Longleftrightarrow x \leqq y, \quad \text{and} \quad x \neq y;\\
&x < y \Longleftrightarrow x_i < y_i \quad \forall\, i = 1, \ldots, n.
\end{aligned}$$

Let $I = [a, b]$ be a real interval and $f : I \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^p$ and $g : I \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be continuously differentiable functions. In order to consider $f(t, x, \dot{x})$, where $x : I \longrightarrow \mathbb{R}^n$ with its derivative $\dot{x}$, denote the $p \times n$ matrices of first partial derivatives of f with respect to x, $\dot{x}$ by f_x and $f_{\dot{x}}$, such that

$$f_{ix} = (\frac{\partial f_i}{\partial x_1}, \ldots, \frac{\partial f_i}{\partial x_n}) \text{ and } f_{i\dot{x}} = (\frac{\partial f_i}{\partial \dot{x}_1}, \ldots, \frac{\partial f_i}{\partial \dot{x}_n}), \quad i = 1, 2, \ldots, p.$$

Similarly, g_x and $g_{\dot{x}}$ denote the $m \times n$ matrices of first partial derivatives of g with respect to x and $\dot{x}$. Let $\mathcal{C}(I, \mathbb{R}^n)$ denote the space of continuously differentiable functions x with norm $\|x\| := \|x\|_\infty + \|Dx\|_\infty$, where the differential operator D is given by

$$u = Dx \Longleftrightarrow x(t) = x(a) + \int_a^t u(s)ds.$$

Therefore, $D = d/dt$ except at discontinuities. We consider the following multiobjective variational problem,

$$\begin{aligned}
\text{(MOP)} \quad \text{Minimize} \quad & \int_a^b f(t, x, \dot{x})dt = (\int_a^b f_1(t, x, \dot{x})dt, \ldots, \int_a^b f_p(t, x, \dot{x})dt)\\
\text{subject to} \quad & x(a) = \alpha,\ x(b) = \beta,\\
& g(t, x, \dot{x}) \leqq 0,\ t \in I.
\end{aligned}$$

Let $K := \{\ x \in \mathcal{C}(I, \mathbb{R}^n),\ x(a) = \alpha,\ x(b) = \beta,\ g(t, x, \dot{x}) \leqq 0,\quad \forall t \in I\ \}$ be the set of feasible solutions of (MOP). Due to the conflicting nature of the objectives, an optimal solution that simultaneously minimizes all the objectives is usually not obtainable. Thus, for the (MOP) problem, the solution is defined in terms of an efficient solution.

DEFINITION 2.1 *A point $x^* \in K$ is said to be an efficient (Pareto optimal) solution of (MOP) if there exists no other $x \in K$ such that*

$$\int_a^b f(t, x, \dot{x})dt \leq \int_a^b f(t, x^*, \dot{x}^*)dt.$$

3. B-type I and generalized B-type I functions

It will be assumed throughout that f is the vector objective function and g is the constraint vector function in problem (MOP). The definition of B-type I for the objective and constraint functions of Bhatia and Mehra (1999) can be generalized to the objective and constraint vectors in the spirit of Aghezzaf and Hachimi (2000).

DEFINITION 3.1 *(f, g) is said to be B-type I at $u \in \mathcal{C}(I, \mathbb{R}^n)$ with respect to b_0, b_1 and η if there exist functions $b_0,\ b_1 : \mathcal{C}(I, \mathbb{R}^n) \times \mathcal{C}(I, \mathbb{R}^n) \longrightarrow \mathbb{R}_+$ and $\eta : I \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that for all $x \in K$,*

$$\begin{aligned} b_0(x, u)\Big[\int_a^b f(t, x, \dot{x})dt - \int_a^b f(t, u, \dot{u})dt\Big] \\ \geqq \int_a^b \eta(t, x, u)^t \Big[f_x(t, u, \dot{u}) - \frac{d}{dt} f_{\dot{x}}(t, u, \dot{u})\Big]dt, \end{aligned} \tag{1}$$

$$\begin{aligned} -b_1(x, u) \int_a^b g(t, u, \dot{u})dt \\ \geqq \int_a^b \eta(t, x, u)^t \Big[g_x(t, u, \dot{u}) - \frac{d}{dt} g_{\dot{x}}(t, u, \dot{u})\Big]dt. \end{aligned} \tag{2}$$

If in the previous definition, (1) is satisfied as a strict inequality, then we say that a pair (f, g) is semistrictly B-type I at $u \in \mathcal{C}(I, \mathbb{R}^n)$ with respect to b_0, b_1 and η.

Now we propose new classes of vector-valued functions called generalized B-type I as follows.

DEFINITION 3.2 *(f, g) is said to be weak B-strictly pseudo-quasi-type I at $u \in \mathcal{C}(I, \mathbb{R}^n)$ with respect to b_0, b_1 and η if there exist functions $b_0,\ b_1 : \mathcal{C}(I, \mathbb{R}^n) \times$*

$\mathcal{C}(I, \mathbb{R}^n) \longrightarrow \mathbb{R}_+$ *and* $\eta : I \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *such that for all* $x \in K$,

$$\int_a^b f(t,x,\dot{x})dt \leq \int_a^b f(t,u,\dot{u})dt$$
$$\Longrightarrow b_0(x,u) \int_a^b \eta(t,x,u)^t \Big[f_x(t,u,\dot{u}) - \frac{d}{dt} f_{\dot{x}}(t,u,\dot{u})\Big] dt < 0,$$
$$-\int_a^b g(t,u,\dot{u})dt \leqq 0$$
$$\Longrightarrow b_1(x,u) \int_a^b \eta(t,x,u)^t \Big[g_x(t,u,\dot{u}) - \frac{d}{dt} g_{\dot{x}}(t,u,\dot{u})\Big] dt \leqq 0.$$

This definition is a slight extension of the B-strictly pseudo-quasi-type I functions of Bhatia and Mehra (1999). This class of functions does not contain the class of B-type I functions, but does contain the class of semistrictly B-type I functions with $b_0 > 0$.

DEFINITION 3.3 (f, g) *is said to be strong* B*-pseudo-quasi-type I at* $u \in \mathcal{C}(I, \mathbb{R}^n)$ *with respect to* b_0, b_1 *and* η *if there exist functions* b_0, $b_1 : \mathcal{C}(I, \mathbb{R}^n) \times \mathcal{C}(I, \mathbb{R}^n) \longrightarrow \mathbb{R}_+$ *and* $\eta : I \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *such that for all* $x \in K$,

$$\int_a^b f(t,x,\dot{x})dt \leq \int_a^b f(t,u,\dot{u})dt$$
$$\Longrightarrow b_0(x,u) \int_a^b \eta(t,x,u)^t \Big[f_x(t,u,\dot{u}) - \frac{d}{dt} f_{\dot{x}}(t,u,\dot{u})\Big] dt \leq 0,$$
$$-\int_a^b g(t,u,\dot{u})dt \leqq 0$$
$$\Longrightarrow b_1(x,u) \int_a^b \eta(t,x,u)^t \Big[g_x(t,u,\dot{u}) - \frac{d}{dt} g_{\dot{x}}(t,u,\dot{u})\Big] dt \leqq 0.$$

Instead of the class of weak B-strictly pseudo-quasi-type I functions, the class of strong B-pseudo-quasi-type I functions does contain the class of B-type I functions with $b_0 > 0$.

We give examples to show that weak B-strictly pseudo-quasi-type I and strong B-pseudo-quasi-type I functions exist. Weak B-strictly-pseudo-quasi-type I functions need not be B-strictly pseudo-quasi-type I for the same b_0, b_1 and η as can be seen from the following example.

EXAMPLE 3.1 Define functions f, g by

$$\begin{array}{rcl} f : [0, \frac{\pi}{2}] \times \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (t, x(t), \dot{x}(t)) & \longmapsto & x(t) \\ g : [0, \frac{\pi}{2}] \times \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (t, x(t), \dot{x}(t)) & \longmapsto & x_1(t) + x_2(t) \end{array}$$

(f, g) is weak B-strictly pseudo-quasi-type I at $u(t) = (-\cos t, -\sin t)$ with respect to functions b_0, $b_1 : \mathcal{C}([0, \frac{\pi}{2}], I\!R^2) \times \mathcal{C}([0, \frac{\pi}{2}], I\!R^2) \longrightarrow I\!R_+$ and $\eta : [0, \frac{\pi}{2}] \times I\!R^2 \times I\!R^2 \longrightarrow I\!R^2$ defined below:

$$\eta(t, x, u) = (x_1(t), x_1(t) + x_2(t) + \frac{4}{\pi}),$$
$$b_0(x, u) = u_1^2(t) + u_2^2(t),$$
$$b_1(x, u) = x_1^2(t) + x_2^2(t) + 1,$$

but (f, g) is not B-strictly pseudo-quasi-type I with respect to the same b_0, b_1 and η at u because for $x(t) = (\cos t, -\sin t - 1)$ and $u(t) = (-\cos t, -\sin t)$;

$$b_0(x, u) \int_0^{\frac{\pi}{2}} \left[\eta(t, x, u)^t f_x(t, u, \dot{u}) + \frac{d}{dt}(\eta(t, x, u))^t f_{\dot{x}}(t, u, \dot{u})\right] dt \geqq 0,$$

but $\int_0^{\frac{\pi}{2}} f(t, x, \dot{x}) dt \not> (-1, -1)$.

Also (f, g) is not B-type I with respect to the same b_0, b_1 and η at u as can be seen by taking $x(t) = (\cos t, -\sin t - 1)$.

Strong B-pseudo-quasi-type I functions need not be B-type I with respect to the same b_0, b_1 and η.

EXAMPLE 3.2 Define functions f, g by

$$\begin{array}{rcl} f : [0, \frac{\pi}{2}] \times I\!R^2 \times I\!R^2 & \longrightarrow & I\!R^2 \\ (t, x(t), \dot{x}(t)) & \longmapsto & x(t) \\ g : [0, \frac{\pi}{2}] \times I\!R^2 \times I\!R^2 & \longrightarrow & I\!R \\ (t, x(t), \dot{x}(t)) & \longmapsto & x_1(t) + x_2(t) \end{array}$$

(f, g) is strong B-pseudo-quasi-type I at $u(t) = (-\cos t, -\sin t)$ with respect to functions b_0, $b_1 : \mathcal{C}([0, \frac{\pi}{2}], I\!R^2) \times \mathcal{C}([0, \frac{\pi}{2}], I\!R^2) \longrightarrow I\!R_+$ and $\eta : [0, \frac{\pi}{2}] \times I\!R^2 \times I\!R^2 \longrightarrow I\!R^2$ defined in the following text

$$\begin{array}{rcl} \eta(t, x, u) & = & (x_2(t) + \frac{2}{\pi}, x_1(t) + x_2(t) + \frac{4}{\pi}), \\ b_0(x, u) & = & u_1^2(t) + u_2^2(t), \\ b_1(x, u) & = & x_1^2(t) + x_2^2(t) + 1, \end{array}$$

but (f, g) is not B-type I with respect to the same b_0, b_1 and η at u as can be seen by taking $x(t) = (-\cos t - 1, -\sin t)$ nor it is weak B-strictly pseudo-quasi-type I with respect to the same b_0, b_1 and η as can be seen by taking $x(t) = (-\cos t - 1, -\sin t)$.

DEFINITION 3.4 *(f,g) is said to be weak B-strictly pseudo-type I at $u \in \mathcal{C}(I,\mathbb{R}^n)$ with respect to b_0, b_1 and η if there exist functions b_0, $b_1 : \mathcal{C}(I,\mathbb{R}^n) \times \mathcal{C}(I,\mathbb{R}^n) \longrightarrow \mathbb{R}_+$ and $\eta : I \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that for all $x \in K$,*

$$\int_a^b f(t,x,\dot{x})dt \leq \int_a^b f(t,u,\dot{u})dt$$
$$\Longrightarrow b_0(x,u)\int_a^b \eta(t,x,u)^t\Big[f_x(t,u,\dot{u}) - \frac{d}{dt}f_{\dot{x}}(t,u,\dot{u})\Big]dt < 0,$$
$$-\int_a^b g(t,u,\dot{u})dt \leqq 0$$
$$\Longrightarrow b_1(x,u)\int_a^b \eta(t,x,u)^t\Big[g_x(t,u,\dot{u}) - \frac{d}{dt}g_{\dot{x}}(t,u,\dot{u})\Big]dt < 0.$$

EXAMPLE 3.3 Define functions f, g by

$$\begin{array}{rcl} f : [0,\frac{\pi}{2}] \times \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (t,x(t),\dot{x}(t)) & \longmapsto & x(t) \\ g : [0,\frac{\pi}{2}] \times \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (t,x(t),\dot{x}(t)) & \longmapsto & (x_1(t)+\frac{2}{\pi}, x_2(t)+\frac{2}{\pi}) \end{array}$$

(f,g) is weak B- strictly pseudo-type I at $u(t) = (-\cos t, -\sin t)$ with respect to functions b_0, $b_1 : \mathcal{C}([0,\frac{\pi}{2}],\mathbb{R}^2) \times \mathcal{C}([0,\frac{\pi}{2}],\mathbb{R}^2) \longrightarrow \mathbb{R}_+$ and $\eta : [0,\frac{\pi}{2}] \times \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined in the following text

$$\begin{array}{rcl} \eta(t,x,u) & = & (x_1(t), x_1(t)+x_2(t)+\frac{2}{\pi}), \\ b_0(x,u) & = & u_1^2(t)+u_2^2(t), \\ b_1(x,u) & = & x_1^2(t)+x_2^2(t)+1, \end{array}$$

but (f,g) is not B-type I with respect to the same b_0, b_1 and η at u as can be seen by taking $x(t) = (-\cos t - \frac{4}{\pi}, -\sin t)$.

4. Sufficient optimality conditions

In the following theorems, we establish various sufficient optimality conditions for (MOP) under generalized B-invexity conditions.

THEOREM 4.1 *Let x^* be a feasible solution for (MOP) and let there exist $\lambda^* \in \mathbb{R}^p, \lambda^* > 0$ and a piecewise smooth function $y^* : I \longrightarrow \mathbb{R}^m$ such that for all*

$t \in I$,

$$\lambda^{*t} f_x(t, x^*, \dot{x}^*) + y^*(t)^t g_x(t, x^*, \dot{x}^*)$$
$$= \frac{d}{dt}\Big(\lambda^{*t} f_{\dot{x}}(t, x^*, \dot{x}^*) + y^*(t)^t g_{\dot{x}}(t, x^*, \dot{x}^*)\Big), \tag{3}$$
$$y^*(t)^t g(t, x^*, \dot{x}^*) = 0, \quad t \in I, \tag{4}$$
$$y^*(t) \geqq 0, \quad t \in I. \tag{5}$$

Further, if $(f, y^*(t)^t g)$ is strong $B-$pseudo-quasi-type I at x^* with respect to functions b_0, b_1, and η with $b_1(x, x^*) > 0$ for all $x \in K$, then x^* is an efficient solution for (MOP).

Proof. If x^* is not an efficient solution for (MOP), then there exists $x \in K$ such that

$$\int_a^b f(t, x, \dot{x})dt \leq \int_a^b f(t, x^*, \dot{x}^*)dt.$$

From (4), we have

$$-\int_a^b y^*(t)^t g(t, x^*, \dot{x}^*)dt = 0.$$

Since $(f, y^*(t)^t g)$ is strong B-pseudo-quasi-type I at x^* with respect to functions b_0, b_1, and η, therefore, for any $x \in K$, we get

$$b_0(x, x^*) \int_a^b \eta(t, x, x^*)^t \Big[f_x(t, x^*, \dot{x}^*) - \frac{d}{dt} f_{\dot{x}}(t, x^*, \dot{x}^*)\Big] dt \leq 0,$$
$$b_1(x, x^*) \int_a^b \eta(t, x, x^*)^t \Big[y^*(t)^t g_x(t, x^*, \dot{x}^*) - \frac{d}{dt} y^*(t)^t g_{\dot{x}}(t, x^*, \dot{x}^*)\Big] dt \leqq 0.$$

Since $b_1(x, x^*)$ is positive, and $\lambda^* > 0$, we get

$$b_0(x, x^*) \int_a^b \eta(t, x, x^*)^t \Big[\lambda^{*t} f_x(t, x^*, \dot{x}^*) - \frac{d}{dt} \lambda^{*t} f_{\dot{x}}(t, x^*, \dot{x}^*)\Big] dt < 0, \tag{6}$$
$$\int_a^b \eta(t, x, x^*)^t \Big[y^*(t)^t g_x(t, x^*, \dot{x}^*) - \frac{d}{dt} y^*(t)^t g_{\dot{x}}(t, x^*, \dot{x}^*)\Big] dt \leqq 0.$$

By $b_0(x, x^*) \geqq 0$, it follows that

$$b_0(x, x^*) \int_a^b \eta(t, x, x^*)^t \Big[y^*(t)^t g_x(t, x^*, \dot{x}^*) - \frac{d}{dt} y^*(t)^t g_{\dot{x}}(t, x^*, \dot{x}^*)\Big] dt \leqq 0. \tag{7}$$

Adding (6) and (7), we obtain

$$b_0(x,x^*)\int_a^b \eta(t,x,x^*)^t\Big[\lambda^{*t}f_x(t,x^*,\dot{x}^*)+y^*(t)^t g_x(t,x^*,\dot{x}^*)$$
$$-\frac{d}{dt}\Big(\lambda^* f_{\dot{x}}(t,x^*,\dot{x}^*)+\frac{d}{dt}y^*(t)^t g_{\dot{x}}(t,x^*,\dot{x}^*)\Big)\Big]dt<0,$$

which contradicts (3). Hence x^* is an efficient solution for (MOP), and the proof is complete. ■

In the next theorem, we replace the strong B-pseudo-quasi-type I by the weak B-strictly-pseudo-quasi-type I of $(f,y^*(t)^t g)$.

THEOREM 4.2 *Let x^* be a feasible solution for (MOP), and let there exist $\lambda^*\in I\!R^p,\lambda^*\geq 0$ and a piecewise smooth function $y^*:I\longrightarrow I\!R^m$ such that for every $t\in I,(x^*,\lambda^*,y^*)$ satisfy (3)-(5) of Theorem (4.1).*

Further, if $(f,y^*(t)^t g)$ is weak B-strictly-pseudo-quasi-type I at x^* with respect to functions b_0, b_1, and η with $b_1(x,x^*)>0$ for all $x\in K$, then x^* is an efficient solution for (MOP).

Proof. Assume that x^* is not an efficient solution for (MOP). Then, there exists $x\in K$ such that

$$\int_a^b f(t,x,\dot{x})dt\leq\int_a^b f(t,x^*,\dot{x}^*)dt.$$

From (4), we have

$$-\int_a^b y^*(t)^t g(t,x^*,\dot{x}^*)dt=0.$$

Since $(f,y^*(t)^t g)$ is weak B-strictly-pseudo-quasi-type I at x^* with respect to functions b_0, b_1, and η we get

$$b_0(x,x^*)\int_a^b \eta(t,x,x^*)^t\Big[f_x(t,x^*,\dot{x}^*)-\frac{d}{dt}f_{\dot{x}}(t,x^*,\dot{x}^*)\Big]dt<0,$$
$$b_1(x,x^*)\int_a^b \eta(t,x,x^*)^t\Big[y^*(t)^t g_x(t,x^*,\dot{x}^*)-\frac{d}{dt}y^*(t)^t g_{\dot{x}}(t,x^*,\dot{x}^*)\Big]dt\leqq 0,$$

and now the proof is similar to that of Theorem (4.1). ■

In our final sufficiency result below, we invoke the weak B-strictly-pseudo-type I of $(f,y^*(t)^t g)$.

THEOREM 4.3 *Let x^* be a feasible solution for (MOP) and let there exist $\lambda^*\in I\!R^p,\lambda^*\geqq 0$ and a piecewise smooth function $y^*:I\longrightarrow I\!R^m$ such that conditions (3)-(5) of Theorem (4.1) are satisfied by (x^*,λ^*,y^*).*

Further, if $(f,y^*(t)^t g)$ is weak B-strictly-pseudo-type I at x^* with respect to functions b_0, b_1, and η, then x^* is an efficient solution for (MOP).

Proof. If x^* is not an efficient solution for (MOP), then there exists $x \in K$ such that

$$\int_a^b f(t,x,\dot{x})dt \leq \int_a^b f(t,x^*,\dot{x}^*)dt.$$

From (4), we obtain

$$-\int_a^b y^*(t)^t g(t,x^*,\dot{x}^*)dt = 0.$$

Since $(f, y^*(t)^t g)$ is weak B-strictly-pseudo-type I at x^* with respect to b_0, b_1 and η we get

$$b_0(x,x^*)\int_a^b \eta(t,x,x^*)^t\Big[f_x(t,x^*,\dot{x}^*) - \frac{d}{dt}f_{\dot{x}}(t,x^*,\dot{x}^*)\Big]dt < 0, \tag{8}$$

$$b_1(x,x^*)\int_a^b \eta(t,x,x^*)^t\Big[y^*(t)^t g_x(t,x^*,\dot{x}^*) - \frac{d}{dt}y^*(t)^t g_{\dot{x}}(t,x^*,\dot{x}^*)\Big]dt < 0. \tag{9}$$

From (8) and (9), we have

$$\int_a^b \eta(t,x,x^*)^t\Big[f_x(t,x^*,\dot{x}^*) - \frac{d}{dt}f_{\dot{x}}(t,x^*,\dot{x}^*)\Big]dt < 0, \tag{10}$$

$$\int_a^b \eta(t,x,x^*)^t\Big[y^*(t)^t g_x(t,x^*,\dot{x}^*) - \frac{d}{dt}y^*(t)^t g_{\dot{x}}(t,x^*,\dot{x}^*)\Big]dt < 0. \tag{11}$$

Because $\lambda^* \geqq 0$, (10) gives

$$\int_a^b \eta(t,x,x^*)^t\Big[\lambda^{*t} f_x(t,x^*,\dot{x}^*) - \frac{d}{dt}\lambda^{*t} f_{\dot{x}}(t,x^*,\dot{x}^*)\Big]dt \leqq 0. \tag{12}$$

Adding (11) and (12), we obtain

$$\int_a^b \eta(t,x,x^*)^t\Big[\lambda^{*t} f_x(t,x^*,\dot{x}^*) + y^*(t)^t g_x(t,x^*,\dot{x}^*) - \frac{d}{dt}\Big(\lambda^{*t} f_{\dot{x}}(t,x^*,\dot{x}^*) + \frac{d}{dt}y^*(t)^t g_{\dot{x}}(t,x^*,\dot{x}^*)\Big)\Big]dt < 0,$$

which contradicts (3). Hence the result. ∎

5. Mixed type duality

Let J_1 be a subset of M, $J_2 = M \backslash J_1$, and e be the vector of $\mathbb{R}^p$ whose components are all ones. We consider the following mixed type dual for (MOP),

$$\text{(XMOP) Maximize } \int_a^b \{f(t,u,\dot{u})dt + [y_{J_1}(t)^t g_{J_1}(t,u,\dot{u})]e\}dt$$

subject to

$$u(a) = \alpha, u(b) = \beta,$$

$$\lambda^t f_x(t,u,\dot{u}) + y(t)^t g_x(t,u,\dot{u}) = \frac{d}{dt}(\lambda^t f_{\dot{x}}(t,u,\dot{u}) + y(t)^t g_{\dot{x}}(t,u,\dot{u})),\ t \in I, \tag{13}$$

$$y_{J_2}(t)^t g_{J_2}(t,u,\dot{u}) \geq 0,\ t \in I, \tag{14}$$

$$y(t) \geq 0,\ t \in I, \tag{15}$$

$$\lambda \in I\!R^p, \lambda \geq 0, \lambda^t e = 1, e = (1,\dots,1) \in I\!R^p. \tag{16}$$

We note that we get a Mond-Weir dual for $J_1 = \emptyset$ and a Wolfe dual for $J_2 = \emptyset$ in (XMOP), respectively.

We shall prove various duality results for (MOP) and (XMOP) under generalized B-type I conditions.

THEOREM 5.1 *(Weak Duality). If, for all feasible x of (MOP) and all feasible (u,λ,y) of (XMOP), any of the following conditions holds*
a) $\lambda > 0$, $(f + y_{J_1}(t)^t g_{J_1} e,\ y_{J_2}(t)^t g_{J_2})$ is strong B- pseudo-quasi-type I at u with respect to b_0, b_1 and η with $b_1(x,u) > 0$,
b) $(f + y_{J_1}(t)^t g_{J_1} e,\ y_{J_2}(t)^t g_{J_2})$ is weak B-strictly pseudo-quasi-type I at u with respect to b_0, b_1 and η with $b_1(x,u) > 0$,
c) $(f + y_{J_1}(t)^t g_{J_1} e,\ y_{J_2}(t)^t g_{J_2})$ is weak B-strictly pseudo-type I at u with respect to b_0, b_1 and η;
then the following cannot hold

$$\int_a^b f(t,x,\dot{x})dt \leq \int_a^b \{f(t,u,\dot{u}) + [y_{J_1}(t)^t g_{J_1}(t,u,\dot{u})]e\}dt.$$

Proof. Let x be feasible for (MOP) and (u,λ,y) be feasible for (XMOP). Suppose that

$$\int_a^b f(t,x,\dot{x})dt \leq \int_a^b \{f(t,u,\dot{u}) + [y_{J_1}(t)^t g_{J_1}(t,u,\dot{u})]e\}dt.$$

Since x is feasible for (MOP) and (u,λ,y) is feasible for (XMOP), we have

$$\int_a^b \{f(t,x,\dot{x}) + [y_{J_1}(t)^t g_{J_1}(t,x,\dot{x})]e\}dt \leq$$
$$\leq \int_a^b \{f(t,u,\dot{u}) + [y_{J_1}(t)^t g_{J_1}(t,u,\dot{u})]e\}dt. \tag{17}$$

From (14), we get

$$-\int_a^b y_{J_2}(t)^t g_{J_2}(t,u,\dot{u})dt \leqq 0.$$

Since $(f + y_{J_1}(t)^t g_{J_1} e, y_{J_2}(t)^t g_{J_2})$ is strong B-pseudo-quasi-type I at u with respect to b_0, b_1 and η, we get

$$b_0(x,u)\int_a^b \eta(t,x,u)^t\Big[f_x(t,u,\dot{u}) + e y_{J_1}(t)^t g_{J_1 x}(t,u,\dot{u}) \\ -\frac{d}{dt}\Big(f_{\dot{x}}(t,u,\dot{u}) + e y_{J_1}(t)^t g_{J_1\dot{x}}(t,u,\dot{u})\Big)\Big]dt \le 0,$$

$$b_1(x,u)\int_a^b \eta(t,x,u)^t\Big[y_{J_2}(t)^t g_{J_2 x}(t,u,\dot{u}) - \frac{d}{dt} y_{J_2}(t)^t g_{J_2\dot{x}}(t,u,\dot{u})\Big)\Big]dt \leqq 0.$$

By $b_1(x,u) > 0$, and $\lambda > 0$, it follows that

$$b_0(x,u)\int_a^b \eta(t,x,u)^t\Big[\lambda^t f_x(t,u,\dot{u}) + y_{J_1}(t)^t g_{J_1 x}(t,u,\dot{u}) \\ -\frac{d}{dt}\Big(\lambda^t f_{\dot{x}}(t,u,\dot{u}) + y_{J_1}(t)^t g_{J_1\dot{x}}(t,u,\dot{u})\Big)\Big]dt < 0, \tag{18}$$

$$\int_a^b \eta(t,x,u)^t\Big[y_{J_2}(t)^t g_{J_2 x}(t,u,\dot{u}) - \frac{d}{dt} y_{J_2}(t)^t g_{J_2\dot{x}}(t,u,\dot{u})\Big)\Big]dt \leqq 0.$$

Since $b_0(x,u) \geqq 0$, we get

$$b_0(x,u)\int_a^b \eta(t,x,u)^t\Big[y_{J_2}(t)^t g_{J_2 x}(t,u,\dot{u}) - \frac{d}{dt} y_{J_2}(t)^t g_{J_2\dot{x}}(t,u,\dot{u})\Big)\Big]dt \leqq 0. \tag{19}$$

Adding (18) and (19), we obtain

$$b_0(x,u)\int_a^b \eta(t,x,u)^t\Big[\lambda^t f_x(t,u,\dot{u}) + y(t)^t g_x(t,u,\dot{u}) \\ -\frac{d}{dt}\Big(\lambda^t f_{\dot{x}}(t,u,\dot{u}) + \frac{d}{dt} y(t)^t g_{\dot{x}}(t,u,\dot{u})\Big)\Big]dt < 0, \tag{20}$$

which contradicts (13).

Now, by hypothesis (b) and from (14) and (17), we get

$$b_0(x,u)\int_a^b \eta(t,x,u)^t\Big[f_x(t,u,\dot{u}) + e y_{J_1}(t)^t g_{J_1 x}(t,u,\dot{u}) \\ -\frac{d}{dt}\Big(f_{\dot{x}}(t,u,\dot{u}) + e y_{J_1}(t)^t g_{J_1\dot{x}}(t,u,\dot{u})\Big)\Big]dt < 0,$$

$$b_1(x,u)\int_a^b \eta(t,x,u)^t\Big[y_{J_2}(t)^t g_{J_2 x}(t,u,\dot{u}) - \frac{d}{dt} y_{J_2}(t)^t g_{J_2\dot{x}}(t,u,\dot{u})\Big)\Big]dt \leqq 0.$$

Since $b_1(x,u) > 0$, $\lambda \geq 0$, and $b_0(x,u) \geqq 0$, we get (20) again contradicting (13).

If (c) holds, then from (14) and (17), we get

$$b_0(x,u)\int_a^b \eta(t,x,u)^t\Big[f_x(t,u,\dot{u}) + ey_{J_1}(t)^t g_{J_1x}(t,u,\dot{u}) \\ -\frac{d}{dt}\Big(f_{\dot{x}}(t,u,\dot{u}) + ey_{J_1}(t)^t g_{J_1\dot{x}}(t,u,\dot{u})\Big)\Big]dt < 0, \tag{21}$$

$$b_1(x,u)\int_a^b \eta(t,x,u)^t\Big[y_{J_2}(t)^t g_{J_2x}(t,u,\dot{u}) - \frac{d}{dt}y_{J_2}(t)^t g_{J_2\dot{x}}(t,u,\dot{u})\Big)\Big]dt < 0. \tag{22}$$

From (21) and (22), we have

$$\int_a^b \eta(t,x,u)^t\Big[f_x(t,u,\dot{u}) + ey_{J_1}(t)^t g_{J_1x}(t,u,\dot{u}) \\ -\frac{d}{dt}\Big(f_{\dot{x}}(t,u,\dot{u}) + ey_{J_1}(t)^t g_{J_1\dot{x}}(t,u,\dot{u})\Big)\Big]dt < 0, \tag{23}$$

$$\int_a^b \eta(t,x,u)^t\Big[y_{J_2}(t)^t g_{J_2x}(t,u,\dot{u}) - \frac{d}{dt}y_{J_2}(t)^t g_{J_2\dot{x}}(t,u,\dot{u})\Big)\Big]dt < 0. \tag{24}$$

Because $\lambda \geq 0$, (23) gives

$$\int_a^b \eta(t,x,u)^t\Big[\lambda^t f_x(t,u,\dot{u}) + y_{J_1}(t)^t g_{J_1x}(t,u,\dot{u}) \\ -\frac{d}{dt}\Big(\lambda^t f_{\dot{x}}(t,u,\dot{u}) + y_{J_1}(t)^t g_{J_1\dot{x}}(t,u,\dot{u})\Big)\Big]dt < 0, \tag{25}$$

adding (24) and (25), we get

$$\int_a^b \eta(t,x,x^*)^t\Big[\lambda^{*t} f_x(t,x^*,\dot{x}^*) + y^*(t)^t g_x(t,x^*,\dot{x}^*) \\ -\frac{d}{dt}\Big(\lambda^{*t} f_{\dot{x}}(t,x^*,\dot{x}^*) + \frac{d}{dt}y^*(t)^t g_{\dot{x}}(t,x^*,\dot{x}^*)\Big)\Big]dt < 0,$$

which contradicts again (13). ■

COROLLARY 5.1 *Let (u^*, λ^*, y^*) be a feasible solution for (XMOP). Assume that $y^*_{J_1}(t)^t g_{J_1}(t,u^*, \dot{u}^*) = 0$ and assume that u^* is feasible for (MOP). If weak duality Theorem* (5.1) *holds between (MOP) and (XMOP), then, u^* is an efficient solution for (MOP) and (u^*, λ^*, y^*) is an efficient solution for (XMOP).*

Proof. Suppose that u^* is not an efficient solution for (MOP), then there exists a feasible x for (MOP) such that

$$\int_a^b f(t,x,\dot{x})dt \leq \int_a^b f(t,u^*,\dot{u}^*)dt.$$

And since $y^*_{J_1}(t)^t g_{J_1}(t, u^*, \dot{u}^*) = 0$, we get

$$\int_a^b f(t,x,\dot{x})dt \leq \int_a^b \{f(t,u^*,\dot{u}^*) + [y^*_{J_1}(t)^t g_{J_1}(t,u^*,\dot{u}^*)]e\}dt.$$

Since (u^*, λ^*, y^*) is feasible for (XMOP) and x is feasible for (MOP), this inequality contradicts the weak duality Theorem (5.1).

Also suppose that (u^*, λ^*, y^*) is not an efficient solution for (XMOP). Then there exists a feasible solution (u, λ, y) for (XMOP) such that

$$\begin{aligned} &\int_a^b \{f(t,u,\dot{u}) + \Big[y_{J_1}(t)^t g_{J_1}(t,u,\dot{u})\Big]e\}dt \\ &\quad \geq \int_a^b \{f(t,u^*,\dot{u}^*) + \Big[y^*_{J_1}(t)^t g_{J_1}(t,u^*,\dot{u}^*)\Big]e\}dt, \end{aligned} \tag{26}$$

and since $y^*_{J_1}(t)^t g_{J_1}(t, u^*, \dot{u}^*) = 0$, (26) reduces to

$$\int_a^b f(t,u,\dot{u}) + \Big[y_{J_1}(t)^t g_{J_1}(t,u,\dot{u})\Big]e\}dt \geq \int_a^b f(t,u^*,\dot{u}^*)dt.$$

Since u^* is feasible for (MOP), this inequality contradicts weak duality Theorem (5.1). Therefore u^* and (u^*, λ^*, y^*) are efficient solutions for their respective programs. ■

THEOREM 5.2 *(Strong duality). Let x^* be an efficient solution for (MOP) at which the Kuhn-Tucker qualification constraint is satisfied, then there exists $\lambda^* \in I\!R^p$, $\lambda^* \geq 0$, $\lambda^{*t}e = 1$ and a piecewise smooth function $y^* : I \longrightarrow I\!R^m$ such that (x^*, λ^*, y^*) is feasible for (XMOP) with $y^*_{J_1}(t)^t g_{J_1}(t, x^*, \dot{x}^*) = 0$.*

If also weak duality Theorem (5.1) holds between (MOP) and (XMOP), then (x^*, λ^*, y^*) is an efficient solution for (XMOP).

Proof. Since x^* is an efficient solution for (MOP) at which the Kuhn-Tucker qualification constraint is satisfied, then there exists $\lambda^* \in I\!R^p$, $\lambda^* \geq 0$, $\lambda^{*t}e = 1$ and piecewise smooth function $y^* : I \longrightarrow I\!R^m$ such that (3)-(5) of Theorem (4.1) hold.

Moreover, $x^* \in K$, hence the feasibility of (x^*, λ^*, y^*) for (XMOP) follows. Also, because weak duality holds between (MOP) and (XMOP), therefore (x^*, λ^*, y^*) is an efficient solution for (XMOP).

If (x^*, λ^*, y^*) is not an efficient solution for (XMOP) then proceeding along the lines similar to that for the Corollary (13), we get a contradiction to the weak duality. ■

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