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On multi-criteria approaches to bandwidth allocation*

by

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Abstract: Modern telecommunication networks face an increasing demand for services. Among these, an increasing number are services that can adapt to available bandwidth, and are therefore referred to as elastic traffic. Nominal network design for elastic traffic becomes increasingly significant.

Typical resource allocation methods are concerned with the allocation of limited resources among competing activities so as to achieve the best overall performance of the system. In the network dimensioning problem for elastic traffic, one needs to allocate bandwidth to maximize service flows and simultaneously to reach a fair treatment of all the elastic services. Thus, both the overall efficiency (throughput) and the fairness (equity) among various services are important.

In such applications, the so-called Max-Min Fairness (MMF) solution concept is widely used to formulate the resource allocation scheme. This approach guarantees fairness but may lead to significant losses in the overall throughput of the network. In this paper we show how the concepts of multiple criteria equitable optimization can be effectively used to generate various fair resource allocation schemes. We introduce a multiple criteria model equivalent to equitable optimization and we develop a corresponding reference point procedure to generate fair efficient bandwidth allocations. The procedure is tested on a sample network dimensioning problem and its abilities to model various preferences are demonstrated.

Keywords: multiple criteria optimization, efficiency, fairness, equity, reference point method, resource allocation, network design, elastic traffic.

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1. Introduction

Resource allocation decisions are concerned with the allocation of limited resources so as to achieve the best system performance. This paper deals with problems of bandwidth allocation within telecommunication networks. The development of the Internet has led to an increased role of the traffic carried by the IP protocol in telecommunication networks. Due to the use of packet switching, the IP protocol can provide greater network utilization (the so-called multiplexing gain). For these reasons, network management can be interested in designing networks which have a high throughput for the IP protocol. Moreover, a fair way of distribution of the bandwidth (or other network resources) among competing demands becomes a key issue in computer networks (Denda et al., 2000) and telecommunications network design, in general (Pióro and Medhi, 2004). Therefore, we focus on the approaches that, while allocating resources, attempt to provide a fair (equal) treatment of all the activities (Luss, 1999; Ogryczak and Śliwiński, 2002).

Note that data traffic carried by the TCP protocol (which is the most frequently used transport protocol in IP networks) has a unique characteristic. The TCP protocol will adapt its throughput to the amount of available bandwidth. It is therefore capable to use the entire available bandwidth, but it will also be able to reduce its throughput in the presence of contending traffic. This type of network traffic has been called *elastic traffic*. Network design today often considers the problem of designing networks that carry elastic traffic. The network design problem reduces to a decision about link capacities and possibly flow routing. Flow sizes are an outcome of the design problem, since it can be assumed that flows adapt to given network resources on a chosen path.

If the network is also used for other types of communication that require guaranteed quality of service, the network design problem can be decomposed into two parts: first, design the network to carry non-elastic traffic in such a way that all demands for that communication are satisfied. Next, use the spare capacity to carry elastic traffic of the IP protocol. Resource allocation models may be used to help to solve such network design problems.

Within a telecommunication network the data traffic is generated by a huge number of nodes exchanging data. In such a network, a relatively small subset of nodes are chosen to serve as hubs which can be used as intermediate switching points or to define the so-called backbone network (Pióro and Medhi, 2004). The hub-based network organization allows the data traffic to be consolidated on the inter-hub links. The problem of network dimensioning with elastic traffic arises when there is a need to design the (inter-hub) link capacities to carry as much traffic as possible between a set of network nodes. This can occur in the case described above, when the network capacity available after considering all non-elastic demands has to be used for elastic traffic, or in another case: when the network capacity is insufficient to carry all non-elastic demands. In such a case, the problem is to determine how much traffic of the non-elastic

demands can be admitted into the network. To do so, the demands can be treated as elastic traffic. The outcome of network design will also specify the limits of traffic to be admitted into the network for each demand (Pióro and Medhi, 2004).

Network management must stay within a budget of expenses for purchasing link bandwidth. Network management will want to have a high throughput of the IP network, to increase the multiplexing gains. This traffic is offered only a best-effort service, and therefore network management is not concerned with offering guaranteed levels of bandwidth to the traffic. Network dimensioning with elastic traffic can therefore be thought of as a search for such network flows that will maximize the network throughput (the sum of all flows in the network) while staying within a budget constraint for the costs of link bandwidth. However, such a problem formulation would lead to the starvation of flows between certain network nodes. Looking at the problem from the user perspective, the network flows between different nodes should be treated as fairly as possible. The users may be interested in high available bandwidth between any two nodes of the network, or in high available bandwidth from all other network nodes to the user's node, or in high available bandwidth from the user's node to all other nodes. Whatever the user preference, it would be expressed in terms of fairness for a certain set of criteria which depend on individual flows. Let us first consider providing fairness for all flows between any two network nodes. Such a goal would clearly lead to lower levels of throughput, since resources must be allocated to distant nodes, which is more expensive than using the entire budget to purchase a high capacity for close nodes.

Therefore, network management must consider two goals: increasing throughput and providing fairness. These two goals are clearly conflicting, if the budget constraint is to be satisfied. Network management could therefore be interested in finding compromise solutions that do not starve network flows, and give satisfying levels of throughput. In particular, the so-called Proportional Fairness method (Kelly et al., 1997) allows for finding solutions which are fair with respect to flows in certain categories depending on the distance between the source and destination of a flow. However, such methods give only one possible compromise solution. The purpose of this work is to show that there exists a methodology that allows the decision maker to explore a set of solutions that could satisfy his preferences with respect to throughput and fairness, and choose the solution which the decision maker finds best. This interactive approach to decision making is superior to a black box approach, when the decision maker has only one solution and cannot express his preferences (Wierzbicki et al., 2000)

The problem of network dimensioning with elastic traffic can be formulated as a Linear Programming (LP) resource allocation problem as follows. Given a network routing topology $G =< V, E >$, consider a set of pairs of nodes as the set I of services. For each service $i \in I$, the elastic flow from source u_i^s to destination u_i^d will be denoted by x_i, which is a variable representing the model outcome. For each service, we are given the information about the routing path

in the network from the source to the destination. This information can be in the form of a matrix $\mathbf{\Delta} = (\delta_{ei})_{e\in E, i\in I}$, which satisfies the relation: $\delta_{ei} = 1$ if link e belongs to the routing path connecting u_i^s with u_i^d (the source and the destination of service i, respectively), otherwise $\delta_{ei} = 0$. Further, for each link $e \in E$, the cost of allocated bandwidth is defined.

In the basic model of network dimensioning it is assumed that any real amount of bandwidth may be installed and marginal costs c_e of link bandwidths is given. Hence, the corresponding link dimensioning function expressing necessary capacity (bandwidth) to meet a required link load (Pióro and Medhi, 2004) is then a linear function (in fact identity function). In this basic model, the cost of the entire path for service i can be directly expressed by the formula:

$$\kappa_i = \sum_{e\in E} c_e \delta_{ei} \quad \text{for } i = 1, \ldots, m. \tag{1}$$

The network dimensioning problem depends on allocating the bandwidth to several links in order to maximize flows of all the services while remaining within available budget B for all link bandwidths. The decisions are usually modeled with (decision) variables: a_e – representing the bandwidth allocated to link $e \in E$. They have to fulfill the following constraints:

$$\sum_{e\in E} c_e a_e \leq B \tag{2}$$

$$\sum_{i} \delta_{ei} x_i \leq a_e \qquad \forall e \in E \tag{3}$$

where (2) represents the budget limit while (3) establishes the relation between service flows and links bandwidth (the quantity $\sum_{i\in I} \delta_{ei} x_i$ is the load of link e). Certainly, all the decision and outcome variables must be nonnegative: $a_e \geq 0$ for all $e \in E$ and $x_i \geq 0$ for all $i \in I$.

Links modularity (bandwidth granulation) is a common feature in communications networks (Pióro and Medhi, 2004). Therefore, in more realistic models, for each link $e \in E$, the minimum unit of bandwidth b_e is specified and the installed capacity a_e must satisfy additional equation:

$$a_e = b_e z_e \quad \forall e \in E \tag{4}$$

where z_e is an integer decision variable representing the number of bandwidth units b_e installed at link e. In the case of modular links (discrete bandwidth units b_e), c_e represents the unit cost. The corresponding link dimensioning function is then a step wise function. Note that one cannot now define directly any cost κ_i of the path (similar to (1)), since this cost depends on possible sharing with other paths of the surplus bandwidth arising due to links modularity.

Thus, in the basic (continuous) case, the model constraints define a linear programming (LP) feasible set while for the case of modular links it turns into a mixed integer linear programming (MILP) feasible set. Constraints (2) and

(3) may be then treated as equations. Together with formula (1) they allow for elimination of variables a_e, thus formulating the problem as a simplified resource allocation model with only one constraint:

$$\sum_{i=1}^{m} \kappa_i x_i = B \tag{5}$$

and variables x_i representing directly decisions. Such a simplification is, certainly, impossible for a modular case, due to additional discrete constraints (4).

The network dimensioning model could have various objective functions, depending on the chosen approach. One may consider two extreme approaches. The first extreme approach is the maximization of the throughput (the sum of flows) $\sum_{i \in I} x_i$. In the basic (continuous) case, due to possible alternative formulation (5), it is apparent that this approach would choose one variable x_{i^o} which has the smallest marginal cost $\kappa_{i^o} = \min_{i=1,\ldots,m} \kappa_i$ and make that flow maximal within the budget limit ($x_{i^o} = B/\kappa_{i^o}$), while limiting all other flows to zero. Alternatively, in the case of not unique i^o, one may give equal values to all flows which have marginal costs equal to the minimal marginal cost. However, all flows that have marginal costs larger than the minimum would have to be zero in a solution that maximizes throughput. In the modular case, the direct formula for a path cost is not available and the step wise link dimensioning function will cause some flows diversification with more small flows. Nevertheless, the main part of solution will be usually generated by one cheapest flow.

The so-called Max-Min Fairness (MMF) solution concept is widely used in formulation of fair resource allocation schemes (Jaffe, 1980; Bertsekas and Gallager, 1987). The worst performance (minimum flow) is there maximized and additionally regularized, if necessary, with the lexicographic (sequential) maximization of the second worst performance, the third worst etc. The MMF concept is consistent with Rawlsian theory of justice (Rawls, 1971; Rawls and Kelly, 2001). Actually, in the basic model with an LP feasible set, due to possible alternative formulation (5), the MMF concept would lead us to a solution that has equal values for all the flows (Ogryczak, 2001):

$$x_i^{MMF} = B / \sum_{i \in I} \kappa_i \qquad \text{for } i = 1, \ldots, m.$$

Again, for the modular case some differentiation of flows usually occurs but for larger budget B it is relatively small. Allocating the resources to optimize the worst performances may cause a large worsening of the overall (mean) performance. Particularly, in the basic model, the MMF throughput ($mB/\sum_{i=1}^{m} \kappa_i$) might be considerably smaller than the maximal throughput ($B/\min_{i=1,\ldots,m} \kappa_i$). In an example built on the backbone network of a Polish ISP, it turned out that the throughput in a perfectly fair solution could be less than 50% of the maximal throughput (Ogryczak, Śliwiński and Wierzbicki, 2003).

Network management can be interested in seeking a compromise between the two extreme approaches discussed above. The approach called *Proportional Fairness* proposed in Kelly et al. (1997) maximizes the sum of logarithms of the flows x_i. Actually, it corresponds to the so-called Nash criterion (Nash, 1950) which maximizes the product of additional utilities compared to the status quo. The use of the logarithmic function makes it impossible to choose zero flows for any pair of nodes, and, on the other hand, makes it not profitable to assign too big flow to any individual demand. The optimization model of the PF method takes the following form:

$$\max \sum_{i=1}^{m} \log(x_i) \tag{6}$$

For the basic (continuous) model of network dimensioning with elastic traffic and unbounded flows, the solution found by the PF method has an interesting property (Pióro et al., 2002). The optimal flows x_i^{PF} are given by the expression:

$$x_i^{PF} = B/\kappa_i \qquad \text{for } i = 1, \ldots, m. \tag{7}$$

This property implies that the optimal flow in the PF model is inversely proportional to the cost of the path that the flow travels in the network. Due to this property, it is not necessary to solve nonlinear models in order to find the PF optimal solution. On the other hand, in the case of modular model one gets a complicated nonlinear optimization problem with integer variables. Moreover, network management could be interested in choosing among a larger set of compromise solutions in order to satisfy their preferences. In the following sections, we shall describe an approach that allows to search for such compromise solutions with multiple linear criteria rather than the use nonlinear objective functions.

2. Fair allocations and equitable efficiency

The generic resource allocation problem may be stated as follows. There is a system dealing with a set I of m services. There is given a measure of services realization within a system. In applications we consider, the measure usually expresses the service flow. However, one may consider such measures as service time, service costs, service delays as well as some more qualitative or subjective measures. Further, there is also given a set A of allocation patterns (allocation decisions). For each service $i \in I$, its measure of realization x_i is a function $x_i = f_i(\mathbf{a})$ of the allocation pattern $\mathbf{a} \in A$. This function, called the individual objective function, represents the outcome (effect) of the allocation pattern for service i. In typical formulations a larger value of the outcome means a better effect (higher service quality or client satisfaction). Otherwise, the outcomes can be replaced with their complements to some large number. Therefore, without loss of generality, we can assume that each individual outcome x_i is to be

maximized, which allows us to view the generic resource allocation problem as a vector maximization model:

$$\max \{(x_1, x_2, \ldots, x_m) \ : \quad \mathbf{x} \in Q\} \tag{8}$$

where $Q = \{(x_1, \ldots, x_m) : x_i = f_i(\mathbf{a}) \quad \text{for } i = 1, \ldots, m, \ \mathbf{a} \in A\}$ denotes the attainable set for outcome vectors $\mathbf{x}$. For the network dimensioning problems, we consider, the set Q is an LP feasible set defined by constraints (2)–(3) in the case of basic model, and a MILP feasible set (2)–(4) in the case of modular model.

Model (8) only specifies that we are interested in maximization of all outcomes x_i for $i \in I = \{1, 2, \ldots, m\}$. In order to make it operational, one needs to assume some solution concept specifying what it means to maximize multiple outcomes. The solution concepts are defined by properties of the corresponding preference model within the outcome space. The preference model is completely characterized by the relation of weak preference (Vincke, 1992), denoted hereafter with $\succeq$. Namely, the corresponding relations of strict preference $\succ$ and indifference $\cong$ are defined by the following formulas

$$\mathbf{x}' \succ \mathbf{x}'' \quad \Leftrightarrow \quad (\mathbf{x}' \succeq \mathbf{x}'' \quad \text{and} \quad \text{not } \mathbf{x}'' \succeq \mathbf{x}') \tag{9}$$

$$\mathbf{x}' \cong \mathbf{x}'' \quad \Leftrightarrow \quad (\mathbf{x}' \succeq \mathbf{x}'' \quad \text{and} \quad \mathbf{x}'' \succeq \mathbf{x}') \tag{10}$$

The preference model related to the standard Pareto–optimal solution concept also assumes that the preference relation $\succeq$ is reflexive

$$\mathbf{x} \succeq \mathbf{x} \tag{11}$$

transitive

$$(\mathbf{x}' \succeq \mathbf{x}'' \quad \text{and} \quad \mathbf{x}'' \succeq \mathbf{x}''') \quad \Rightarrow \quad \mathbf{x}' \succeq \mathbf{x}''' \tag{12}$$

and strictly monotonic

$$\mathbf{x} + \varepsilon \mathbf{e}_i \succ \mathbf{x} \quad \text{for } \varepsilon > 0, \ i \in I \tag{13}$$

where $\mathbf{e}_i$ denotes the i–th unit vector in the outcome the space. The last assumption expresses the fact that for each individual outcome the larger value is better (maximization). The preference relations satisfying axioms (11)–(13), called hereafter rational preference relations, allow us to formalize the Pareto-efficient solution concept with the following definitions. An outcome vector $\mathbf{x}'$ rationally dominates $\mathbf{x}''$ ($\mathbf{x}' \succ_r \mathbf{x}''$), iff $\mathbf{x}' \succ \mathbf{x}''$ for all rational preference relations $\succeq$. In other words, an outcome vector $\mathbf{x}''$ is dominated by $\mathbf{x}'$, if no rational decision maker prefers $\mathbf{x}''$ to $\mathbf{x}'$. If $\mathbf{x} = \mathbf{f}(\mathbf{a})$ is rationally nondominated, then the allocation pattern $\mathbf{a} \in A$ is called Pareto-efficient (Pareto-optimal).

The relation of weak rational dominance $\succeq_r$ may be expressed in terms of the vector inequality : $\mathbf{x}' \succeq_r \mathbf{x}''$ iff $x_i' \geq x_i''$ for all $i \in I$. This leads to the commonly used definition of the Pareto-optimal solutions as feasible solutions

for which one cannot improve any criterion without worsening another (Steuer, 1986). However, the axiomatic definition of the rational preference relation allows us to introduce additional properties of the preferences related to fairness concepts. The concept of fairness has been studied in various areas beginning from political economics problems of fair allocation of consumption bundles to abstract mathematical formulation (Steinhaus, 1949). In order to ensure fairness in a system, all system entities have to be equally well provided with the system's services. This leads to concepts of fairness expressed by the equitable rational preferences (Ogryczak, 1997; Kostreva and Ogryczak, 1999). First of all, the fairness requires impartiality of evaluation, thus focusing on the distribution of outcome values while ignoring their ordering. This means that in the multiple criteria problem (8) we are interested in a set of outcome values without taking into account which outcome is taking a specific value. Hence, we assume that the preference model is impartial (anonymous, symmetric). In terms of the preference relation this may be written as the following axiom

$$(x_{\tau(1)}, x_{\tau(2)}, \ldots, x_{\tau(m)}) \cong (x_1, x_2, \ldots, x_m) \tag{14}$$

for any permutation τ of I. Further, fairness requires equitability of outcomes which causes that the preference model should satisfy the (Pigou–Dalton) principle of transfers. The principle of transfers states that a transfer of any small amount from an outcome to any other relatively worse-off outcome results in a more preferred outcome vector. As a property of the preference relation, the principle of transfers takes the form of the following axiom

$$x_{i'} > x_{i''} \quad \Rightarrow \quad \mathbf{x} - \varepsilon \mathbf{e}_{i'} + \varepsilon \mathbf{e}_{i''} \succ \mathbf{x} \quad \text{for } 0 < \varepsilon < x_{i'} - x_{i''} \tag{15}$$

The preference relations satisfying all axioms (11)–(15) will be called hereafter *fair (equitable) rational preference relations*. Note that according to any fair rational preference relation a solution generating all three outcomes equal to 2 is considered better than any solution generating individual outcomes: 4, 2 and 0 (due to principle of transfers), while it remains worse than a solution generating one outcome 4 and two other equal to 2 (due to the monotonicity).

The fair rational preference relations allow us to define the concept of fairly (equitably) efficient solution, similar to the standard efficient (Pareto-optimal) solution defined with the rational preference relations. We say that outcome vector $\mathbf{x}'$ *fairly dominates* $\mathbf{x}''$ ($\mathbf{x}' \succ_e \mathbf{x}''$), iff $\mathbf{x}' \succ \mathbf{x}''$ for all fair rational preference relations $\succeq$. An allocation pattern $\mathbf{a} \in A$ is called *equitably efficient* if $\mathbf{x} = \mathbf{f}(\mathbf{a})$ is fairly nondominated. Note that each fairly efficient solution is also Pareto-efficient, but not vice verse.

Typical solution concepts for multiple criteria problems are defined by aggregation functions $g : Y \to R$ to be maximized. Thus the multiple criteria problem (8) is replaced with the maximization problem

$$\max \{g(\mathbf{x}) : \quad \mathbf{x} \in Q\} \tag{16}$$

In order to guarantee the consistency of the aggregated problem (16) with the fair (equitable) maximization of all individual outcomes in the original multiple criteria problem, the preference relation induced by the aggregation function maximization

$$\mathbf{x}' \succeq_g \mathbf{x}'' \quad \Leftrightarrow \quad g(\mathbf{x}') \geq g(\mathbf{x}'')$$

must be a fair rational preference relation.

The simplest aggregation functions commonly used for the multiple criteria problem (8) are defined as the sum of outcomes

$$g(\mathbf{x}) = \sum_{i=1}^{m} x_i \tag{17}$$

or the worst outcome

$$g(\mathbf{x}) = \min_{i=1,\ldots,m} x_i. \tag{18}$$

In the network dimensioning problem, the former represents throughput maximization while the latter corresponds to the (simplified) MMF model. Both functions are symmetric and thereby their relations $\succeq_g$ satisfy the impartiality requirement (14) but they do not satisfy the equitability requirement (15) (although satisfying the weak form of this requirement; Ogryczak, 1997). Hence, these aggregations do not guarantee the fairness of solutions. It turns out, however, that for any strictly concave, increasing function $s : R \to R$, the function $g(\mathbf{x}) = \sum_{i=1}^{m} s(x_i)$ generates the fair rational preference relation $\succeq_g$. This defines a family of the fair aggregations according to the following corollary (Kostreva and Ogryczak, 1999):

COROLLARY 2.1 *For any strictly concave, increasing function $s : R \to R$, the optimal solution of the problem*

$$\max \{\sum_{i=1}^{m} s(x_i) \; : \quad \mathbf{x} \in Q\} \tag{19}$$

is a fair solution for resource allocation problem (8).

In the case of the outcomes restricted to positive values, one may use logarithmic function thus resulting in the proportional fairness model (6) or various root functions:

$$g(\mathbf{x}) = \sum_{i=1}^{m} (x_i)^{\alpha} \quad \text{for } 0 < \alpha < 1.$$

For a common case of upper bounded outcomes $x_i \leq u^*$ one may use power functions:

$$g(\mathbf{x}) = -\sum_{i=1}^{m} (u^* - x_i)^p \quad \text{for } 1 < p,$$

which corresponds to the minimization of of the well-known Hölder p-norm distances from the upper bound. Various other concave functions s can be used to define fair aggregations (19) and the resulting resource allocation schemes. However, the problem of network dimensioning we consider is originally an LP model and a MILP in the case of link modularity. Therefore, it is important if various fair allocation schemes can be generated with LP tools.

The theory of majorization (Marshall and Olkin, 1979) includes the results which allow us to express the relation of fair (equitable) dominance as a vector inequality on the cumulative ordered outcomes (Kostreva and Ogryczak, 1999). This can be mathematically formalized as follows. First, introduce the ordering map $\Theta : R^m \to R^m$ such that $\Theta(\mathbf{x}) = (\theta_1(\mathbf{x}), \theta_2(\mathbf{x}), \ldots, \theta_m(\mathbf{x}))$, where $\theta_1(\mathbf{x}) \leq \theta_2(\mathbf{x}) \leq \cdots \leq \theta_m(\mathbf{x})$ and there exists a permutation τ of the set I such that $\theta_i(\mathbf{x}) = x_{\tau(i)}$ for $i = 1, \ldots, m$. Next, apply to ordered outcomes $\Theta(\mathbf{x})$ a linear cumulative map thus resulting in the *cumulative ordering map* $\bar{\Theta}(\mathbf{x}) = (\bar{\theta}_1(\mathbf{x}), \bar{\theta}_2(\mathbf{x}), \ldots, \bar{\theta}_m(\mathbf{x}))$ defined as

$$\bar{\theta}_i(\mathbf{x}) = \sum_{j=1}^{i} \theta_j(\mathbf{x}) \quad \text{for } i = 1, \ldots, m. \tag{20}$$

The coefficients of vector $\bar{\Theta}(\mathbf{x})$ express, respectively: the smallest outcome, the total of the two smallest outcomes, the total of the three smallest outcomes, etc. The theory of majorization allow us to derive the following theorem (Kostreva and Ogryczak, 1999):

THEOREM 2.1 *Outcome vector* $\mathbf{x}'$ *fairly dominates* $\mathbf{x}''$, *if and only if* $\bar{\theta}_i(\mathbf{x}') \geq \bar{\theta}_i(\mathbf{x}'')$ *for all* $i \in I$ *where at least one strict inequality holds.*

Vector $\bar{\Theta}(\mathbf{y})$ can be viewed graphically with the absolute Lorenz curve which can be mathematically formalized as follows. First, we introduce the right-continuous cumulative distribution function:

$$F_{\mathbf{x}}(\xi) = \sum_{i=1}^{m} \frac{1}{m} \delta_i(\xi) \quad \text{where} \quad \delta_i(\xi) = \begin{cases} 1 & \text{if } x_i \leq \xi \\ 0 & \text{otherwise} \end{cases}$$

which for any real value ξ provides the measure of outcomes smaller or equal to ξ. Next, we introduce the quantile function $F_{\mathbf{x}}^{(-1)}$ as the left-continuous inverse of the cumulative distribution function $F_{\mathbf{x}}$:

$$F_{\mathbf{x}}^{(-1)}(\nu) = \inf \{\xi : F_{\mathbf{x}}(\xi) \geq \nu\} \quad \text{for } 0 < \nu \leq 1$$

By integrating $F_{\mathbf{x}}^{(-1)}$ one gets $F_{\mathbf{x}}^{(-2)}(0) = 0$ and

$$F_{\mathbf{x}}^{(-2)}(\nu) = \int_0^{\nu} F_{\mathbf{x}}^{(-1)}(\alpha) d\alpha \quad \text{for } 0 < \nu \leq 1.$$

Graphs of functions $F_{\mathbf{x}}^{(-2)}(\nu)$ (with respect to ν) take the form of convex curves (Fig. 1), the *absolute Lorenz curves*. In our case of m outcomes,

$F_{\mathbf{x}}^{(-2)}(i/m) = \frac{1}{m}\bar{\theta}_i(\mathbf{x})$ for $i = 1, \ldots, m$ and the absolute Lorenz curve is a piece wise linear curve connecting point (0,0) and points $(i/m, \bar{\theta}_i(\mathbf{x})/m)$ for $i = 1, \ldots, m$. The fair dominance $\mathbf{x}' \succeq_e \mathbf{x}''$ means then that $F_{\mathbf{x}'}^{(-2)}(\nu) \geq F_{\mathbf{x}''}^{(-2)}(\nu)$ for all $0 \leq \nu \leq 1$. We will use absolute Lorenz curves to demonstrate and compare various allocations patterns.

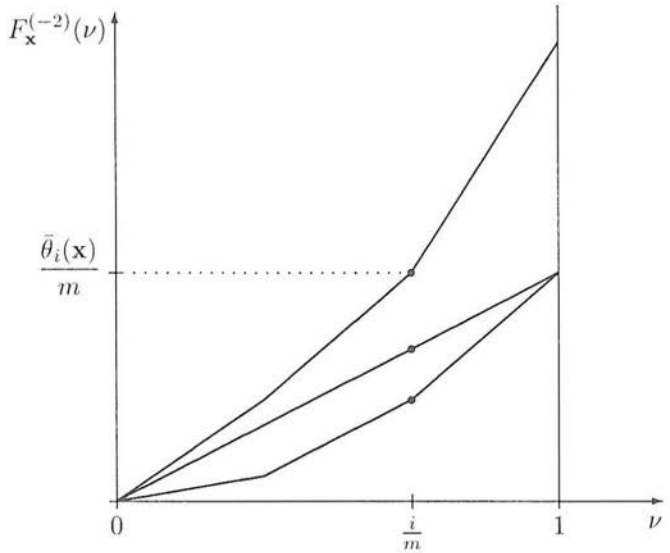

Figure 1. Vectors $\bar{\Theta}(\mathbf{x})$ as the absolute Lorenz curves.

Note that Theorem 2.1 permits one to express fair solutions of problem (8) as Pareto-efficient solutions to the multiple criteria problem with objectives $\bar{\Theta}(\mathbf{x})$

$$\max \{(\eta_1, \eta_2, \ldots, \eta_m) \; : \quad \eta_i = \bar{\theta}_i(\mathbf{x}) \quad \text{for } i = 1, \ldots, m, \; \mathbf{x} \in Q\}. \tag{21}$$

Moreover, the multiple criteria problem (21) may serve as a source of fair allocation schemes. Although the definitions of quantities $\bar{\theta}_k(\mathbf{x})$, used as criteria in (21), are very complicated, the quantities themselves can be modeled with simple auxiliary variables and constraints. It is commonly known that the worst (smallest) outcome may be defined by the following optimization: $\bar{\theta}_1(\mathbf{x}) = \max \{t : t \leq x_i \quad \text{for } i = 1, \ldots, m\}$, where t is an unrestricted variable. It turns out that this approach can be generalized to provide an effective modeling technique for quantities $\bar{\theta}_k(\mathbf{x})$ with arbitrary k (Ogryczak and Tamir, 2003). Namely, for a given outcome vector $\mathbf{x}$ the quantity $\bar{\theta}_k(\mathbf{x})$ may be found by solving the following linear program:

$$\begin{aligned} \bar{\theta}_k(\mathbf{x}) \quad &= \quad \max \; kt - \sum_{i=1}^{m} d_i \\ &\text{s.t.} \quad t - x_i \leq d_i, \; d_i \geq 0 \quad \text{for } i = 1, \ldots, m \end{aligned} \tag{22}$$

where t is an unrestricted variable while nonnegative variables d_i represent, for several outcome values x_i, their downside deviations from the value of t.

Formula (22) allows us to formulate problem (21) as the following multiple criteria optimization problem:

$$\begin{aligned} &\max\ (\eta_1, \eta_2, \ldots, \eta_m) && (23)\\ &\text{subject to} \quad \mathbf{x} \in Q \\ &\eta_k = kt_k - \sum_{i=1}^{m} d_{ik} \quad \text{for } k = 1, \ldots, m && (24)\\ &t_k - d_{ik} \leq x_i, \quad d_{ik} \geq 0 \quad \text{for } i, k = 1, \ldots, m. && (25) \end{aligned}$$

Note that problem (23)–(25) adds only linear constraints to the original attainable set Q. Hence, for the basic network dimensioning problems with the set Q defined by constraints (2)–(3), the resulting formulation (23)–(25) remains in the class of (multi-criteria) linear programs. Certainly, the problem becomes a MILP for the modular dimensioning model with the attainable set (2)–(4).

3. Multiple criteria analysis

Theorem 2.1 allows one to generate fairly efficient solutions of (8) as efficient solutions of problem (21). The aggregation maximizing the sum of outcomes, corresponds to maximization of the last (m–th) objective in problem (21). Similarly, the maximin scalarization corresponds to maximization of the first objective in (21). For better understanding of the multiple criteria problem (21), one may consider normalized objective functions:

$$M_{\frac{k}{m}}(\mathbf{x}) = \frac{1}{k}\bar{\theta}_k(\mathbf{x}), \quad \text{for } k = 1, \ldots, m \tag{26}$$

thus representing for each k the mean outcome of the k worst-off services, called the worst conditional mean. Note that for $k = 1$, $M_{\frac{1}{m}}(\mathbf{x}) = \bar{\theta}_1(\mathbf{x}) = \theta_1(\mathbf{x}) = M(\mathbf{x})$ thus representing the minimum outcome, and for $k = m$, $M_{\frac{m}{m}}(\mathbf{x}) = \frac{1}{m}\bar{\theta}_m(\mathbf{x}) = \frac{1}{m}\sum_{i=1}^{m} \theta_i(\mathbf{x}) = \frac{1}{m}\sum_{i=1}^{m} x_i = \mu(\mathbf{x})$ which is the mean outcome. Formula (22) allows us to maximize the worst conditional means for various intermediate values k and it can be effectively applied to network traffic engineering problems (Ogryczak and Śliwiński, 2002).

For modeling larger gamut of fair preferences one may use some combinations of criteria in model (21). In particular, for the weighted sum on gets

$$\sum_{i=1}^{m} w_i \eta_i = \sum_{i=1}^{m} w_i \bar{\theta}_i(\mathbf{x}), \quad w_i > 0 \quad \text{for } i = 1, \ldots, m. \tag{27}$$

Note that due to the definition of map $\bar{\Theta}$ with (20), the above function can be expressed in the form with weights $v_i = \sum_{j=i}^{m} w_j$ $(i = 1, \ldots, m)$ allocated to coordinates of the ordered outcome vector. Such an approach to aggregation of

outcomes was introduced by Yager (1988) as the so–called Ordered Weighted Averaging (OWA). The OWA aggregation is obviously a piece wise linear function since it remains linear within every area of the fixed order of arguments. If weights v_i are strictly decreasing and positive (as for strictly positive w_i in (27)), then the OWA problem

$$\max \{\sum_{i=1}^{m} v_i \theta_i(\mathbf{x}) \ : \quad \mathbf{x} \in Q\} \tag{28}$$

is LP solvable with respect to given values x_i (Ogryczak and Śliwiński, 2003).

When differences among weights tend to infinity, the OWA aggregation approximates the lexicographic ranking of the ordered outcome vectors. That means, as the limiting case of the OWA problem (28), we get the lexicographic problem:

$$\text{lexmax } \{(\theta_1(\mathbf{x}), \theta_2(\mathbf{x}), \ldots, \theta_m(\mathbf{x})) \ : \quad \mathbf{x} \in Q\} \tag{29}$$

which represents the MMF (lexicographic maximin) approach to the original resource allocation problem (8). Problem (29) is a regularization of the standard maximin optimization, but in the former, in addition to the worst outcome, we maximize also the second worst outcome (provided that the smallest one remains as large as possible), maximize the third worst (provided that the two smallest remain as large as possible), and so on.

If weights v_i are strictly decreasing and positive, i.e., the corresponding weights w_i in (27) are strictly positive, then each optimal solution of the OWA problem (28) is a fair solution of (8). Moreover, in the case of LP models, as the basic network dimensioning one, every fair allocation scheme can be identified as an optimal solution to some OWA problem with appropriate monotonic weights (Kostreva and Ogryczak, 1999). While equal weights define linear aggregation, several decreasing sequences of weights provide us with various piece wise linear aggregations. Indeed, our earlier experience with application of the OWA criterion to the basic (continuous) problem of network dimensioning with elastic traffic (Ogryczak, Śliwiński and Wierzbicki, 2003) showed that we were able to generate easily allocations representing the classical fairness models. On the other hand, in order to find a larger variety of new compromise solutions we needed to incorporate some scaling techniques originating from the reference point methodology. Actually, it is a common flaw of the weighting approaches that they provide poor controllability of the preference modeling process and in the case of multiple criteria problems with discrete (or more general nonconvex) feasible sets, they may fail to identify several compromise efficient solutions (Steuer, 1986). Better controllability and the complete parameterization of nondominated solutions can be achieved with the direct use of the reference point methodology.

The reference point method was introduced by Wierzbicki (1982) and later extended leading to efficient implementations of the so-called aspira-

tion/reservation based decision support (ARBDS) approach with many successful applications (Lewandowski and Wierzbicki, 1989; Wierzbicki et al., 2000). The ARBDS approach is an interactive technique allowing the DM to specify the requirements in terms of aspiration and reservation levels, i.e., by introducing acceptable and required values for several criteria. Depending on the specified aspiration and reservation levels, a special scalarizing achievement function is built which may be directly interpreted as expressing utility to be maximized. Maximization of the scalarizing achievement function generates an efficient solution to the multiple criteria problem. The solution is accepted by the DM or some modifications of the aspiration and reservation levels are introduced to continue the search for a better solution. The ARBDS approach provides a complete parameterization of the efficient set to multi-criteria optimization. Exactly, all properly efficient solutions with bounded trade-offs can be identified with this approach (Kaliszewski, 1994), while in LP and MILP problems with bounded feasible sets that we consider, the approach covers the entire efficient set (Ogryczak, 1997). Further, when applying the ARBDS methodology to the ordered cumulated criteria in (21), one may generate all (fairly) equitably efficient solutions of the original resource allocation problem (8).

While building the scalarizing achievement function the following properties of the preference model are assumed. First of all, for any individual outcome η_i more is preferred to less (maximization). To meet this requirement the function must be strictly increasing with respect to each outcome. Second, a solution with all individual outcomes η_i satisfying the corresponding reservation levels is preferred to any solution with at least one individual outcome worse (smaller) than its reservation level. Next, provided that all the reservation levels are satisfied, a solution with all individual outcomes η_i equal to the corresponding aspiration levels is preferred to any solution with at least one individual outcome worse (smaller) than its aspiration level. That means, the scalarizing achievement function maximization must enforce reaching the reservation levels prior to further improving of criteria. In other words, the reservation levels represent some soft lower bounds on the maximized criteria. When all these lower bounds are satisfied, then the optimization process attempts to reach the aspiration levels.

The generic scalarizing achievement function takes the following form (Wierzbicki, 1982):

$$\sigma(\eta) = \min_{1 \le i \le m} \{\sigma_i(\eta_i)\} + \varepsilon \sum_{i=1}^{m} \sigma_i(\eta_i) \tag{30}$$

where ε is an arbitrary small positive number and σ_i, for $i = 1, 2, \ldots, m$, are the partial achievement functions measuring actual achievement of the individual outcome η_i with respect to the corresponding aspiration and reservation levels (η_i^a and η_i^r, respectively). Thus, the scalarizing achievement function is, essentially, defined by the worst partial (individual) achievement but addition-

ally regularized with the sum of all partial achievements. The regularization term is introduced only to guarantee the solution efficiency in the case when the maximization of the main term (the worst partial achievement) results in a non-unique optimal solution.

The partial achievement function σ_i can be interpreted as a measure of the DM's satisfaction with the current value (outcome) of the i-th criterion. It is a strictly increasing function of outcome η_i with value $\sigma_i = 1$ if $\eta_i = \eta_i^a$, and $\sigma_i = 0$ for $\eta_i = \eta_i^r$. Thus, the partial achievement functions map the outcomes values onto a normalized scale of the DM's satisfaction. Various functions can be built meeting those requirements (Wierzbicki et al., 2000). We use the piece wise linear partial achievement function introduced in Ogryczak (1997). It is given by

$$\sigma_i(\eta_i) = \begin{cases} \gamma(\eta_i - \eta_i^r)/(\eta_i^a - \eta_i^r), & \text{for } \eta_i \le \eta_i^r \\ (\eta_i - \eta_i^r)/(\eta_i^a - \eta_i^r), & \text{for } \eta_i^r < \eta_i < \eta_i^a \\ \beta(\eta_i - \eta_i^a)/(\eta_i^a - \eta_i^r) + 1, & \text{for } \eta_i \ge \eta_i^a \end{cases} \tag{31}$$

where β and γ are arbitrarily defined parameters satisfying $0 < \beta < 1 < \gamma$.

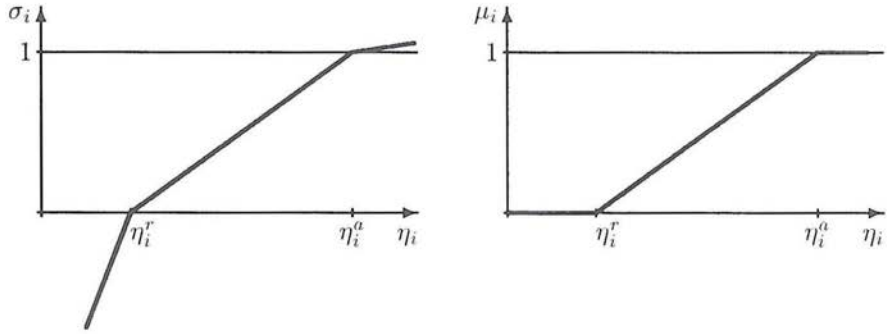

Figure 2. Partial achievement function σ_i and fuzzy membership function μ_i

For outcomes between the reservation and the aspiration levels, the partial achievement function σ_i can be interpreted as a membership function μ_i for a fuzzy target. However, such a membership function would be neither strictly monotonic nor concave. In the partial achievement function (31), parameter β represents additional increase of the DM's satisfaction over level 1 when a criterion generates outcomes better than the corresponding aspiration level, while parameter $\gamma > 1$ represents dissatisfaction connected with outcomes worse than the reservation level. Hence, the partial achievement function can be viewed as an extension of the fuzzy membership function to a strictly monotonic and concave utility function (Fig. 2). In other words, maximization of the scalarizing achievement function (30) is consistent with the fuzzy methodology in the case of not attainable aspiration levels and satisfable all reservation levels while modeling a reasonable utility for any values of aspiration and reservation levels.

Under the assumption that the parameters β and γ satisfy inequalities $0 < \beta < 1 < \gamma$, the partial achievement function (31) is strictly increasing and concave. Hence, it can be expressed in the form:

$$\sigma_i(\eta_i) = \min \left\{ \gamma \frac{\eta_i - \eta_i^r}{\eta_i^a - \eta_i^r},\ \frac{\eta_i - \eta_i^r}{\eta_i^a - \eta_i^r},\ \beta \frac{\eta_i - \eta_i^a}{\eta_i^a - \eta_i^r} + 1 \right\}$$

which guarantees LP computability with respect to outcomes η_i. Finally, maximization of the entire scalarizing achievement function (30) can be implemented by the following auxiliary LP constraints:

$$\max\ \underline{s} + \varepsilon \sum_{i=1}^{m} s_i$$

$$\begin{array}{lrcll} \text{s.t.} & s_i & \geq & \underline{s} & \text{for } i = 1, \ldots, m \\ & s_i & \leq & \gamma\ (\eta_i - \eta_i^r)/(\eta_i^a - \eta_i^r) & \text{for } i = 1, \ldots, m \\ & s_i & \leq & (\eta_i - \eta_i^r)/(\eta_i^a - \eta_i^r) & \text{for } i = 1, \ldots, m \\ & s_i & \leq & \beta\ (\eta_i - \eta_i^a)/(\eta_i^a - \eta_i^r) + 1 & \text{for } i = 1, \ldots, m \end{array}$$

where s_i for $i = 1, \ldots, m$ and $\underline{s}$ are unbounded variables introduced to represent values of several partial achievement functions and their minimum, respectively.

Recall that in our model outcomes η_k represent cumulative ordered flows x_i, i.e. $\eta_k = \sum_{i=1}^{k} \theta_i(\mathbf{x})$. Hence, the reference vectors (aspiration and reservation) represent, in fact, some reference distributions of outcomes (flows). Moreover, due to the cumulation of outcomes, while considering equal flows $x_i = \alpha$ for $i = 1, \ldots, m$ as the reference (aspiration or reservation) distribution, one needs to set the corresponding levels as $\eta_i = i\alpha$.

4. Computational results

The methods described in preceding sections have been tested on a sample network dimensioning problem with elastic traffic. Recall that in the case of elastic traffic, the outcome of the network dimensioning procedure are the capacities of links in a given network, and that the flows will adapt to the bandwidth available on the links in the designed network. The input to a network dimensioning problem with elastic traffic consists of a network topology (without specified link capacities), of pairs of nodes that specify sources and destinations of flows, of sets of network paths that could be used for each flow, and of optional constraints on the capacities of links or on flow sizes. The user must also specify a budget for purchasing link capacity, prices of a unit of link capacity (possibly different for each link), and may specify module sizes and prices for a link.

The network topology of the presented problem (Fig. 3) is patterned after the backbone network of a Polish ISP (Ogryczak, Śliwiński and Wierzbicki, 2003). The network consists of 12 nodes and 18 links. Flows between any pair of different nodes have been considered (therefore, there can be $144 - 12 = 132$

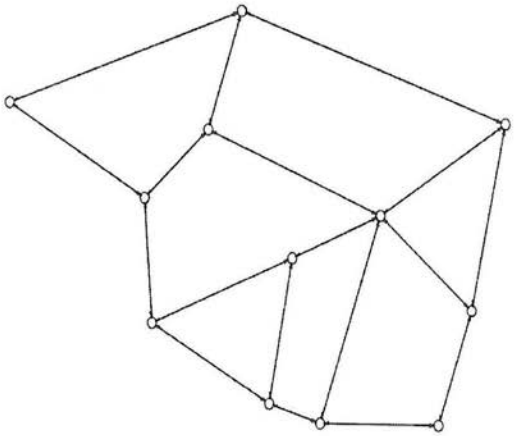

Figure 3. Sample network topology.

flows), and all flows use the shortest network path for transport. All links have unit costs equal to one, and the budget for link bandwidth is $B = 1000$. Since all links have equal costs of one, path costs are equal to the path length (1, 2, 3 or 4 in the illustrative topology). All flows are unbounded. However, it is clear that due to the budget constraint no flow can exceed B.

The presented problem has been studied without additional constraints and with equal link costs, since in such a case it was simple to understand the best choices with respect to fairness and overall throughput. The fairest outcome would have all flows of equal sizes. On the other hand, the best network throughput could be achieved by purchasing link capacity only for the cheapest flows (with path costs equal to 1), at the expense of starvation of some other flows.

Additionally, a modular version of the problem was considered. The size of a link capacity module was set to 5 (typical outcomes had most link capacities in the range of 20 to 40). For each link, an integer variable has been introduced (thus there are 18 integer variables in the modular version of the model).

The final input to the model consisted of the reservation and aspiration levels for the sums of ordered criteria. For simplicity, all aspiration levels were set close to the optimum values of the criteria, and only reservation levels were used to control the outcomes. One of the most significant parameters was the reservation level for the sum of all criteria (the network throughput). This value will be denoted by η_m^r.

The other reservation levels were chosen in such a way that they formed a linearly increasing sequence with slope (step) r for the ordered criteria $\theta_i(\mathbf{x})$. Hence, for the final criteria $\eta_i = \bar{\theta}_i(\mathbf{x})$ representing the sums of ordered outcomes in model (23)–(25), the sequence of reservation levels increased quadratically.

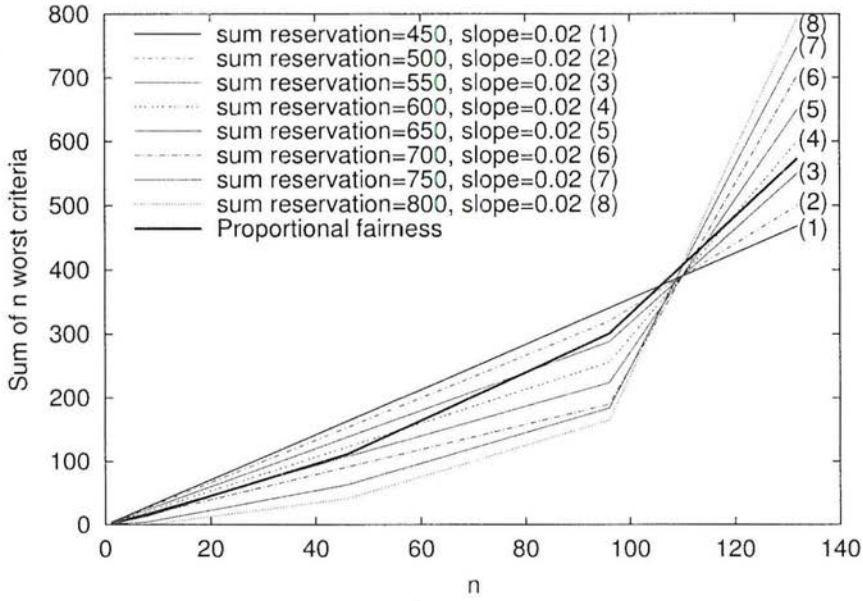

Figure 4. Varying throughput reservation for continuous link capacities.

Thus, only two parameters were used to control the outcomes: the reservation level η_m^r for the total throughput and the slope r for the linearly increasing sequence.

The first experiment consisted of a search for compromise solutions that traded off fairness against efficiency. The throughput reservation was varied from 450 to 1000. For values of η_m^r above 800, some flows were starved, and therefore these outcomes were not considered further. The linear increase of the other reservation levels was varied as well. For $r = 0$, all outcomes divided flows into at most two groups (in one group, all flows were equal). For larger values of r, some outcomes (especially for large throughput reservations) divided flows into four groups that were determined by the prices of the shortest paths that were used to transport the flows. The results of the experiment for $r = 0.02$ are shown in Fig. 4. For higher values of r, the increase of the throughput reservation above 750 resulted in flow starvation.

Note that the throughput reservation was effectively used to find outcomes with the desired network throughput. On the other hand, the optimization procedure automatically found outcomes that divided flows into categories according to their path costs. This shows that the presented methodology is cost-aware, and that it is possible to guarantee fairness to all flows with the same path cost. For the lowest throughput reservation of $\eta_m^r = 450$, the outcome was a perfectly fair distribution. For comparison, the solution obtained

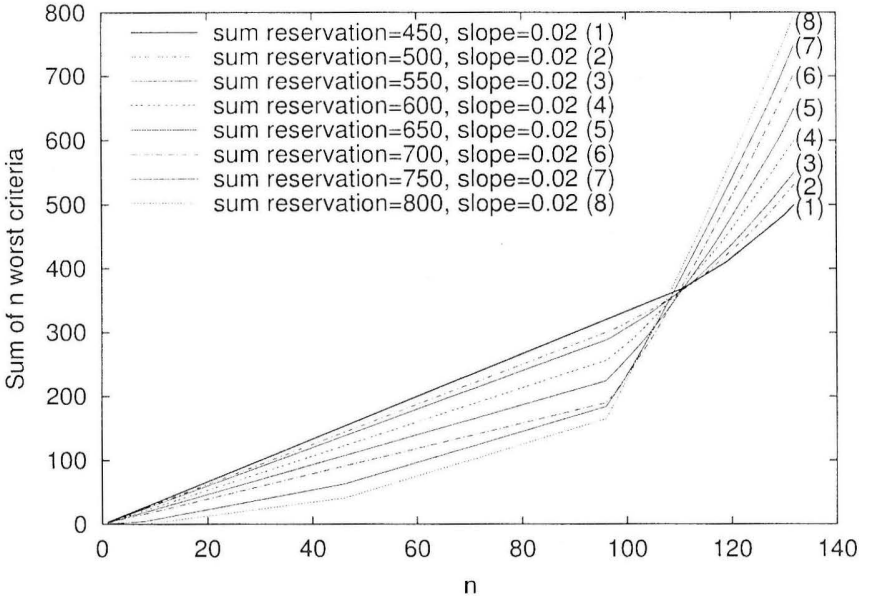

Figure 5. Varying throughput reservation for modular link capacities.

by Proportional Fairness is also shown in Fig. 4. Note that the outcome for $\eta_m^r = 550$ is very close to the outcome obtained by Proportional Fairness. Using the methodology described in this paper, the user can choose from a large number of different outcomes and control the tradeoff between fairness and efficiency.

The second experiment repeated the search for compromise solutions for modular link capacities using the same parameter configurations as in the first experiment. Here the choice of the reference point methodology should allow the user to find solutions closest to his preferences. Predictably, the introduction of modular link capacities makes it more difficult to find fair solutions. The results shown in Fig. 5 indicate that the excess capacities of modules were used by the cheapest flows, leading to a higher network throughput than in the case of a problem without modular link capacities. On the other hand the cheapest flows were not equal for some outcomes. Note that in the second experiment, the perfectly fair solution was not found for $\eta_m^r = 450$.

Overall, the experiments on the sample network topology demonstrated the versatility of the described methodology for equitable optimization. The use of reservation levels, controlled by a small number of simple parameters, allowed to search for solutions best fitted to the preferences of a network designer. The obtained solutions divided flows into categories determined by flow cost. For modular solutions, the cheapest flows consumed the excess link capacity. These

characteristics demonstrate that the model is cost-aware and fulfills the axioms of equitable optimization.

5. Concluding remarks

While designing systems which serve many users, like the telecommunications networks, there is a need to respect the fairness rules, i.e. to allocate resources equitably among the competing services. Allocating the resources to optimize the worst performance may cause a large worsening of the overall (mean) performance. Therefore, several other fair allocation schemes are searched and analyzed.

Our earlier computational experiments with application of the OWA criterion to the basic (continuous) problem of network dimensioning with elastic traffic (Ogryczak, Śliwiński and Wierzbicki, 2003) showed that we were able to generate easily allocations representing the classical fairness models. On the other hand, in order to find a larger variety of new compromise solutions we needed to incorporate some scaling techniques originating from the reference point methodology. Actually it is a common flaw of the weighting approaches that they provide poor controllability of the preference modeling process and in the case of multiple criteria problems with discrete (or more general nonconvex) feasible sets, they may fail to identify several compromise efficient solutions (Steuer, 1986).

In standard multiple criteria optimization, good controllability and the complete parameterization of nondominated solutions can be achieved with the direct use of the reference point methodology. While looking for fairly efficient bandwidth allocation the reference point methodology can be applied to the cumulated ordered outcomes. Our initial experiments with such an approach to the problem of network dimensioning with elastic traffic have confirmed the theoretical advantages of the method. We were able able to generate easily various (compromise) fair solutions for both continuous and modular problems.

The search for fairly efficient compromise solutions was controlled by only two parameters. One of these parameters was a reservation level for network throughput. The network designer could therefore specify how much throughput was required, while the more expensive flows were treated as fairly as possible. The second parameter allowed the network designer to control the difference in throughputs of cheaper and more expensive flows. Still, flows with the same cost were always treated fairly. Thus, the use of the reference point method for equitably fair optimization can be made simplex for less experienced users, while at the same time the model fully exploits the theoretical advantages of these methods for network design.

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