Euler-Poincaré reduction
of externally forced rigid body motion

by

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Abstract: If a mechanical system experiences symmetry, the Lagrangian becomes invariant under a certain group action. This property leads to substantial simplification of the description of movement. The standpoint in this article is a mechanical system affected by an external force of a control action. Assuming that the system possesses symmetry and the configuration manifold corresponds to a Lie group, the Euler-Poincaré reduction breaks up the motion into separate equations of dynamics and kinematics. This becomes of particular interest for modeling, estimation and control of mechanical systems. A control system generates an external force, which may break the symmetry in the dynamics. This paper shows how to model and to control a mechanical system on the reduced phase space, such that complete state space asymptotic stabilization can be achieved. The paper comprises a specialization of the well-known Euler-Poincaré reduction to a rigid body motion with forcing. An example of satellite attitude control illustrates usefulness of the Euler-Poincaré reduction in control engineering. This work demonstrates how the energy shaping method applies for Euler-Poincaré equations.

Keywords: modelling, control, rigid body motion, Lagrange mechanics, energy shaping.

1. Introduction

A description of a mechanical system with forcing is addressed in this paper. It focuses on modelling of a particular system, a rigid body. The issue has been exhaustively analyzed in the literature of classical mechanics. This gives freedom to treat it from a Hamiltonian or a Lagrangian point of view, as a motion on: Riemannian, symplectic or Poisson manifold. The standard references on this subject are Goldstein (1980), Abraham and Marsden (1978), Marsden and Ratiu (1999). It is the variational principles that are assumed in this article as axioms and the equations of motion are derived therefrom. Let $I \subseteq \mathbb{R}$ be an open interval. A motion in a set S denotes a smooth curve $\gamma : I \to S$. The equations of motion are differential equations, whose flow lines correspond to motions. If the configuration manifold is a Lie group and the Lagrangian becomes invariant under a group action, in this work the left translation, the motion can be transformed using Euler-Poincaré reduction into two sets of equations: kinematics and dynamics; Marsden and Ratiu (1999, Ch. 13.6). This description is of particular interest for modelling in Morton (1994), control in Wiśniewski and Kulczycki (2003), and estimation in Bak (1999).

The work merges two known techniques: the Euler-Poincaré reduction of classical mechanics and the energy shaping of control engineering. Main focus in the literature on mechanics is on reducing differential equations describing the motion of a mechanical system, which are invariant under the action of a Lie group. Hence, one obtains equations with fewer coordinates or even a globally defined differential operator on a quotient manifold; Marsden and Sheurle (1993), Marsden and Ratiu (1999), Koiller (1992). Control of mechanical systems with symmetry was treated before e.g. in Bloch, Leonard, Marsden (2000), Bloch et al. (1996). In these works the internal forces gave rise to the control action, however, the effect of general forces was not discussed. The energy shaping method will be applied in this paper. In its most common formulation it gives a control action, being the sum of the gradient of potential energy and the dissipation force; Nijmeijer and van der Schaft (1990, Ch. 12) and Koditschek (1989). In this article the energy shaping method will be adopted to a mechanical system with symmetry. It is shown that the reduction of the motion of a mechanical system can be used for feedback synthesis, despite the symmetry breaking property of the control action.

The article constitutes a tutorial on modelling the motion of a rigid body. Relevant notions of classical mechanics are recalled first. Subsequently, the article introduces the Euler-Poincaré reduction for a mechanical system with forcing, which is then implemented for the rotary motion of a rigid body. Two configuration manifolds are of interest, the special orthogonal group SO_3 of particular interest in robotics, and the group of unit quaternions Sp_1 used in aerospace for a global representation of attitude. An example of satellite attitude control, wherein the Euler-Poincaré description of the rigid body motion

In this work M stands for a C^∞ n-manifold with smooth structure $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \mathcal{U}}$. The system $\pi_{TG} : TM \to M$ defines the tangent bundle, and $\pi_{T^*G} : T^*M \to M$ the cotangent bundle of M. The main concern of this work will be the motion of a system with forcing.

DEFINITION 1 *A force field on a configuration manifold M is a fiber preserving map, $F : TM \to T^*M$ over the identity. It means that for each U_α, $\alpha \in \mathcal{U}$ the following diagram commutes*

$$
\begin{array}{ccc}
TU_\alpha & \xrightarrow{\ F\ } & T^*U_\alpha \\
\downarrow{\scriptstyle \pi_{TG}} & & \downarrow{\scriptstyle \pi_{T^*G}} \\
U_\alpha & \xrightarrow[\ \text{id}\ ]{} & U_\alpha.
\end{array}
$$

The Lagrange-d'Alembert principle is in the sequel stated in terms of the variational calculus. If $\gamma : [a, b] \to M$ denotes a piecewise smooth curve, a variation of γ means a family $\Gamma : [-\epsilon, \epsilon] \times [a, b] \to M$ of piecewise smooth curves such that $\Gamma_0(t) = \gamma(t)$ for all $t \in [a, b]$. It is called a proper variation if, in addition, $\Gamma_s(a) = \gamma(a)$ and $\Gamma_s(b) = \gamma(b)$ for all $s \in [-\epsilon, \epsilon]$. A variation field $\delta\gamma$ of the variation Γ means the vector field along γ, $\delta\gamma : [a, b] \to T_{\gamma(t)}M$ defined by

$$
\delta\gamma(t) = (d\Gamma_t)_0 \left(\frac{\partial}{\partial s} \right) = \left. \frac{\partial \Gamma(s, t)}{\partial s} \right|_{s=0},
$$

where $(d\Gamma_t)_s : T_s \mathbb{R} \to T_{\Gamma_t(s)} M$ denotes the differential of Γ_t at s, and $\frac{\partial}{\partial s}$ stands for the basis of $T_s \mathbb{R}$. A vector field V along γ is proper if it vanishes at the endpoints, i.e. $\delta\gamma(a) = \delta\gamma(b) = 0$. Thus, the variation field of a proper variation is proper. For details, refer to Lee (1997).

The next definition expresses the Lagrange-d'Alembert principle. It is an axiom stating the conditions for a mechanical system, with a given Lagragian and known external forces, to follow a motion $(\gamma, \dot\gamma) \in TM$.

DEFINITION 2 (7.8.4 IN MARSDEN AND RATIU, 1999) *Given a Lagrangian $L : TM \to \mathbb{R}$ and a force field $F : TM \to T^*M$, the integral Lagrange-d'Alembert principle for a curve $\gamma(t)$ with the proper variation $\Gamma_s(t)$ is*

$$
\left. \frac{\partial}{\partial s} \right|_{s=0} \int_a^b L(\Gamma_s(t), \dot\Gamma_s(t)) dt + \int_a^b F(\gamma(t), \dot\gamma(t))(\delta\gamma(t)) dt = 0. \tag{1}
$$

The motion appears particularly simple for the configuration manifold being a finite dimensional Lie group G. The emphasis is put in this work on this class of configuration manifolds. The Lie algebra T_eG of G is denoted by $\mathfrak{g}$. Every group element $a \in G$ defines a left translation $\mathcal{L}_a : G \to G$, $g \mapsto ag$. It also

DEFINITION 3 (2.10 IN BRÖCKER AND DIECK, 1985) *The adjoint representation is a homomorphism*

$$Ad : G \to Aut(\mathfrak{g}), \quad g \mapsto (dc_g)_e = Ad_g,$$

where Ad_g means the differential of c_g at the unit element $(dc_g)_e : \mathfrak{g} \to \mathfrak{g}$. *The adjoint representation* Ad *induces a homomorphism of Lie algebras*

$$ad : \mathfrak{g} \to End(\mathfrak{g}), \quad X \mapsto (dAd_X)_e = ad_X,$$

where $Ad_X : G \to \mathfrak{g}, \quad g \mapsto Ad_g X$.

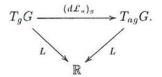The map ad sends X to the homomorphism $Y \mapsto [X, Y]$. Thus

$$[X, Y] = ad_X Y.$$

As mentioned before, the Lagrange-d'Alembert principle gives the condition for a curve on the tangent bundle TG to represent a motion. However, if the Lagrangian $L : TG \to \mathbb{R}$ turns out to be invariant under the left translation, the equations of motion are particularly simple. They break up into two separate equations: the kinematics and the dynamics, hence the motion corresponds to a curve $I \to G \times \mathfrak{g}$. This constitutes the contents of Section 2. Rotary motion of rigid body comprises an important example of the above. Its motion is defined on a linear Lie group. Section 3. addresses the case of the special orthogonal group SO_3, and Section 4. treats the group of unit quaternions Sp_1. Section 5. gives an example of a control application. It shows that the energy shaping method applies to systems modeled by the Euler-Poincaré equations, and a controller for three-axis stabilization of a rigid body is synthesized.

2. Euler-Poincaré motion

The Euler-Poincaré equation with forcing will be formulated in this section. A mechanical system may experience a certain symmetry, expressed in the sequel by the invariance of the Lagrangian under the left translation.

DEFINITION 4 *The Lagrangian $L : TG \to \mathbb{R}$ is left invariant if the following diagram commutes*

$$
\begin{array}{ccc}
T_g G & \xrightarrow{(d\mathcal{L}_a)_g} & T_{ag} G. \\
& \searrow{\scriptstyle L} \quad \swarrow{\scriptstyle L} & \\
& \mathbb{R} &
\end{array}
$$

Assuming the Lagrangian invariant under the left translation, the objective is to consider independently the dynamics, i.e. the motion on the Lie algebra $\mathfrak{g}$ and the kinematics, the motion on the Lie group G. For this purpose, the translation of the variation vector field will be examined. Namely, the differential of the

PROPOSITION 1 (5.1 IN BLOCH ET AL., 1996) *Let* $\Gamma(s,t) : U \subset \mathbb{R}^2 \to G$ *be a variation of a curve* $\gamma(t)$ *on a Lie group* G, *and denote* $\Xi, \Lambda : U \to \mathfrak{g}$ *by*

$$\Xi(s,t) = \left(d\mathcal{L}_{\Gamma(s,t)^{-1}}\right)_{\Gamma(s,t)} \left(\frac{\partial \Gamma(s,t)}{\partial t}\right) \tag{2}$$

and

$$\Lambda(s,t) = \left(d\mathcal{L}_{\Gamma(s,t)^{-1}}\right)_{\Gamma(s,t)} \left(\frac{\partial \Gamma(s,t)}{\partial s}\right). \tag{3}$$

Then

$$\frac{\partial \Xi(s,t)}{\partial s} - \frac{\partial \Lambda(s,t)}{\partial t} = [\Xi(s,t), \Lambda(s,t)]. \tag{4}$$

Conversely, if U *is simply connected and* $\Xi, \Lambda : U \to \mathfrak{g}$ *are smooth functions satisfying (4) then there exists a smooth function* $\Gamma : U \to G$ *satisfying (2) and (3).*

The tangent space $T_{\Xi(s,t)}\mathfrak{g}$ in Proposition 1 is isomorphic to the Lie algebra $\mathfrak{g}$, and through the rest of the paper $T_{\Xi(s,t)}\mathfrak{g}$ and $\mathfrak{g}$ are canonically identified with $\mathbb{R}^n$, where n denotes the dimension of the manifold G. The theorem below states the main results.

THEOREM 1 *Let* G *be a Lie group with Lie algebra* $\mathfrak{g}$, $L : TG \to \mathbb{R}$ *be a left invariant Lagrangian,* $l : \mathfrak{g} \to \mathbb{R}$ *be its restriction to the Lie algebra and* $F : TG \to T^*G$ *a force field. For a curve* $\gamma : [a,b] \to G$, *let* $\xi : [a,b] \to \mathfrak{g}$, $\xi(t) = \left(d\mathcal{L}_{\gamma(t)^{-1}}\right)_{\gamma(t)} \dot{\gamma}(t)$.

Then the integral Lagrange-d'Alembert principle

$$\left.\frac{\partial}{\partial s}\right|_{s=0} \int_a^b L(\Gamma_s(t), \dot{\Gamma}_s(t))dt + \int_a^b F(\gamma(t), \dot{\gamma}(t))(\delta\gamma(t))dt = 0 \tag{5}$$

holds for all proper variations, and is equivalent to the Euler-Poincaré equation with forcing

$$\frac{d}{dt}dl_{\xi(t)} = ad^*_{\xi(t)}dl_{\xi(t)} + \left(d\mathcal{L}_{\gamma(t)}\right)^*_e F(\gamma(t), \dot{\gamma}(t)) \tag{6}$$

$$\dot{\gamma}(t) = \left(d\mathcal{L}_{\gamma(t)}\right)_e \xi(t). \tag{7}$$

Equation (6) denotes the dynamics and (7) the kinematics.
Proof of Theorem 1. Vector fields $\Xi, \Lambda : U \to \mathfrak{g}$ are defined as in Proposition (1)

$$\Xi(s,t) = \left(d\mathcal{L}_{\Gamma(s,t)^{-1}}\right)_{\cdots} \left(\frac{\partial \Gamma(s,t)}{\cdots}\right)$$

$$\Lambda(s,t) = \left(d\pounds_{\Gamma(s,t)^{-1}}\right)_{\Gamma(s,t)} \left(\frac{\partial \Gamma(s,t)}{\partial s}\right)$$

$$\xi(t) = \Xi(0,t) \text{ and } \lambda(t) = \Lambda(s,t).$$

Since L is left invariant, meaning

$$L\left(\Gamma(s,t), \frac{\partial \Gamma(s,t)}{\partial t}\right) = L\left(\pounds_{\Gamma(s,t)^{-1}}\Gamma(s,t), \left(d\pounds_{\Gamma(s,t)^{-1}}\right)_{\Gamma(s,t)} \frac{\partial \Gamma(s,t)}{\partial t}\right)$$
$$= L(e, \Xi(s,t),$$

the first part of (5) becomes

$$\left.\frac{\partial}{\partial s}\right|_{s=0} \int_a^b L(\Gamma_s(t), \dot\Gamma(s,t))\, dt = \left.\frac{\partial}{\partial s}\right|_{s=0} \int_a^b l(\Xi(s,t))\, dt$$
$$= \int_a^b (dl)_{\xi(t)}(\delta\xi(t))\, dt. \tag{8}$$

In (8) the chain rule was used

$$\frac{\partial(l \circ \Xi(s,t))}{\partial s} = (d(l \circ \Xi_t(s)))_s \left(\frac{\partial}{\partial s}\right) = (dl)_{\Xi(s,t)}(d\Xi_t(s))_s \left(\frac{\partial}{\partial s}\right)$$
$$= (dl)_{\Xi(s,t)} \frac{\partial\Xi(s,t)}{\partial s}.$$

According to Proposition (1) the variation field of $\Xi(s,t)$ is of the form

$$\delta\xi(t) = \left.\frac{\partial\Xi(s,t)}{\partial s}\right|_{s=0} = \frac{\partial\lambda(t)}{\partial t} + ad_{\xi(t)}\lambda(t). \tag{9}$$

Substituting (9) into (8) and using integration by parts gives

$$\int_a^b (dl)_{\xi(t)}(\delta\xi(t))\, dt = \int_a^b (dl)_{\xi(t)} \left(\frac{\partial\lambda(t)}{\partial t} + ad_{\xi(t)}\lambda\right) dt$$
$$= \int_a^b \left(-\frac{d}{dt}(dl)_{\xi(t)} + ad^*_{\xi(t)}(dl)_{\xi(t)}\right)(\lambda(t))\, dt. \tag{10}$$

The right hand side of (5) can be rewritten as

$$\int_a^b F(\gamma(t), \dot\gamma(t))(\delta\gamma(t))\, dt = \int_a^b F(\gamma(t), \dot\gamma(t))(((d\pounds_{\gamma(t)^{-1}})_{\gamma(t)})^{-1}\lambda(t))\, dt$$
$$= \int_a^b F(\gamma(t), \dot\gamma(t))((d\pounds_{\gamma(t)})_e\lambda(t))\, dt$$
$$\int_a^b {}_{(d\pounds\quad)^* F(\gamma(t),\ \dot\gamma(t))(\lambda(t))}\, dt \tag{11}$$

Comparing (10) and (11) with (5) and using the fundamental lemma of calculus of variations, we obtain the Euler-Poincaré equation (6). ∎

Theorem 1 gives a general expression of motion on a Lie group. The next two sections address equations of motion for a particular mechanical system, a rigid body.

3. Reduction on SO_3

The objective of this section is to derive equations of motion for a rigid body. The special orthogonal group

$$G = SO_3 = \{A \in GL_3(\mathbb{R}) : A^T A = I \text{ and } det(A) = 1\}$$

comprises the configuration manifold. The Lie algebra of SO_3 will be first identified, and its properties will be subsequently examined. The section concludes with formulation of the equation of motion for the rigid body with forcing.

The Lie algebra of SO_3 consists of all skew symmetric matrices

$$\mathfrak{so}_3 = T_e SO_3 = SS_3 = \{A \in GL_3(\mathbb{R}) : A^T = -A\}$$

and it is spanned by E_1, E_2 and E_3

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The following isomorphism of vector spaces shall be introduced

$$s : \mathbb{R}^3 \to SS_3, \quad (x_1, x_2, x_3) \mapsto x_1 E_1 + x_2 E_2 + x_3 E_3 = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

The map s can be used to represent the cross product $a \times b = s(a)b$. This makes s a Lie algebra isomorphism

$$s : (\mathbb{R}^3, \times) \to (SS_3, [\cdot, \cdot]),$$

taking $a \times b$ to $[s(a), s(b)]$. Since SO_3 is a subgroup of $GL_3(\mathbb{R})$ the multiplication of matrices describes the differential of the left translation, i.e.

$$(d\mathcal{L}_A)_B : T_B SO_3 \to T_{AB} SO_3, \quad C \mapsto AC.$$

The kinematics for a matrix group follows

$$\dot{\gamma}(t) = (d\mathcal{L}_{\gamma(t)})_e \, \xi(t) = \gamma(t)\xi(t). \tag{12}$$

Equation (12) defines relation between the velocity $\dot{\gamma}(t) \in T_{\gamma(t)} SO_3$ and $\xi(t)$,
an element of the Lie algebra

Define an angular velocity as $\omega(t) = s^{-1}(\xi(t))$ and the Lagrangian $\tilde{l} = l \circ s : \mathbb{R}^3 \to \mathbb{R}$. The Lagragian comprises the kinetic energy only

$$\tilde{l}(\omega) = T(\omega) = \frac{1}{2}\omega^{\mathrm{T}}J\omega,$$

where J denotes the inertia matrix. The Lagrangian turns out to be left invariant and the assumption of Theorem 1 is satisfied. To establish the equations of motion, the differential of the Lagrangian

$$d\tilde{l}_\omega = J\omega$$

and an explicit expression for $ad_\xi^* dl_\xi(t)$

$$ad_\xi^* dl_\xi(X) = dl_\xi([\xi, X]), \tag{13}$$

where $X \in \mathfrak{so}_3$, are provided. Since s is the Lie algebra isomorphism, Eq. (13) becomes

$$ad_\omega^* d\tilde{l}_\omega(s^{-1}(X)) = d\tilde{l}_\omega \cdot (\omega \times s^{-1}(X)) = (d\tilde{l}_\omega \times \omega) \cdot s^{-1}(X).$$

Concluding,

$$ad_\omega^* d\tilde{l}_\omega = d\tilde{l}_\omega \times \omega,$$

and the dynamics follows

$$\frac{d}{dt}(J\omega(t)) = J\omega(t) \times \omega(t) + s^{-1}(\gamma(t)^* F(\gamma(t), \dot{\gamma}(t))). \tag{14}$$

Equation (14) is indeed the celebrated equation of the rigid body dynamics, where the second summand corresponds to the external torque. However, it appears central for this work that the torque can be computed explicitly from the force field. Thus, the control algorithms derived from the Lagrangian or Hamiltonian formalism, which provide the control force field, can be directly implemented for an Euler-Poincaré system. In particular, the energy shaping method in Section 5. applies for control of a rigid body.

4   Reduction on unit quaternions

Alternatively, a group of all unit quaternions could be taken as the configuration manifold. This attitude representation plays an important role in aerospace and robotics. Quaternions owe their significance due to simple physical interpretation of an angle and an axis of rotation. For small angles the three components of the vector part of a quaternion approximate pitch, roll and yaw. Furthermore there is a variety of estimation algorithms based on quaternionic representation

It is vital for this exposition to examine its geometric and algebraic proper-
ties. The unit quaternions can be viewed as a three sphere imbedded in $\mathbb{R}^4$ or
more conveniently for computation as a complex matrix group. Both interpre-
tations are treated in this section.

The quaternion algebra $\mathbb{H}$ will be defined first. The $\mathbb{R}$-algebra $\mathbb{H}(+, \cdot)$ is the
division algebra of 2 by 2 complex matrices of the form

$$\mathbb{H} = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : a, b \in \mathbb{C} \right\},$$

with matrix addition and multiplication. Another definition of quaternions is
the algebra $\mathbb{R}^4(+, \cdot)$ with standard addition in $\mathbb{R}^4$ and a product given by the
following formula:

$$x \cdot y = Q(x)y, \tag{15}$$

where

$$Q(x) = \begin{bmatrix} x^0 & -x^1 & -x^2 & -x^3 \\ x^1 & x^0 & -x^3 & x^2 \\ x^2 & x^3 & x^0 & -x^1 \\ x^3 & -x^2 & x^1 & x^0 \end{bmatrix}.$$

The algebras $\mathbb{H}(+, \cdot)$ and $\mathbb{R}^4(+, \cdot)$ are isomorphic with a ring isomorphism given
by

$$\tilde{w} : \mathbb{R}^4 \to \mathbb{H}, \quad (x_0, \ x_1, \ x_2, \ x_3) \mapsto \begin{bmatrix} x_0 - ix_3 & -x_2 - ix_1 \\ x_2 - ix_1 & x_0 + ix_3 \end{bmatrix}.$$

Since a configuration manifold of a Lie group is in focus, only the group
properties of $\mathbb{H}$ will be further exploited. Specifically, the quaternions with the
norm

$$N\left(\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \right) = |a|^2 + |b|^2$$

equal one, are of interest. The unit quaternions form a group

$$Sp_1 = \{x \in \mathbb{H} : N(x) = 1\},$$

with the product inherited from $\mathbb{H}$. In fact Sp_1 is the same as the special unitary
group

$$SU_2 = \{A \in GL_2(\mathbb{C}) : A^*A = I \text{ and } det(A) = 1\},$$

and makes up a subgroup of the Lie group $GL_2(\mathbb{C})$. The matrix group SU_2

The three-sphere constitutes the second interpretation of the unit quaternion. The differential manifold SU_2 becomes indeed diffeomorphic to the three-sphere $S^3 = \{x \in \mathbb{R}^4 : ||x|| = 1\}$ with a diffeomorphism

$$w : S^3 \to SU_2, \ (x_0, \ x_1, \ x_2, \ x_3) \mapsto \begin{bmatrix} x_0 - ix_3 & -x_2 - ix_1 \\ x_2 - ix_1 & x_0 + ix_3 \end{bmatrix}.$$

It appears useful to treat the three-sphere as a Lie subgroup of $(\mathbb{R}^3, \cdot)$, then the map $w : (S^3, \cdot) \to (SU_2, \cdot)$ is a group isomorphism, and $x \cdot y = w^{-1}(w(x)w(y))$.

The Lie algebra of SU_2 consists of the 2 by 2 skew-Hermitian traceless matrices $\mathfrak{su}_2 \subset \mathbb{H}$

$$\mathfrak{su}_2 = T_e SU_2 = \{A \in GL_2(\mathbb{C}) : A = A^* \text{ and } tr(A) = 0\}.$$

It shall be noted that the Pauli spin matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \ \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

span $\mathfrak{su}_2$. The map

$$r : \mathbb{R}^3 \to \mathfrak{su}_2, \ (x_1, \ x_2, \ x_3) \mapsto \frac{1}{2i}(x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3) = \frac{1}{2} \begin{bmatrix} -ix_3 & -x_2 - ix_1 \\ x_2 - ix_1 & ix_3 \end{bmatrix},$$

defines a Lie algebra isomorphism $(\mathbb{R}^3, \times) \to (\mathfrak{su}_2, [\cdot, \cdot])$ taking $X \times Y$ to $[r(X), r(Y)]$. It will be useful to write r as follows

$$r = \frac{1}{2}\tilde{w} \circ i, \text{ where } i : \mathbb{R}^3 \hookrightarrow \mathbb{R}^4, \ (x_1, \ x_2, \ x_3) \mapsto (0, \ x_1, \ x_2, \ x_3),$$

then its left inverse becomes

$$r^{-1} = 2\pi \circ \tilde{w}^{-1}|_{\mathfrak{su}_2} \text{ where } \pi : \mathbb{R}^4 \to \mathbb{R}^3, \ (x_0, \ x_1, \ x_2, \ x_3) \mapsto (x_1, \ x_2, \ x_3).$$

The remaining part of this section relies on Theorem 1 and the equations of motion for the rigid body are formulated. Since SU_2 is a subgroup of $GL_2(\mathbb{C})$, the multiplication of matrices gives the differential of the left translation

$$(d\mathcal{L}_A)_B : T_B SU_2 \to T_{AB} SU_2, \ C \mapsto AC.$$

The kinematics follows

$$\dot{\gamma}(t) = (d\mathcal{L}_{\gamma(t)})_e \ \xi(t) = \gamma(t)\xi(t).$$

Upon defining the angular velocity as $\omega(t) = r^{-1}(\xi(t))$ and $q(t) = w^{-1}(\gamma(t))$ the kinematics takes the familiar form

$$\dot{q}(t) = (dw)^{-1}_{\gamma(t)}\dot{\gamma}(t) = \tilde{w}^{-1}(w(q(t))r(\omega(t))) = \frac{1}{2}\tilde{w}^{-1}(\tilde{w}(q(t))\tilde{w}(i(\omega(t))))$$

Consider a Lagragian $\tilde{l} = l \circ r : \mathbb{R}^3 \to \mathbb{R}$. Now the Euler-Poincaré motion can be written

$$\frac{d}{dt} dl_\omega(t) = ad^*_\omega dl_\omega(t) + (dr)^*_e (d\pounds_{\gamma(t)})^*_e \, F(\gamma(t), \dot{\gamma}(t)).$$

Each term of the equation above will be computed separately in the sequel. As in Section 3. the Lagragian corresponds to the kinetic energy only

$$\tilde{l}(\omega) = T(\omega) = \frac{1}{2}\omega^{\mathrm{T}} J\omega,$$

where J denotes the inertia matrix. The Lagrangian is left invariant and Theorem 1 applies. As in the case of SO_3, the differential of the Lagrangian equals

$$d\tilde{l}_\omega = J\omega,$$

and the expression for $ad^*_\xi dl_\xi(t)$ takes on the form

$$ad^*_\xi dl_\xi(X) = dl_\xi([\xi, X]), \tag{16}$$

where $X \in \mathfrak{su}_2$. Since r is the Lie algebra isomorphism, (16) becomes

$$ad^*_\omega d\tilde{l}_\omega(r^{-1}(X)) = d\tilde{l}_\omega \cdot (\omega \times r^{-1}(X)) = (d\tilde{l}_\omega \times \omega) \cdot r^{-1}(X),$$

which gives

$$ad^*_\omega d\tilde{l}_\omega = d\tilde{l}_\omega \times \omega.$$

The external forcing is formulated as

$$(dr)^*_e ((d\pounds_\gamma)^*_e \, F(\gamma, \dot{\gamma}))(V) = (d\pounds_\gamma)^*_e \, F(\gamma, \dot{\gamma})((dr)_e(V))$$

$$= \gamma^* F(\gamma, \dot{\gamma})((dr)_e(V)) = \gamma^* F(\gamma, \dot{\gamma})(r(V)) = \frac{1}{2}\gamma^* F(\gamma, \dot{\gamma})(\tilde{w} \circ i(V))$$

$$= \frac{1}{2}\pi \circ \tilde{w}^*(\gamma^* F(\gamma, \dot{\gamma})(V)),$$

where $V \in T_e S^3 \cong T_e(T_e S^3)$. With a definition $f(q, \omega) = \tilde{w}^*(F(\gamma, \dot{\gamma}))$ the torque becomes

$$(dr)^*_e ((d\pounds_\gamma)^*_e \, F(\gamma, \dot{\gamma})) = \frac{1}{2}\pi \circ \tilde{w}^*(\gamma^* F(\gamma, \dot{\gamma}))$$

$$= \frac{1}{2}\pi \circ \tilde{w}^* \left((w^*)^{-1}(q^*)(w^*)^{-1}(f)\right) = \frac{1}{2}\pi \circ \tilde{w}^* \left((w^*)^{-1}(q^* \cdot f)\right)$$

$$= \frac{1}{2}\pi \circ q^* \cdot f(q, \omega) = \frac{1}{2}\pi(Q^*(q)f(q, \omega)).$$

The dynamics of the rigid body follows

$$\frac{d}{dt}(J\omega(t)) = J\omega(t) \times \omega(t) + \frac{1}{2}\pi(Q^{\mathrm{T}}(q(t))f(q(t), \omega(t))). \tag{17}$$

The second summand in (17) gives an explicit expression for the external torque. This form appears particularly useful for control synthesis. The energy shaping technique will be applied in the next section for computing the control

5. Control synthesis

The energy shaping has been formulated for a general mechanical system in Koditschek (1989) and Bullo and Murray (1999). The idea is to produce a control input consisting of a term contributing to potential energy and a part providing dissipation. In a simplest case, if a system functions in $\mathbb{R}^n$ and has potential energy $U : \mathbb{R}^n \to \mathbb{R}$, the energy shaping puts forward a feedback control of the form $-\frac{\partial V(q)}{\partial q} + M_d$, where $V : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function. The term M_d denotes a dissipative force. Assuming that the time derivative of its work $\dot{W} = M_d^T \dot{q}$ is negative definite, and the minimum of the potential energy $U + V$ is reached at a point p, the control law makes the system asymptotically stable to the equilibrium point $(q(t), \dot{q}(t)) = (p, 0)$. The name "shaping" comes form the property of the feedback that shapes the potential energy of the system to the desired form using the controller contribution V.

The energy shaping has its generalization for an arbitrary manifold G. Again, the control consists of a differential of a potential function $\phi : G \to \mathbb{R}$ and a dissipative force field $f_d : TG \to T^*G$ as indicated in the following equation:

$$f(\gamma, \dot{\gamma}) = -d\phi(\gamma) + f_d(\gamma, \dot{\gamma}). \tag{18}$$

The dissipative force field f_d satisfies $f_d(v)(v) < 0$ for all nonzero $v \in TG$. If p is a local minimum of ϕ, then according to Theorem 1 in Koditschek (1989), $(p, 0)$ becomes asymptotically stable equilibrium state of the closed loop system.

It follows from Section 2. that the control law (18) applies to the systems described by the Euler-Poincaré form. The control input becomes

$$M(\gamma(t), \dot{\gamma}(t)) = -(d\mathcal{L}_{\gamma(t)})_e^* d\phi(\gamma(t)) + (d\mathcal{L}_{\gamma(t)})_e^* f_d(\gamma(t), \dot{\gamma}(t)). \tag{19}$$

The first component in (19) will be called the conservative force and is denoted by M_c, whereas the second one constitutes the dissipative force, M_d.

An illustration of the energy shaping for the Euler-Poincaré system will be given in the remaining part of the article. Consider a rigid body, e.g. a spacecraft, to be stabilized in the inertial coordinate system with use of gas jets. The task is to design a suitable control law. For this purpose quaternionic parametrization of the attitude will be applied.

Consider the inclusion $j : S^3 \hookrightarrow \mathbb{R}^4$, and let the potential function ϕ parameterize through some smooth function $\tilde{\phi} : \mathbb{R}^n \to \mathbb{R}$, i.e. $\phi = \tilde{\phi} \circ j$. Since $(d\phi)_q = (d\tilde{\phi})_{j(q)}|_{T_q S^3}$, the differential $(d\phi)_q$ is

$$(d\phi)_q = Q(q) i\pi Q^*(q)(d\tilde{\phi})_{j(q)}, \tag{20}$$

where q_i are the canonical coordinate functions in $\mathbb{R}^n$, and $(d\tilde{\phi})_{j(q)} = \sum_{k=0}^{3} \frac{\partial \tilde{\phi}}{\partial q_i} dq_i$.

Making use of (17) and (20), we obtain the conservative force as equal

$$M_c = -\frac{1}{2} Q^T(q) \frac{\partial \phi(q)}{\partial q} - \frac{1}{2} [d^1 \phi, d^2 \phi, d^3 \phi]^T \tag{21}$$

where

$$[d^0\phi(q) \quad d^1\phi(q) \quad d^2\phi(q) \quad d^3\phi(q)] = \frac{\partial\phi(q)}{\partial q}Q(q).$$

Taking a dissipative force field

$$f_d = -D\dot{q},$$

where D indicates a positive definite matrix, and combining Eqs. (19), (21), the control law follows

$$M = -\frac{1}{2}[d^1\phi \ d^2\phi \ d^3\phi]^T - \frac{1}{2}\pi Q^T(q)D\dot{q}.$$

It was shown in Wiśniewski and Kulczycki (2003) that for a particular choice of $D = 4k_d E_{4\times4}$ and the potential function $\phi(q) = k_p(1 - q_0)$ having the global minimum at the identity e and the maximum at $-e$ the differential $d\phi(q)$ equals

$$[d^0\phi \ d^1\phi \ d^2\phi \ d^3\phi] = k_p [1 - q^0 \quad q^1 \quad q^2 \quad q^3]. \tag{22}$$

Now the control law reduces to the well known PD form

$$M = -k_p[q_1 \ q_2 \ q_3]^T - k_d\omega. \tag{23}$$

This shows that the energy shaping approach presented in this paper agrees with the previous results on the 3-axis attitude control summarized in Ting-Yung Wen and Kreutz-Delgado (1991). For other examples of potential functions used in guidance one is referred to Wiśniewski and Kulczycki (2004).

6. Conclusion

This work applied the calculus of variations to derive the Euler-Poincaré equations of motion with forcing. It showed that if the Lagrangian $L : TG \to \mathbb{R}$ was invariant under the left translation, the equations of motion broke up into two separate expressions: the kinematics and the dynamics. The rigid body motion considered comprised an illustrative example. The paper focused on two configuration manifolds: the special orthogonal group and the group of unit quaternions. It showed that the energy shaping method could be applied for the Euler-Poincaré system. The findings were applied for the rigid body stabilization in three axes. The resulting control consisted of the sum of the conservative and the dissipative force fields.

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