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The stability radius of an efficient solution in minimax Boolean programming problem

by

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Abstract: We consider a vector minimax Boolean programming problem. The problem consists in finding the set of Pareto optimal solutions. When the problem's parameters vary then the optimal solution of the problem obtained for some initial parameters may appear non-optimal. We calculate the maximal perturbation of parameters which preseves the optimality of a given solution of the problem. The formula for the stability radius of the given Pareto optimal solution was obtained.

Keywords: sensitivity analysis, stability radius, Boolean programming, Pareto optimal solution.

1. Introduction

Stability theory is an integral part of any traditional section of mathematics. J. Hadamard included the stability condition in the concept of a well-posed mathematical problem on a par with conditions of existence and uniqueness of solution (see Hadamard, 1902). In optimization the question of stability of a problem arises in the case when the set of feasible solutions and (or) the choice function depend on parameters, for which the area of change is known only. The presence of such parameters in optimization models is caused by inaccuracy of initial data, non-adequacy of models to real processes, errors of numerical methods, errors of approximation and other factors. So, it appears important to the classes of problems in which small changes of input data lead to small changes of the result. The problems with such properties are called stable. It is obvious that any optimization problem arising in practice can not be correctly formulated and solved without the use of results of the stability theory.

The common presence of discrete optimization models in economy, management and design caused a great interest of many specialists in the questions of stability, sensitivity and postoptimality analysis of combinatorial optimization problems (see, e.g., Sotskov et al., 1995, 1998; Sergienko et al., 1995; Libura 1996, 2000; Greenberg, 1998; Hoesel and Wagelmans, 1999).

A vector optimization problem is usually understood as the problem of finding a set of efficient solutions, i.e. of choosing from the set of feasible solutions the alternatives which satisfy a given optimality principle. In the case where the partial criteria of the problem have an equal importance, the Pareto optimality principle (see Pareto, 1909) is most often used. Investigation of the stability of a vector optimization problem means usually the study of behavior of the set of efficient solutions under changing problem's parameters.

One of the methods of sensitivity analysis is related to the finding of the so called stability radius (see e.g. Sotskov et al., 1995; Chakravarti and Wagelmans, 1999), defined as the limiting level of perturbations of problem's parameters for which the initially optimal solution remains optimal.

2. Statement of the problem

The problem of stability in the minimax Boolean programming problem can be formulated as follows. Let $C = (c_{ij}) \in \mathbf{R}^{n\times m}$, $n, m \in \mathbf{N}$, $m \geq 2$, $C_i = (c_{i1}, c_{i2}, ..., c_{im})$, $\mathbf{E}^m = \{0,1\}^m$, and T be the non-empty subset of the permutations set S_m, which is defined on the set $N_m = \{1, 2, ..., m\}$. On the set of feasible solutions (i.e. Boolean vectors) $X \subseteq \mathbf{E}^m$, $|X| > 1$, we define the vector criterion

$$f(x, C) = (f_1(x, C_1), f_2(x, C_2), ..., f_n(x, C_n)) \longrightarrow \min_{x \in X}.$$

The components (partial criteria) are functions

$$f_i(x, C_i) = \max_{t\in T} \sum_{j \in N(x)} c_{it(j)},\ i \in N_n,$$

where

$$t = \left[\begin{array}{cccc} 1 & 2 & \dots & m \\ t(1) & t(2) & \dots & t(m) \end{array} \right],$$

$$N(x) = \{j \in N_m :\ x_j = 1\},\ x = (x_1, x_2, ..., x_m)^T.$$

Suppose that $C_i[t] = (c_{it(1)}, c_{it(2)}, ..., c_{it(m)})$. Then we can rewrite the partial criteria in the following form

$$f_i(x, C_i) = \max C_i[t]x,\ i \in N_n.$$

The problem of finding the set of efficient solutions (the Pareto set)

$$P^n(C) = \{x \in X : \ \pi(x, C) = \emptyset\}$$

is called a vector minimax Boolean programming problem and denoted $Z^n(C)$, with

$$\pi(x, C) = \{x' \in X : \ q(x, x', C) \geq 0_{(n)}, \ q(x, x', C) \neq 0_{(n)}\},$$
$$q(x, x', C) = (q_1(x, x', C_1), \ q_2(x, x', C_2), \ \dots \ , q_n(x, x', C_n)),$$
$$q_i(x, x', C_i) = f_i(x, C_i) - f_i(x', C_i), \ i \in N_n, \ 0_{(n)} = (0, 0, \dots, 0) \in \mathbf{R}^n.$$

The number

$$\rho^n(x^0, C) = \begin{cases} \sup \Omega & \text{if } \Omega \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

is called the stability radius of the efficient solution x^0, where

$$e\Omega = \{\varepsilon > 0 : \ \forall C' \in \Re(\varepsilon) \ (x^0 \in P^n(C + C'))\},$$
$$\Re(\varepsilon) = \{C' \in \mathbf{R}^{n \times m} : \| C' \|_\infty < \varepsilon\}.$$

3. Basic results

For any $x^0 \neq x$ and any permutation $t \in T$ we introduce the following notations:

$$\begin{aligned} T(x^0, x) &= \{t \in T : \ \forall t' \in T \ (N(x^0, t) \neq N(x, t'))\}, \\ N(x, t) &= \{t(j) : \ j \in N_m, \ x_j = 1\}, \ \bar{T}(x^0, x) = T \backslash T(x^0, x). \end{aligned}$$

LEMMA 3.1. *Assume that $x^0 \neq x$, $t^0 \in \bar{T}(x^0, x)$. Then $C_i[t^0]x^0 \leq f_i(x, C_i)$ for any index $i \in N_n$ and matrix $C \in \mathbf{R}^{n \times m}$.*

The efficient solution x^0 is called trivial if the set $T(x^0, x)$ is empty for any $x \in X \backslash \{x^0\}$ and non-trivial otherwise.

THEOREM 3.1. *The stability radius $\rho^n(x^0, C)$ of any trivial solution x^0 is infinite.*

Proof. Let $x^0 \in P^n(C)$. Since x is trivial, the equality $T = \bar{T}(x^0, x)$ is true for any $x \in X \backslash \{x^0\}$. By Lemma 3.1, the inequality $(C + C')_i[t^0]x^0 \leq f_i(x, C_i + C_i')$ holds for any $x \in X \backslash \{x^0\}$, $t^0 \in T$, $i \in N_n$, $C' \in \mathbf{R}^{n \times m}$. Hence $q(x^0, x, C + C') \leq 0_{(n)}$. So, the solution $x^0 \in P^n(C)$ preserves efficiency for any independent pertubation of matrix C. Thus, $\rho^n(x^o, C) = \infty$. Theorem 3.1 is proved. ∎

LEMMA 3.2. *Let x^0 be non-trivial, $\varphi > 0$. Suppose that for any matrix $C' \in \Re(\varphi)$ and $x \in X \backslash \{x^0\}$ there exists an index $i \in N_n$ such that $q_i(x, x^0, C_i + C_i') > 0$. Then, $x^0 \in P^n(C + C')$ for any matrix $C' \in \Re(\varphi)$.*

For any non-trivial solution x^0 put

$$\varphi^n(x^0, C) = \min_{x \in Q(x^0)} \max_{i \in N_n} \min_{t^0 \in T(x^0,x)} \max_{t \in T} \frac{C_i[t]x - C_i[t^0]x^0}{\sigma(x^0, t^0, x, t)},$$

where

$$\sigma(x^0, t^0, x, t) = |(N(x^0, t^0) \cup N(x, t)) \backslash (N(x^0, t^0) \cap N(x, t))|,$$
$$Q(x^0) = \{x \in X \backslash \{x^0\} : \ T(x^0, x) \neq \emptyset\}.$$

The following statements are true

$$C_i[t]x - C_i[t^0]x^0 + ||C_i||_\infty \sigma(x^0, t^0, x, t) \geq 0, \ i \in N_n, \tag{1}$$
$$t^0 \in \bar{T}(x, x^0) \ \longrightarrow \ \exists t \in T \ (\sigma(x^0, t^0, x, t) = 0). \tag{2}$$

It is easy to see that $0 \leq \varphi^n(x^0, C) < \infty$.

THEOREM 3.2. *The stability radius $\rho^n(x^0, C)$ of any non-trivial solution x^0 is expressed by the formula*

$$\rho^n(x^0, C) = \varphi^n(x^0, C).$$

Proof. First let us prove that $\rho^n(x^0, C) \geq \varphi := \varphi^n(x^0, C)$. For $\varphi = 0$, there is nothing to prove. Let $\varphi > 0$. Then, for any $x \in X \backslash \{x^0\}$ there exists an index $i \in N_n$ such that

$$\min_{t^0 \in T(x^0,x)} \max_{t \in T} \frac{C_i[t]x - C_i[t^0]x^0}{\sigma(x^0, t^0, x, t)} \geq \varphi.$$

Using (1) and (2) we get the following statements

$$\begin{aligned} q_i(x, x^0, C_i + C_i') &= \max_{t \in T} (C_i + C_i')[t]x - \max_{t^0 \in T} (C_i + C_i')[t^0]x^0 = \\ &= \min_{t^0 \in T} \max_{t \in T} (C_i[t]x - C_i[t^0]x^0 + C_i'[t]x - C_i'[t^0]x^0) \geq \\ &\geq \min_{t^0 \in T} \max_{t \in T} (C_i[t]x - C_i[t^0]x^0 - ||C_i'||_\infty \sigma(x^0, t^0, x, t)) = \\ &= \min_{t^0 \in T(x^0,x)} \max_{t \in T} (C_i[t]x - C_i[t^0]x^0 - ||C_i'||_\infty \sigma(x^0, t^0, x, t)) > \\ &> \min_{t^0 \in T(x^0,x)} \max_{t \in T} (C_i[t]x - C_i[t^0]x^0 - \varphi\sigma(x^0, t^0, x, t)) \geq 0. \end{aligned}$$

By Lemma 3.2, we obtain that any non-trivial solution x^0 preserves efficiency for any perturbing matrix $C' \in \Re(\varphi)$ (i.e $\rho^n(x^0, C) \geq \varphi$). It remains to check that $\rho^n(x^0, C) \leq \varphi$. According to the definition of φ, there exists $x \in X \backslash \{x^0\}$ such that for any $i \in N_n$

$$\varphi \geq \min_{t^0 \in T(x^0,x)} \max_{t \in T} \frac{C_i[t]x - C_i[t^0]x^0}{\sigma(x^0, t^0, x, t)} = \max_{t \in T} \frac{C_i[t]x - C_i[\tilde{t}]x^0}{\sigma(x^0, \tilde{t}, x, t)}. \tag{3}$$

Let $\varepsilon > 0$. Consider the following perturbing matrix C_i^*. Every row of this matrix consists of the elements

$$c_{ij}^* = \begin{cases} \alpha & \text{if } j \in N(x^0, \tilde{t}) \\ -\alpha & \text{otherwise} \end{cases}$$

where $\varphi < \alpha < \varepsilon$. Using (3) we get the following expressions:

$$\begin{aligned} & q_i(x, x^0, C_i + C_i^*) = \max_{t \in T}(C_i + C_i^*)[t]x - \max_{t \in T}(C_i + C_i^*)[t]x^0 \leq \\ & \max_{t \in T}(C_i + C_i^*)[t]x - (C_i + C_i^*)[\tilde{t}]x^0 = (C_i + C_i^*)[\hat{t}]x - (C_i + C_i^*)[\tilde{t}]x^0 = \\ & = C_i[\hat{t}]x - C_i[\tilde{t}]x^0 - \alpha\sigma(x^0, \tilde{t}, x, \hat{t}) < C_i[\hat{t}]x - C_i[\tilde{t}]x^0 - \varphi\sigma(x^0, \tilde{t}, x, \hat{t}) \leq \\ & \leq C_i[\hat{t}]x - C_i[\tilde{t}]x^0 - \sigma(x^0, \tilde{t}, x, \hat{t}) \max_{t \in T} \frac{C_i[t]x - C_i[\tilde{t}]x^0}{\sigma(x^0, \tilde{t}, x, t)} \leq \\ & \leq C_i[\hat{t}]x - C_i[\tilde{t}]x^0 - \sigma(x^0, \tilde{t}, x, \hat{t}) \frac{C_i[\hat{t}]x - C_i[\tilde{t}]x^0}{\sigma(x^0, \tilde{t}, x, \hat{t})} = 0. \end{aligned}$$

Hence, x^0 is not an efficient solution of the perturbed problem $Z^n(C + C^*)$, where $C^* \in \Re(\varphi)$. It means that $\rho^n(x^0, C) \geq \varphi$. That completes the proof of Theorem 3.2. ■

4. Supplementary remarks

In a particular case, when $|T| = 1$, $t = \begin{bmatrix} 1 & 2 & \dots & m \\ 1 & 2 & \dots & m \end{bmatrix}$, we have the usual vector Boolean programming problem with linear partial criteria:

$$f_i(x, C_i) = C_i x, \; i \in N_n.$$

Then, from Theorem 3.2 we conclude that the stability radius of any solution x^0 is expressed by the formula (see Emelichev et al., 2003)

$$\rho^n(x^0, C) = \min_{x \in X \setminus \{x^0\}} \max_{i \in N_n} \frac{C_i x - C_i x^0}{\sigma(x^0, x)},$$

where

$$\sigma(x^0, x) = \sum_{j=1}^{m} |x - x^0| = \sum_{j=1}^{m} |x| + \sum_{j=1}^{m} |x^0| - 2 < x, x^0 > .$$

For the case of a scalar problem ($n = 1$), our formula for the stability radius transforms into (compare with Libura, 1993)

$$\rho^n(x^0, C) = \min_{x \in X \setminus \{x^0\}} \frac{C_i x - C_i x^0}{\sigma(x^0, x)}. \tag{4}$$

At the end of this paper we give a small example.

Let $n = m = 2, X = \{x_1, x_2\},\ x_1 = (0,1),\ x_2 = (1,1),\ T = \{t_1, t_2\}$,

$$t_1 = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix},\ t_2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.$$

Then

$$f(x_1, C) = (1,2),\ f(x_2, C) = (2,3), P^2(C) = \{x_1\}.$$

By applying Theorem 3.2 we get $\rho^2(x_1, C) = \frac{1}{2}$. If we remove t_2 from the set T, then

$$f(x_1, C) = (1,1),\ f(x_2, C) = (2,3), P^2(C) = \{x_1\}.$$

Using (4) we get $\rho^2(x_1, C) = 1$.

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