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A generalization of the Zionts-Wallenius multiple criteria decision making algorithm

by

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Abstract: In multicriteria problem solving, much can be learned by observing the decision-making process. Some, if not many, of the theoretical constructs used in some academically-generated models are simply not necessary. Taking this into account, we generalize the Zionts-Wallenius Multiple Criteria Decision Making Algorithm. We generalize the approach so that it can solve general convex problems. We do this by drawing from other methods, and by incorporating what we have learned in our work. To deal with the class of convex problems we face, we broaden the concept of tradeoff, and use global tradeoffs. Theory is developed, and then a method incorporating the theory is presented. A small example is included. We discuss how our development enriches decision-making tools currently available. We discuss applications in finance and technology.

Keywords: multiple criteria decision making, convex problems, tradeoff, portfolio selection.

1. Introduction

Consider a problem for which the solution process is assisted by a facilitator or analyst. An important question among researchers who study decision making is how much input can reasonably be demanded from a problem's decision maker (DM) to help her[1] identify her *most preferred decision*.

At one extreme, "purists" argue that the only methodologically-justified approach is to elicit from the DM her complete preference structure *a priori*. We identify her *criteria* in the process. Then, a *value* (we limit ourselves to deterministic problems) *function* consistent with the DM's preferences is constructed.

[1]We use feminine pronouns throughout the paper for the decision maker.

Assuming that there are no errors in the process, all that is needed is to determine the most desirable *feasible* decision. Formally elegant, this approach is impractical. Many real DMs (in contrast to DMs in simulated experiments) cannot or may not want to reveal their value functions. Such a procedure leaves no role for the DM to play during the selection process; it makes decision making a black-box procedure. Also, the above approach is demanding with respect to the extent of information required of a DM. The existence of a function, which represents DM's preferences, is itself a strong assumption; verification of the function would add considerable extra effort. This approach remains an area of theoretical interest with limited practical appeal.

An alternative to constructing a general value function is to construct a proxy value function. One proxy value function is linear, a weighted sum of the multiple criteria. Though the responsibility to come up with weights remains with the DM, that process is simpler than it may sound. It does require significant input of the DM.

To further relieve the DM of the burden of providing complex information about her preferences, pair-wise comparisons of alternate decisions may be used. In its simplest form pair-wise comparisons consist of sequentially comparing decisions, each time discarding the less preferred, until all decisions have been considered. The last remaining decision is the most preferred. A more sophisticated variation is to use results of pair-wise comparisons to discard subsets of non-explicitly considered decisions. Pair-wise comparisons fit the interactive scheme (admit learning loop), in contract to the first two approaches, which operate in "batch" mode. This approach is less demanding on the DM - at successive iterations she is supposed only to choose the more preferred decision of two.

Our general observation is that complex decision-making models and complex decision-making support algorithms are less frequently used in solving practical decision problems than simple ones. This is certainly true in Multiple Criteria Decision Making (MCDM). If the complexity of such models and/or algorithms were weighted by the frequency of applications, then simple decision tools would be given highest scores.

One motivation for this work is to provide a contribution to MCDM of a potentially high score on the complexity - frequency of applications scale. To achieve this aim we base our work on the viability of interactive MCDM methods documented in the popular press. For example, the approach of T. Saaty in his Analytic Hierarchy Process implemented in his Expert Choice software has achieved great results. See, for example, his write-up in *Fortune Magazine* (1999).

Our approach is to use the Zionts-Wallenius algorithm (1976, 1983) that implements pair-wise comparison of decisions and uses a linear proxy value function for implicit discarding subsets of non-specifically considered decisions. The Zionts-Wallenius algorithm distinguishes itself from other interactive methods by explicitly specifying two ways of comparing decisions: "1. values of criteria of alternate decisions are compared", and "2. tradeoffs of criteria of alternate decisions are compared".

The Zionts-Wallenius algorithm is designed to solve problems whose underlying formal models consist of linear constraints, linear objective functions, and pseudo-concave value functions. Our purpose is to generalize the Zionts-Wallenius algorithm to a broader class of problems, namely to convex (as opposed to linear) problems. We claim that the two ways of comparing decisions used by the Zionts-Wallenius algorithm is a significant improvement compared with methods that only compare values of criteria. Tradeoff relations are complementary to relations of values of criteria. The explicit use of tradeoffs in decision making provides a new dimension to help differentiate between decisions whenever criteria values are not useful.

In applications of the Zionts-Wallenius algorithm, tradeoffs have turned out to be of marginal practical importance. This can be explained by the fact that the algorithm searches only those feasible decisions represented by vertices of a polyhedral set. In that case there is not much difference between the two ways of comparing decisions, except for the form in which related information is presented (tradeoffs are usually presented in the form of ratios). The situation changes when decisions are no longer restricted to vertices; a different definition of tradeoff and tools to calculate them are required. Appropriate extensions and results are presented in this paper.

The need for trade-off information being exploited in MCDM has been implicit in many papers. A good example of this is the paper by Makowski et al. (1996). In solving a water management multiple criteria model, the authors discovered that by slightly relaxing a constraint on water quality and then searching the efficient frontier for alternative solutions, they were able to find acceptable solutions with significantly reduced costs. (This may argue for using soft constraints, rather than hard constraints.) Similar results can be obtained using the approach we present.

Our contribution extends the well-known and widely applied Zionts-Wallenius algorithm in two significant ways:

- allowing for more complex underlying models,
- providing for multiple ways of expressing preferences among decisions.

Our approach is an extension of a classical algorithm, which has been applied successfully to a wide range of decision problems reported in publications. With the notion of tradeoff we simply activate a dimension, inherently present in multiple criteria decision making but until now restricted by the lack of simple tools to handle it. The tool we provide is simple and imposes virtually no additional computational burden.

An overview of the paper follows. In the next section we formulate the problem and introduce basic concepts. In Section 3 we recall how the Zionts-Wallenius algorithm generates improved successive trial solutions and uses a DM's partial preferences revealed in the course of an interactive decision process. In Section 4 we recall two definitions of tradeoffs and discuss their role in interactive MCDM. In Section 5 we show how to derive tradeoff information when generating successive trial solutions. In Section 6 we describe the proposed algorithm and in Section 7 we apply the algorithm to the classic portfolio selection problem. Section 8 is devoted to a discussion of the applicability of our development. Some concluding remarks are presented in Section 9.

2. Problem definition and basic concepts

A formulation of the multicriteria decision problem is as follows:

$$\begin{aligned} &\max f(x) \\ &\text{subject to } x \in X_0 \subseteq X, \end{aligned} \tag{1}$$

where $f : X \to R^k$, $k \geq 2$, $f = (f_1, f_2, \dots, f_k)$ is a vector of objective functions, $f_i : X \to R$, $i = 1, \dots, k$, X_0 is the set of feasible decisions (solutions), and "max" stands for the operator of determining all efficient decisions of X_0.

In what follows we shall be interested in the properties of elements $f(x)$ of the set $f(X_0)$. Using the notation $f(x) = y$ and $f(X_0) = Z$, elements y are called *outcomes* and Z is called the *outcome set.*

Let $\bar{y} \in Z$. The following are commonly accepted definitions of various types of efficiency. The outcome $\bar{y} \in Z$ is:

weakly efficient if there is no y, $y \in Z$, such that $y_i > \bar{y}_i$, $i = 1, \dots, k$,

efficient if $y_i > \bar{y}_i$, $i = 1, \dots, k$, $y \in Z$, implies $y = \bar{y}$,

properly efficient if it is efficient and there exists a finite number $M > 0$ such that for each i we have

$$\frac{y_i - \bar{y}_i}{\bar{y}_j - y_j} \leq M$$

for some j such that $y_j < \bar{y}_j$ whenever $y \in Z$ and $y_i > \bar{y}_i$.

3. Handling DM preferences in the Zionts-Wallenius algorithm

As mentioned before, the Zionts-Wallenius algorithm is applicable to MCDM problems in which objective functions as well as constraints are linear and the DM has an implicit pseudo-concave value function.

The algorithm directs the DM interactively towards decisions maximizing that implicit function. The Zionts-Wallenius algorithm asks the DM to compare adjacent efficient extreme outcomes to a trial efficient outcome, so long as an adjacent efficient extreme outcome is sufficiently distinct from a current trial outcome to make a comparison. The DM makes the comparison by comparing values of criteria. If an adjacent efficient extreme outcome is not sufficiently distinct, the DM is asked to assess tradeoff information relevant to the current trial outcome.

To convert a DM preference into a mechanism for generating improving outcomes, the method generates constraints on weights in a proxy linear value function. Let the proxy linear value function be

$$\sum_l \lambda_l \, y_l. \tag{2}$$

If the DM prefers the current trial outcome y^{tr} to an adjacent outcome y^a (one that is adjacent to the trial outcome, efficient, and extreme) then the coefficients λ_l should be such that

$$\sum_l \lambda_l \, y_l^{tr} > \sum_l \lambda_l \, y_l^a. \tag{3}$$

If y^a is preferred to y^{tr} the coefficients λ_l should be such that

$$\sum_l \lambda_l \, y_l^{tr} < \sum_l \lambda_l \, y_l^a. \tag{4}$$

If y^a and y^{tr} are not distinct enough to permit a comparison, the DM evaluates the vector $y^a - y^{tr}$. If the DM likes changing the criteria in the proportion indicated by this vector, then $y^a - y^{tr}$ should satisfy

$$\sum_l \lambda_l \, (y_l^a - y_l^{tr}) > 0. \tag{5}$$

If she does not, then $y^a - y^{tr}$ should satisfy

$$\sum_l \lambda_l \, (y_l^a - y_l^{tr}) < 0. \tag{6}$$

Because (5) and (6) reduce to (3) and (4), respectively, relations (3), (4) constitute a representation of DM assessments of two different types of information.

Vector $y^a - y^{tr}$ is efficient (i.e. all elements of $\alpha(y^a - y^{tr})$, $0 \le \alpha \le 1$ are efficient) and it defines point-to-point tradeoffs (see Section 4).

Constraints of the form (3) and (4) are successively added for each pair of considered outcomes y^{tr}, y^a to constrain the set of vectors λ and to narrow the search in the space of weights. Successive trial outcomes y^{tr} are generated using vectors λ from the constrained set and the linear function (2).

4. Tradeoffs

A (desirable) tradeoff is defined in the following manner by *Webster's New World Dictionary of the English Language* (Simon and Schuster, New York, 1980): "It is an exchange, especially a giving up of one benefit or advantage in order to gain another regarded as more desirable". It is defined technically as some specific (usually local) property of the explicit or implicit value function (Kuhn and Tucker, 1951; Chankong and Haimes, 1979; Sakawa and Yano, 1990). It is also a measure of the benefits and costs of moving from one scenario to any other scenario measured by values of relative changes in objective functions (Zionts and Wallenius, 1976, 1983; Wierzbicki, 1990; Halme, 1992; Henig and Buchanan, 1997; Kaliszewski, 1993, 1994; Kaliszewski and Michalowski, 1994, 1995, 1997). We use the latter meaning of tradeoff.

We consider two types of tradeoffs: a *point-to-point* tradeoff is defined for a selected pair of scenarios; a *global* tradeoff is defined for a particular outcome $\bar{y}$ of Z. A tradeoff specifies an amount by which one (or more) criterion value increases (a *gain*) while another (one or more) decreases (a *loss*) when moving from one outcome to another.

A point-to-point tradeoff is a tradeoff between two scenarios; it is effectively a direction or gradient. It gives the relative change of each objective. It may be useful to choose one of the objectives as a numeraire (or reference) and use it as a denominator for all the others.

A global tradeoff for a given $\bar{y}$ is defined as a limit of tradeoffs. It may not be and usually is not achievable. Mathematically, it is calculated as a supremum of all point-to-point tradeoffs defined for such pairs of outcomes $\bar{y}$, y, $y \in Z$, that for y all components except component j have values greater or equal to components of $\bar{y}$ and for y the component j has a value less than that of $\bar{y}$. In other words, a global tradeoff specifies the least upper bound on an increase in one criterion relative to a unit decrease in another criterion occurring while moving from a particular outcome in a direction where all the remaining criteria do not decrease (see the definition of sets $Z_i^<(\bar{y})$, $i = 1, \ldots, k$, below). In what follows we refer to a global tradeoff simply as a tradeoff. A formal definition of tradeoff is given later in this section.

Simply calculating the supremum over all point-to-point tradeoffs (which is the definition of a *gain-to-loss ratio*, Kaliszewski, 1994) is obviously not equivalent to determining a tradeoff. In many instances a finite gain-to-loss ratio does not exist whereas a tradeoff does (Kaliszewski, 1994). Therefore, a tradeoff may be used as a universal construct to convey relative information.

In contrast to point-to-point tradeoffs which are defined by the components of two given outcomes, determining a global tradeoff for an outcome $\bar{y}$ requires calculations that relate to outcome set Z. So far, most of the research on tradeoffs is focused on deriving or assessing tradeoffs for a given efficient outcome. Wierzbicki (1990), Halme (1992), Henig and Buchanan (1997), and Kaliszewski (1993, 1994) addressed this problem. The problem of generating efficient outcomes with *a priori* set bounds on tradeoffs was investigated by Kaliszewski (2000), and Kaliszewski and Michalowski (1995, 1997). In the proposed algorithm we make use of a relation between weighting coefficients in a linear scalarization (2) and bounds imposed on tradeoffs (Theorem 5.2, Section 5).

Let $\bar{y} \in Z$, $Z \subseteq R^k$. For $i = 1, \ldots, k$ we denote:

$$Z_i^<(\bar{y}) = \{y \in Z \,|\, y_i < \bar{y}_i,\, y_l \geq \bar{y}_l,\, l = 1, \ldots, k,\, l \neq i\}.$$

DEFINITION 4.1 *Let $\bar{y} \in Z$.* **Tradeoff** $T_{ij}^G(\bar{y})$ *(the superscript G stands for global) involving the objective functions i and j, $i, j = 1, \ldots, k, i \neq j$ is defined as*

$$\sup_{y \in Z_j^<(\bar{y})} \frac{y_i - \bar{y}_i}{\bar{y}_j - y_j}.$$

In other words, the tradeoff is the smallest number which bounds from above all point-to-point tradeoffs involving a given outcome $\bar{y}$, an outcome y, $y \in Z_j^<(\bar{y})$, and a pair of indices.

We adopt the convention that if $Z_j^<(\bar{y}) = \emptyset$ then $T_{ij}^G(\bar{y}) = -\infty$, $i = 1, \ldots, k$, $i \neq j$. A method of calculating tradeoffs, that avoids finding the supremum of a hyperbolic function, was given in Kaliszewski (1993, 1994).

In contrast to other definitions of tradeoffs, we do not require in Definition 4.1 that an outcome $\bar{y}$ for which a tradeoff is defined, be efficient. It is easy to show that, if Z is convex and $\bar{y}$ is not weakly efficient, then finite tradeoffs do not exist. For non-convex outcome sets, tradeoffs can exist for outcomes which are not weakly efficient, as demonstrated by the case of finite sets.

As we see from the above definition, a global tradeoff generally differs from a point-to-point tradeoff. However, it is a limit, as was indicated earlier. Fig. 1 explains the role of sets $Z_\bullet^<(\bar{y})$ in Definition 4.1. Observe that the same construction is valid for a set which is not polyhedral.

5. Deriving tradeoff information

If an outcome set is convex, then as shown by the following three results, we can use a linear function to generate efficient outcomes and simultaneously elicit relative tradeoff information.

THEOREM 5.1 *(Geoffrion, 1968). Assume Z is convex. An outcome $\bar{y} \in Z$ is properly efficient if and only if there exists a vector λ such that $\bar{y}$ solves the problem*

$$\max_{y \in Z} \sum_l \lambda_l y_l \tag{7}$$

for some $\lambda > 0$.

The above theorem is also applicable to a more general case, in which Z is not convex but is R_+^k-convex, i.e. $Z - R_+^k$ is convex, where R_+^k is the non-negative orthant of R^k.

As follows from the next two theorems, assessments of tradeoffs for outcomes generated by problem (7) are available at no cost.

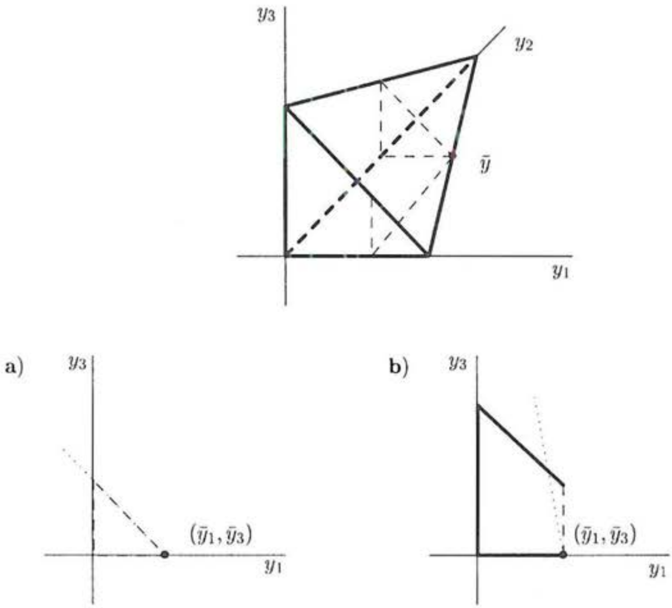

Figure 1. The role of the set $Z_1^<(\bar{y})$ in Definition 4.1. For Z given as above: a) calculation of $T_{31}^G(\bar{y})$ is equivalent in this case to calculation of the supremum of $\frac{y_3-\bar{y}_3}{\bar{y}_1-y_1}$ over the set $\{y \in Z \mid y_2 - \bar{y}_2 = 0\}$, the supremum exists;
b) calculation of the supremum of $\frac{y_3-\bar{y}_3}{\bar{y}_1-y_1}$, where $y_3 - \bar{y}_3 \geq 0,\ \bar{y}_1 - y_1 > 0$, over Z is equivalent in this case to calculation of the supremum of $\frac{y_3-\bar{y}_3}{\bar{y}_1-y_1}$ over the set $\{y \in Z \mid y_2 = 0,\ y_1 \leq \bar{y}_1,\ y_3 \geq \bar{y}_3\}$, the supremum does not exist.

THEOREM 5.2 *(Kaliszewski, 2000). Assume $Z - R_+^k$ is convex. Let $\bar{y}$ solve problem (7) for some $\lambda > 0$. Then*

$$T_{ij}^G(\bar{y}) \leq \frac{\lambda_j}{\lambda_i} \tag{8}$$

for all $i, j = 1, \ldots, k,\ i \neq j$.

Proof. If $\bar{y}$ solves (7) for some $\lambda > 0$ then $\sum_l \lambda_l\, y_l \leq \sum_l \lambda_l\, \bar{y}_l$ for all $y \in Z$. Suppose $y \in Z_j^<(\bar{y})$ for some $j = 1, \ldots, k$. We have

$$\sum_{l \neq j} \lambda_l (y_l - \bar{y}_l) \leq \lambda_j (\bar{y}_j - y_j),$$
$$\lambda_i (y_i - \bar{y}_i) \geq 0 \quad \text{for all} \quad i = 1, \ldots, k,\ i \neq j,$$
$$\lambda_j (\bar{y}_j - y_j) > 0.$$

Hence, $\lambda_i(y_i-\bar{y}_i) \leq \lambda_j(\bar{y}_j-y_j)$ for all $i=1,\ldots,k$, $i \neq j$, and $(y_i-\bar{y}_i)/(\bar{y}_j-y_j) \leq \lambda_j/\lambda_i$ for all $y \in Z_j^{<}(\bar{y})$ and $i=1,\ldots,k$, $i \neq j$. Consequently, $T_{ij}^G(\bar{y}) \leq \frac{\lambda_j}{\lambda_i}$ for all $i=1,\ldots,k$, $i \neq j$. The same argument holds for any $j=1,\ldots,k$. ■

THEOREM 5.3 *Assume $Z - R_+^k$ is convex. If, for some $\lambda > 0$, $\sum_l \lambda_l y_l$ is a unique (up to a scalar multiplier) hyperplane such that*

$$\max_{y \in Z} \sum_l \lambda_l y_l = \sum_l \lambda_l \bar{y}_l$$

for some $\bar{y} \in Z$, then

$$T_{ij}^G(\bar{y}) = \frac{\lambda_j}{\lambda_i}$$

for all $i,j=1,\ldots,k$, $i \neq j$.

Proof. Since $\sum_l \lambda_l y_l$ is a unique (up to a scalar multiplier) hyperplane such that

$$\max_{y \in Z} \sum_l \lambda_l y_l = \sum_l \lambda_l \bar{y}_l$$

for any $\lambda' > 0$ such that $\lambda' = \beta\lambda$ for no scalar multiplier β, we have

$$\sum_l \lambda'_l y_l > \sum_l \lambda'_l \bar{y}_l$$

for some $y \in Z$, $y \neq \bar{y}$. If $y \in Z_j^{<}(\bar{y})$ then

$$\lambda'_i(y_i - \bar{y}_i) > \lambda'_j(\bar{y}_j - y_j) + \sum_{l \neq i,j} \lambda'_l(\bar{y}_l - y_l),$$

$$(y_i - \bar{y}_i)/(\bar{y}_j - y_j) > \lambda'_j/\lambda'_i + (\sum_{l \neq i,j} (\lambda'_l/\lambda'_i)(\bar{y}_l - y_l))/(\bar{y}_j - y_j)$$

and, by Theorem 5.2,

$$\lambda_j/\lambda_i \geq T_{ij}^G(\bar{y}) \geq (y_i-\bar{y}_i)/(\bar{y}_j-y_j) > \lambda'_j/\lambda'_i + (\sum_{l \neq i,j} (\lambda'_l/\lambda'_i)(\bar{y}_l-y_l))/(\bar{y}_j-y_j).$$

The supremum of the rightmost term over $y \in Z_j^{<}(\bar{y})$ and all $\lambda' > 0$, $\lambda' \neq \lambda$, equals λ_j/λ_i, therefore

$$T_{ij}^G(\bar{y}) = \frac{\lambda_j}{\lambda_i}$$ ■

DM preferences with respect to criteria (encapsulated by the relations (3) and (4)) define preferred outcomes as solutions of problem (7).

Similarly, DM preferences with respect to maximal admissible tradeoffs, (captured by relations (8)) define preferred outcomes as solutions of problem (7).

Assuming R^k_+-convexity of Z, we distinguish three classes of problem (1):

1. Z is polyhedral; this is the case addressed by the Zionts-Wallenius algorithm,
2. is "Pareto-smooth", i.e. at each efficient outcome $\bar{y}$ there is exactly one tangent hyperplane of the form

$$\sum_l \lambda_l y_l = \sum_l \lambda_l \bar{y}_l \tag{9}$$

which defines (by Theorem 5.3) all the tradeoffs

$$T^G_{ij}(\bar{y}) = \frac{\lambda_j}{\lambda_i},\ i, j = 1, \ldots, k,\ i \neq j, \tag{10}$$

3. the general case in which tangent hyperplanes (9) provide bounds on tradeoffs rather than their exact values, as stated by Theorem 5.2.

It is noteworthy that formally the function (2) is in general not a special case of the proxy function exploited in Kaliszewski (2000). This implies that formula (8) of Theorem 5.2 is in general not a special case of the tradeoff bounding formula derived in Kaliszewski (2000).

6. The generalized algorithm

With Theorem 5.2 we are in a position to propose a generalization of the Zionts-Wallenius algorithm. The generalized algorithm applies to cases in which the outcome set is R^k_+-convex and the concept of "adjacent vertices" does not apply. It is composed of two basic elements, namely a method for elicitation and handling DM preferences with respect to absolute information, based on Theorem 5.1, and a method for handling DM preferences with respect to relative information, based on Theorem 5.2.

Any relation (3) or (4) set by the DM to express her preferences relating to criteria indirectly establishes, in the light of Theorem 5.2, bounds on tradeoffs of preferred outcomes. Vice-versa, any bound set by the DM on tradeoffs in the form

$$\frac{\lambda_s}{\lambda_t} \leq b_{st} \quad \text{for some} \quad s, t = 1, \ldots, k, s \neq t, \tag{11}$$

and a resulting selection of values λ_i, $i = 1, \ldots, k$, satisfying that bound, indirectly form a preference structure with respect to criteria. In this way the two decision-making paradigms (i.e. absolute information paradigm versus relative information paradigm) interrelate.

It is up to the DM to follow these paradigms. She can even stick to both paradigms. If she treats them independently, there is the possibility of inconsistency. It is obvious that the DM has to make one of the paradigms a leading paradigm; otherwise she must compromise. At least, at each step of a decision making process, the DM can be informed as to how her priority paradigm relates to its complement.

In our generalized approach, for each outcome, tradeoffs can be calculated (Henig, Buchanan, 1997; Kaliszewski, 1994) or, if calculations turn out to be too expensive, approximated. In the algorithm the decision maker responds to questions regarding whether she likes or dislikes tradeoffs. By doing so, by virtue of Theorem 5.2, she provides bounds on the weights in the proxy value function. The rationale for doing so is as follows. Large tradeoffs for an outcome (similar to large values of point-to-point tradeoffs in the Zionts-Wallenius algorithm) mean that some other outcome can be more attractive than the given outcome. (A "large" gain in one criterion can be achieved by a "small" loss in another at no loss in the remaining criteria). Hence, outcomes with limited tradeoffs (and consequently, with limited point-to-point tradeoffs) are potential candidates for "the most preferred" outcome. The argument given here is purely qualitative. The above argument in a broader decision making context was originally presented in Kaliszewski and Michalowski (1999).

The proposed algorithm *GIDMA-Convex (Generalized Interactive Decision Making Algorithm-Convex)* is as follows.

GIDMA-Convex

1. Find an efficient trial outcome y^{tr}.
2. Derive a set of reference outcomes $\{y^{ref}\}$.
3. Ask the DM to evaluate each pair y^{ref}, y^{tr} in terms of y_i, $i = 1, \ldots, k$, and to express her preferences; ask the DM to evaluate y^{tr} in terms of (maximal) tradeoffs and to express her preferences.
4. Derive an outcome y satisfying preferences defined in Step 3. Define it as y^{tr}. Go to Step 2.

Now we shall discuss the steps of GIDMA-Convex algorithm in more detail.

1. The algorithm makes use of (7) to generate efficient outcomes. The vector λ should be specified to provide a good starting outcome. If we do not have a suitable starting set of weights, a vector with all components equal (provided that objectives have first been scaled) may be used.
2. We derive reference outcomes using vectors λ from the preference set (which is built by successive constraints of the type (3), (4), or (11); at the beginning this set is composed of all vectors $\lambda > 0$), and problem (7). These can be generated using a well-dispersed set of weights according to the approach of Steuer (1986).
3. In this step the DM is free to express her preferences with respect to values of the components of y, tradeoffs, or both. In the first and third case she is

supplied with a small number of reference outcomes to compare with the current y^{tr}. As an alternative, outcomes y^{tr} and y^{ref} from the previous iterations which satisfy the current preferences with respect to values of components and tradeoffs can be also used as reference outcomes (we do not exploit this alternative in the numerical example of Section 7).

DM preferences with respect to values of the components of y are expressed by constraints of type (3), or (4), one for each trial outcome.

If the DM chooses to refer to tradeoffs, she can evaluate exact tradeoffs for y^{tr} (which have to be calculated) or evaluate bounds on tradeoffs resulting from vectors λ used to derive y^{tr}. On the basis of the analysis, she sets bounds on tradeoffs that trial solutions should satisfy at the next step.

DM preferences with respect to tradeoffs are expressed in the form of condition (11) which constrain the selection of vectors λ from the preference set.

The DM can be supplied with additional information on what is the least bound on tradeoffs for outcomes generated with vectors λ from the current preference set. If this bound is too large, the preference set can be further constrained.

The DM can also be supplied with such information as the maximal values of selected criteria for outcomes generated with vectors λ from the current preference set. If the DM is not satisfied with the maximal values, then she may relax some constraints of the preference set to get a larger maximum. As a result, DM preferences previously revealed may be modified.

4. To derive trial outcomes, vectors λ are selected from the preference set. Though we do not propose a method, we might use a middlemost set according to a scheme of Köksalan et al. (1984), by finding the set of weights farthest from the nearest constraint. Given a set of λ's, we then generate an outcome by solving problem (7).

The DM is therefore able to express her preferences following two distinct decision making paradigms. She is absolutely free to structure a hierarchy of the paradigms, with the option of changing her hierarchy in the course of a decision process. She may also decide to put more or less relative stress on one paradigm, and even to ignore (possibly temporarily) one of the paradigms at any stage of the process.

The algorithm proposed by Roy and Wallenius (Roy, Wallenius, 1991, 1992) is also an extension of the Zionts-Wallenius algorithm. Elaborating on the concept of basic and nonbasic variables, Roy and Wallenius were able to expand the simplex method framework to nonlinear (mainly convex) multiple criteria programming. Consequently, the number of tradeoffs to be considered in their approach is at most $n - m$ (only efficient tradeoffs are to be considered), where n is the number of variables and m the number of equality constraints. To make the approach operational, in addition to convexity/concavity requirements, complex differentiability assumptions must apply.

In contrast, in the present paper we operate only in the criteria space. Accordingly, the concepts of basic and non-basic variables are not appropriate. The methodology we propose relies only on a relationship between criteria over the outcome set. As a consequence, the number of tradeoffs to be considered is at most $k \times (k-1)$, where k is the number of criteria. Moreover, since we do not exploit variable-criteria relations, and no assumption except R^k_+-convexity of the outcome set is made, our approach is conceptually, technically, and practically less complex than that proposed by Roy and Wallenius.

Let us now consider the question of convergence. The convergence of the Zionts-Wallenius method relies on successively adding constraints (3) or (4) which shrink the set of weights (the preference set). The same principle applies to the Roy and Wallenius algorithm, as well as to the algorithm of Dell and Karwan (Dell, Karwan, 1990), another major extension of the Zionts-Wallenius algorithm. The practical stopping rule is DM's inability to distinguish between two successive outcomes.

Since the Zionts-Wallenius algorithm and Roy-Wallenius algorithm exploited the polyhedral structure of the set of feasible solutions and restricted themselves to vertex solutions only, for bounded feasible solution sets they both converge in a finite number of steps. In both algorithms the polyhedrality of feasible solution sets, under the condition of pseudo-concavity on DM's implicit value function enabled for an optimality condition. In contrast, the Dell-Karwan algorithm produces, in principle, an infinite sequence of solutions and offers no optimality conditions.

As in all the algorithms mentioned above, convergence of our algorithm is ensured by shrinking the set of weights (the preference set) with constraints (3) and (4). Constraints on weights coming from DM's preferences with respect to tradeoffs, provided they do not lead to inconsistency (i.e. they do not cause the constraints on the set of weights to be inconsistent) can only strengthen convergence of the algorithm. As with the Dell-Karwan algorithm, we provide no optimality conditions.

No formal results on convergence ratio for interactive MCDM algorithms can be derived in general, because the DM and her preferences constitute an unpredictable factor. In numerical experiments reported in Zionts and Wallenius (1983), and Dell and Karwan (1991), observed convergence was high (see also Section 7 for a numerical example solved).

7. A numerical example

To illustrate the operation of the *GIDMA-Convex* algorithm, we solve a small numerical example. Before doing this we feel it is necessary to comment on the merits of our example.

The mechanics of the algorithm need not be revealed to the DM. An analogy is that of driving a car; the driver need not know the theory of internal combustion engines. The DM wants a solution. She evaluates outcomes and eventually stops the decision process; the rest is the responsibility of a facilitator. In other words, the DM "part" of *GDIM-Convex* follows the generic interactive MCDM approach, which relies on evaluating a sequence of successive feasible decisions. An "informed" DM may wish to take some or all responsibilities of the facilitator. Communication between the DM and the facilitator should be made in a customary problem-oriented manner. The form of communication used in the example seems to be acceptable in its specific economic context but it is by no means indicative for a range of other possible applications.

We now illustrate the *GIDMA-Convex* algorithm using a simple example of an important finance problem: the Markowitz mean-variance portfolio model. In the model a portfolio is selected from a group of stocks to maximize a linear function of expected portfolio return and minimize portfolio variance. The Markowitz model assumes that all capital is fully invested. By normalizing the amount of capital to be unity, the individual stock investment is represented as the fraction of the portfolio invested in each stock. Short sales of stock are permitted, and are indicated as negative investments.

It is interesting to observe that the outcome set of (12) is not convex but R^k_+-convex (see e.g. Elton, Gruber, 1995).

The model is as follows.

$$\begin{aligned} &\text{(maximize portfolio expected return)} && \max \sum_{i=1}^{n} e_i x_i \\ &\text{(minimize portfolio variance)} && \min \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} x_i x_j \qquad (12) \\ &\text{(the "fully-invested" constraint)} && \sum_{i=1}^{n} x_i = 1, \end{aligned}$$

where ρ_{ij} denotes the covariance matrix coefficient for the stock i and the stock j, and e_i denotes the expected return for the stock i.

In the finance literature this model is often solved using a single objective function that minimizes variance, subject to a constraint specifying a minimal acceptable expected return, namely:

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} x_i x_j$$

$$\sum_{i=1}^{n} e_i x_i \geq \bar{y}_e,$$

$$\sum_{i=1}^{n} x_i = 1$$

where $\bar{y}_e$ is the minimum acceptable level of expected return of the portfolio.

Observe that if in the first constraint the inequality sign is changed to an equality sign the problem can be solved analytically (see, e.g. Dahl et al., 1993) but any constraint $x_i \geq 0$ prohibiting short sale of a particular stock makes this approach invalid.

It is certainly more appropriate to solve the portfolio selection problem (12) using two criteria. We accordingly seek a satisfactory compromise on the levels of expected return and variance. We repeat here the argument about the rationale for using tradeoffs as preference indicators, presented in Section (6), this time in the portfolio selection context.

If, at a given outcome, some point-to-point tradeoffs are attractive, then the corresponding solution may not be 'the most preferred'. Solutions for which no point-to-point tradeoff is attractive can be 'most preferred'. Consequently, outcomes with unattractive (limited) tradeoffs are also potential candidates for 'the most preferred' outcome.

To illustrate, we shall solve a numerical example with data taken from the well-known "three-stock" Markowitz example (see Markowitz, 1959). However, in contrast to the finance literature, rather than solving the problem by minimizing risk subject to minimal acceptable return levels, we shall solve it in interactive manner.

We add a third objective to maximize the earnings-to-price (minimize the price-to-earnings) ratio of the portfolio. We do this to depart from the simplicity of two-objective problems in which sets $Z_i^<(\bar{y})$ reduce to $Z_i^<(\bar{y}) = \{y \in Z \mid y_1 < \bar{y}_1, y_2 \geq \bar{y}_2\}\}$ and $Z_2^<(\bar{y}) = \{y \in Z \mid y_2 < \bar{y}_2, y_1 \geq \bar{y}_1\}$. Our third objective is therefore to maximize:

$$\sum_{i=1}^{n} p_i x_i,$$

where p_i is the reciprocal of the price-to-earnings ratio P/E of a stock, a commonly-used finance measure of stock value. This measure enables what is commonly called "portfolio tilting" (for an extensive survey of publications on that topic see Ziemba, 1994).

The problem we solve below for $n = 3$ is small but illustrative. The algorithm works the same for any n. The data for the example are shown below (all data, except those for P/E, which we randomly generated, are the original Markowitz data). There are three stocks: ATT, GM, USX, characterized by a covariance matrix and expected returns over the investment period:

	ATT	GM	USX	
ATT	.01080754	.01240721	.01307513	
GM	.01240721	.05839170	.05542639	**Covariance matrix**
USX	.01307513	.05542639	.09422681	
	.0890833	.213667	.234583	**Expected returns**
	0.24	0.12	0.06	**E/P**

Since all objectives are to be maximized, we use the negative of variance as an objective.

To simulate the DM's behavior in terms of $y_i, i = 1, \ldots, k$, we assume that her (unknown) value function is [2]

$$f(y_e, y_\sigma) = -30y_v^2 + y_e^2 + y_p^2 = -30y_{-v}^2 + y_e^2 + y_p^2,$$

where $y_v = \sum_{i=1}^n \sum_{j=1}^n \rho_{ij} x_i x_j$, $y_v = -y_{-v}$, $y_e = \sum_{i=1}^n e_i x_i$ and $y_p = \sum_{i=1}^n p_i x_i$. Given the constraint that all capital must be invested, $\sum_{i=1}^n x_i = 1$, this function attains its maximum $f^{\max} = 1.392$ at $\tilde{y}^{\max} = (\tilde{y}_{-v}^{\max}, \tilde{y}_e^{\max}, \tilde{y}_p^{\max}) = (-0.054, 1.240, 0.115)$ ($x^{\max} = (0.060, 0.737, 0.0203)$, where x_1 corresponds to ATT, x_2 corresponds to GM, x_3 corresponds to USX).

We now use the *GIDMA-Convex* algorithm to assist the DM in finding a portfolio that has a satisfactory compromise on expected return, variance, and E/P. We assume that the DM is consistent with her value function.

The steps of the algorithm with the simulated behavior of DM are as follows. Without loss of generality we assume that $\lambda_e + \lambda_{-v} + \lambda_p = 1$. The initial preference set is

$$\Lambda^1 = \{(\lambda_{-v}, \lambda_e, \lambda_p) | \lambda_{-v} + \lambda_e + \lambda_p = 1,\ \lambda_{-v} > 0,\ \lambda_e > 0,\ \lambda_p > 0\}.$$

Additional information that may be made available to the DM at the beginning of the process includes the following:

- the maximal y_{-v} over efficient outcomes of $Z : y_{-v}^{\max} = -0.011$;
- $(y_{-v}^{\max},\ y_e(y_{-v}^{\max}), y_p(y_{-v}^{\max})) = (-0.011,\ 1.084,\ 0.246)$;
- the maximal y_e over efficient outcomes of Z : $y_e^{\max}$ - unbounded;
- the maximal y_p over efficient outcomes of Z : $y_p^{\max}$ - unbounded;

(The value of the DM value function at $(y_{-v}^{\max},\ y_e(y_{-v}^{\max}), y_p(y_{-v}^{\max}))$: $f(y_{-v}^{\max},\ y_e(y_{-v}^{\max}), y_p(y_{-v}^{\max})) = -0.003$.

<u>*GIDMA-Convex*</u>

1. For $\lambda^{tr(1)} = (\lambda_{-v}, \lambda_e, \lambda_p) = (0.500, 0.400, 0.100) \in \Lambda^1$ we get a trial outcome $y^{tr(1)} = (-0.044, 1.194, 0.134)$ ($x^{tr(1)} = (0.174, 0.713, 0.112)$). In the example the choices from Λ^i are arbitrarily chosen - we have not normalized the objectives but they are roughly of the same magnitude. (The value of the DM value function at $y^{tr(1)} : f(y^{tr(1)}) = 1.387$).

Iteration 1

2. The following vectors are selected from the set Λ^1 :
$\lambda^{ref(1,1)} = (0.700, 0.200, 0.100)$, $\lambda^{ref(1,2)} = (0.600, 0.300, 0.100)$, $\lambda^{ref(1,3)} =$

[2]Observe that the algorithm follows the *man-machine* interactive scheme. The scheme, one of the cornerstones of *cybernetics*, was proposed to account for unpredictability of *man* (the DM). Consequently, no assumption on DM's behavior is made here, and the only way to get quantitative results is simulation.

(0.400, 0.500, 0.100) (the first superscript index denotes the iteration number). Solving (7) with these vectors yields

$$y^{ref(1,1)} = (-0.013, 1.109, 0.222) \quad (x^{ref(1,1)} = (0.831, 0.202, -0.032)),$$
$$y^{ref(1,2)} = (-0.021, 1.145, 0.185) \quad (x^{ref(1,2)} = (0.557, 0.415, 0.028)),$$
$$y^{ref(1,3)} = (-0.103, 1.268, 0.058) \quad (x^{ref(1,3)} = (-0.400, 1.161, 0.239)).$$

(The value of the DM value function at $y^{ref(1,1)}, y^{ref(1,2)}, y^{ref(1,3)} : f(y^{ref(1,1)}) = 1.275, f(y^{ref(1,2)}) = 1.332, f(y^{ref(1,3)}) = 1.297$).

3. *Evaluation of* $y^{tr(1)}$ and $y^{ref(1,1)}, y^{ref(1,2)}, y^{ref(1,3)}$ *in terms of* y_{-v}, y_e *and* y_p.

The DM prefers $y^{tr(1)}$ to $y^{ref(1,1)}$ as well as to $y^{ref(1,3)}$ because she is not willing to accept such a low return as represented by $y^{ref(1,1)}$ and such a high risk as represented by $y^{ref(1,3)}$. By this we have

$$\lambda y^{ref(1,1)} < \lambda y^{tr(1)} \text{ and } \lambda y^{ref(1,3)} < \lambda y^{tr(1)}.$$

The DM is uncertain about her preference between $y^{tr(1)}$ and $y^{ref(1,2)}$ because they are so close in value.

Evaluation of $y^{tr(1)}$ *in terms of tradeoffs*

The DM states that she will not accept any outcome y as the final choice as long as it shows $T^G_{e,-v}(y) > 2$ (for such a y there is a potential to improve at least by two units of expected return at the expense of one unit of variance at no loss in P/E ratio) or $T^G_{-v,e}(y) > 2$ (for such a y there is a potential to improve at least by two units on variance at the expense of one unit of expected return at no loss in P/E ratio). To comply with this requirement we set $\frac{\lambda_{-v}}{\lambda_e} \leq 2$ and $\frac{\lambda_e}{\lambda_{-v}} \leq 2$. The preference set is now

$$\Lambda^2 = \Lambda^1 \cup \{\lambda | \lambda y^{ref(1,1)} < \lambda y^{tr(1)}, \lambda y^{ref(1,3)} < \lambda y^{tr(1)}, \frac{\lambda_{-v}}{\lambda_e} \leq 2, \frac{\lambda_e}{\lambda_{-v}} \leq 2\}$$

or

$$\lambda_{-v} + \lambda_e + \lambda_p = 1,$$
$$0.0312\lambda_{-v} - 0.085\lambda_e + 0.088\lambda_p < 0,$$
$$-0.0588\lambda_{-v} + 0.074\lambda_e - 0.076\lambda_p < 0,$$
$$\frac{\lambda_{-v}}{\lambda_e} \leq 2,$$
$$\frac{\lambda_e}{\lambda_{-v}} \leq 2,$$
$$\lambda_{-v} > 0,\ \lambda_e > 0,\ \lambda_p > 0,$$

4. For $\lambda^{tr(2)} = (\lambda_{-v}, \lambda_e, \lambda_p) = (0.450, 0.377, 0.173) \in \Lambda^2$ we get a trial outcome $y^{tr(2)} = (-0.030, 1.166, 0.167)\,(x^{tr(2)} = (0.382, 0.646, -0.029))$ (The value of the DM value function at $y^{tr(2)} : f(y^{tr(2)}) = 1.360$).

Iteration 2

2. The following vectors are selected from the set Λ^2 :
$\lambda^{ref(2,1)} = (0.440, 0.377, 0.183)$, $\lambda^{ref(2,2)} = (0.550, 0.333, 0.117)$, $\lambda^{ref(2,3)} = (0.600, 0.333, 0.067)$. Solving (7) with these vectors gives

$$y^{ref(2,1)} = (-0.029, 1.163, 0.170) \quad (x^{ref(2,1)} = (0.400, 0.648, -0.048)),$$
$$y^{ref(2,2)} = (-0.025, 1.156, 0.175) \quad (x^{ref(2,2)} = (0.472, 0.500, 0.029)),$$
$$y^{ref(2,3)} = (-0.029, 1.166, 0.162) \quad (x^{ref(2,3)} = (0.400, 0.505, 0.095)).$$

(The value of the DM value function at $y^{ref(2,1)}, y^{ref(2,2)}, y^{ref(2,3)} : f(y^{ref(2,1)}) = 1.357, f(y^{ref(2,2)}) = 1.348, f(y^{ref(2,3)}) = 1.361$).

Observe that we could use $y^{tr(1)}$ as a reference outcome for it satisfies the current preferences with respect to values of the components and tradeoffs. Arbitrarily, we avoid doing this.

3. *Evaluation of* $y^{tr(2)}$ and $y^{ref(2,1)}, y^{ref(2,2)}, y^{ref(2,3)}$ *in terms of* y_{-v}, y_e *and* y_p

The DM prefers $y^{tr(2)}$ to $y^{ref(2,2)}$ because she is not willing to accept the low expected return represented by $y^{ref(2,2)}$. By this we have

$$\lambda\, y^{ref(2,2)} < \lambda\, y^{tr(2)}$$

The DM is uncertain about her preference among $y^{tr(2)}$, $y^{ref(2,1)}$ and $y^{ref(2,3)}$ in terms of values of criteria.

Evaluation of $y^{tr(2)}$ *in terms of tradeoffs*

For $y^{tr(2)}$ we have

$$T^G_{-v,p}(y) = \frac{0.173}{0.450} = 0.38$$
$$T^G_{e,p}(y) = \frac{0.173}{0.377} = 0.46\ .$$

The DM decides that these numbers show a significant potential to improve variance and expected return at the expense of P/E ratio. She also decides that for any outcome to be considered for the final choice, it should show a lower potential and sets $T^G_{-v,p}(y^{tr(2)}) \leq 0.2$ and $T^G_{e,p}(y^{tr(2)}) \leq 0.2$ which amounts to constraints $\frac{\lambda_p}{\lambda_{-v}} \leq 0.2$ and $\frac{\lambda_p}{\lambda_e} \leq 0.2$. The preference set is now

$$\Lambda^3 = \Lambda^2 \cup \{\lambda | \lambda y^{ref(2,2)} < \lambda y^{tr(2)}, \frac{\lambda_p}{\lambda_{-v}} \leq 0.2, \frac{\lambda_p}{\lambda_e} \leq 0.2\}$$

or

$$\lambda_{-v} + \lambda_e + \lambda_p = 1,$$

$$0.0312\lambda_{-v} - 0.085\lambda_e + 0.088\lambda_p < 0,$$
$$-0.0588\lambda_{-v} + 0.074\lambda_e - 0.076\lambda_p < 0,$$
$$0.005\lambda_{-v} + 0.012\lambda_e - 0.008\lambda_p < 0,$$
$$\frac{\lambda_p}{\lambda_{-v}} \leq 0.2,$$
$$\frac{\lambda_p}{\lambda_e} \leq 0.2,$$
$$\lambda_{-v} > 0,\ \lambda_e > 0,\ \lambda_p > 0.$$

4. For $\lambda^{tr(3)} = (\lambda_{-v}, \lambda_e, \lambda_p) = (0.480, 0.453, 0.067)$ we get a trial outcome $y^{tr(3)} = (-0.070, 1.233, 0.093)$ $(x^{tr(3)} = (-0.118, 0.900, 0.217))$. (The value of the DM value function at $y^{tr(3)} : f(y^{tr(3)}) = 1.380$).

Iteration 3

2. The following vectors are selected from the set Λ^3 : $\lambda^{ref(3,1)} = (0.420, 0.540, 0.040)$, $\lambda^{ref(3,2)} = (0.440, 0.480, 0.080)$, $\lambda^{ref(3,3)} = (0.460, 0.460, 0.080)$. Solving (7) with these vectors gives

$$y^{ref(3,1)} = (-0.142, 1.305, 0.016) \quad (x^{ref(3,1)} = (-0.671, 1.228, 0.383)),$$
$$y^{ref(3,2)} = (-0.087, 1.252, 0.074) \quad (x^{ref(3,2)} = (-0.269, 1.032, 0.237)),$$
$$y^{ref(3,3)} = (-0.073, 1.237, 0.090) \quad (x^{ref(3,3)} = (-0.149, 0.940, 0.209)).$$

(The value of the DM value function at $y^{ref(3,1)}, y^{ref(3,2)}, y^{ref(3,3)} : f(y^{ref(3,1)}) = 1.099$, $f(y^{ref(3,2)}) = 1.347$, $f(y^{ref(3,3)}) = 1.375$).

3. *Evaluation of $y^{tr(3)}$ in terms of y_{-v}, y_e and y_p*

The DM prefers $y^{tr(3)}$ to $y^{ref(3,1)}$, $y^{ref(3,2)}$ and $y^{ref(3,3)}$ because she is not willing to accept the high variance. We then have

$$\lambda y^{ref(3,1)} < \lambda y^{tr(3)},$$
$$\lambda y^{ref(3,2)} < \lambda y^{tr(3)},$$
$$\lambda y^{ref(3,3)} < \lambda y^{tr(3)}.$$

Evaluation of $y^{tr(3)}$ in terms of tradeoffs

The full tradeoff matrix for $y^{tr(3)}$ is as follows:

	$-v$	e	p
$-v$	*	$T^G_{-v,e}(y^{tr(3)}) = 0.944$	$T^G_{-v,p}(y^{tr(3)}) = 0.140$
e	$T^G_{e,-v}(y^{tr(3)}) = 1.060$	*	$T^G_{e,p}(y^{tr(3)}) = 0.148$
p	$T^G_{p,-v}(y^{tr(3)}) = 7.164$	$T^G_{p,e}(y^{tr(3)}) = 6.761$	*

DM decides that these numbers show no significant potential to improve on any criterion relative to other criteria.

The algorithm terminates (by any stopping rule such as: DM satisfaction with $y^{tr(i)}$, time limit, iteration limit, or "volume" of Λ^i related constraint).

This example is peculiar in that our starting outcome $y^{tr(1)}$ is extremely close to the underlying optimal $y^{\max}$ (it is not true, however, for the starting solution $x^{tr(1)}$ and the underlying optimal $x^{\max}$). We had thought of changing it in such a way that the starting solution be not so close, but decided to keep it the way we had done it. Obviously, we were lucky. Further, the most preferred solution $x^{tr(3)}$ may look strange, because the amounts invested in different stocks are of markedly different magnitudes. Our solution prescribes a negative amount of (short sells) stock number 1. Given the data, the solution gives the optimal amounts to invest; that is all we can say. Minimum or maximum amounts for each stock can be enforced by using additional constraints.

8. Discussion

The novelty and potential of the proposed algorithm lies in the ability to support two distinct decision paradigms in the course of an interactive decision making process in case of R^k_+-convex outcome sets. This gives the algorithm a new dimension absent in other classes of decision-making algorithms exploiting principles different from the prototyping Zionts-Wallenius approach. Those two paradigms are interrelated and we have shown how to trace their relationship and exploit it in a decision process. Besides the prototypical Zionts-Wallenius algorithm the same two-paradigm approach was proposed in Kaliszewski et al. (1997), Kaliszewski, Michalowski (1999), and Kaliszewski (2000), whereas only the last paper (Kaliszewski, 2000), can be considered as a generalization of the Zionts-Wallenius algorithm.

In Kaliszewski (2000) an algorithm (GIDMA) was proposed in which tradeoff information is derived from a Tchebycheff proxy value function. Similar to formula (8) of Section 5 of this paper, the tradeoff information in Kaliszewski (2000) is in the form of upper bounds on tradeoffs. Because there is no a priori indicator of bound tightness (bound tightening requires additional computing (Kaliszewski, 1994), one can consider the approach justified only if some special case considerations do not apply.

In contrast to Kaliszewski (2000), the focal point of this paper is on the result of Theorem 5.3 (but of course instances where tradeoff information in the form of the inequality (8) can be provided are also covered by virtue of Theorem 5.2) which applies when the set $Z - R^k_+$ is "smooth" (i.e. roughly speaking: it admits no vertices). The information is exact, i.e. the bounds on tradeoffs are tight.

A popular belief among MCDM researchers is that the vast majority of practical applications of MCDM methodologies involve linear models (linear constraints, linear objective functions). It is probably time to modify this belief and adjust it to reality. Currently more and more importance is attributed to so-called quadratic programming models (linear constraints, quadratic and linear objective functions) for such models can encompass risk (see classical work of Markowitz, 1959; also Zenios, 1993; Dahl et al., 1993; Elton, Gruber, 1995). Models involving risk have become the standard for financial industry worldwide (*Basel Capital Accord*, 1988, and *New Basel Capital Accord*, 2004) form a part of regulatory framework for financial institutions. As the basic principle of investing is to maximize profit, risk is similarly to be minimized. As a result, we have multiple criteria. Risk is captured via the notions of random variables, variance and correlations of investments.

Financial models are routinely solved in leading banking and investment institutions but as a rule by one-step optimization. Recently, they have become accessible to individual investors via internet access to specialized service providers. An interested reader can consult e.g. the website www.riskgrades.com, it is noteworthy that optimization options available there are either maximization of return under constrained risk or minimization of risk under constrained return but nothing "in between", not to mention interactive solving option. These models generally have some form of mean-variance portfolio selection model as outlined in Section 7. What we have proposed is an enrichment of the model capabilities and model solving options. Accordingly, we need not validate the resulting algorithm numerically. Our proposed algorithm offers a way of enriching existing models, even for realistic-sized problems.

As with any preference capturing technique, the algorithm we propose is particularly useful for incremental optimization problems. For example, in the world of finance the problem of portfolio dynamic adjustment is crucial. Because of the transaction costs, adjustments of portfolio are limited to a small subset of total assets held in a portfolio. Limited changes to the composition of portfolio cause limited changes in the shape of the Pareto set (this is common wisdom based on practical observations; in theory degenerate counterexamples can be constructed). Thus, DM's preferences captured in the course of an interactive decision making process can be applied in one-step incremental optimization for a certain period of portfolio adjustments. (See Chen et al., 1971, for additional information on portfolio revision).

Recently Fliege and Heseler (Fliege, Heseler, 2002) reported on solving bi-criteria quadratic programming problems in connection with power generation. A sequence of slightly modified problems is to be solved every fifteen minutes. Each feasible solution represents a possible variant of power plant dispatch; efficient solutions represent economically effective variants. Since problems are to be solved so often and sets $Z - R^k_+$ are "smooth", preference capture technique offered by our algorithm can be especially attractive in this application.

9. Concluding remarks

The relation between bounds on tradeoffs and weighting (scalarizing) parameters are more general than those exploited in this paper. To generate outcomes in problems where outcome sets are not convex, instead of linear scalarization one has to apply a Tchebycheff scalarization, see Dell, Karwan (1990), Kaliszewski (1987, 1994, 1995), Steuer (1986), Wierzbicki (1986, 1990). Also for Tchebycheff scalarizations relations exist between tradeoffs and scalarization parameters. Those relations can form the basis for further research on more general algorithms for interactive decision making taking into account more than one decision paradigm. Some results in this field were described in papers by Kaliszewski (2000), Kaliszewski, Michalowski (1995, 1997, 1999), and Kaliszewski et al. (1997).

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