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Stability and accuracy functions in multicriteria combinatorial optimization problem with Σ-MINMAX and Σ-MINMIN partial criteria

by

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Abstract: We consider a vector generic combinatorial optimization problem in which initial coefficients of objective functions are subject to perturbations. For Pareto and lexicographic principles of efficiency we introduce appropriate measures of quality of a given feasible solution from the point of view of its stability. These measures correspond to so-called stability and accuracy functions defined earlier for scalar optimization problems. Then we study properties of such functions and calculate the maximal norms of perturbations for which an efficient solution preserves the efficiency.

Keywords: multicriteria combinatorial optimization, sensitivity analysis, stability and accuracy functions, Pareto and lexicographic optima.

1. Introduction

Many real-life optimization models which arise in different areas, e.g. in scheduling, vehicle routing, location modeling and design, must be stated as multicriteria problems. Their solving consists in finding feasible decisions which provide a compromise between multiple objectives (see e.g. Sawaragi et al., 1985, Steuer, 1986, Ehrgott, 1997). An immanent property of real-life problems is also uncertainty of data which can be handled by different approaches, like stability and sensitivity analysis (see e.g. Sotscov et al., 1995, 1998, Chakravarty and Wagelmans, 1999), stochastic programming (see e.g. Kall and Wallace, 1994), robust optimization (see e.g. Kouvelis and Yu, 1997, Ben-Tal and Nemirowski, 1998, Bertsimas and Sim, 2002) etc.

This paper concerns stability analysis for multicriteria optimization problems. Recently we observe a growing stream of papers devoted to this direction. Most of them concentrate on finding maximal perturbations of the problem data, for which the optimality (efficiency) of a given solution can be preserved (see e.g. Emelichev, 2002). An important drawback of this approach consists in the fact that such maximal perturbations appear to be very small or, frequently, equal to zero. Therefore it is necessary to analyze what is happening with a particular solution in the case when data perturbations destroy its efficiency. In that case we want to know what is the value of relative quality measure for this solution, which can be defined in a special appropriate way depending on the considered particular optimality principle. Such quality measures lead to the concepts of stability and accuracy functions.

The first attempt to analyze the quality of a solution in single objective case for facility location problem was done in Labbe' et al. (1991). In Libura (1999, 2000) explicit formulae of stability and accuracy functions were obtained for scalar linear combinatorial optimization problems. These functions under multiobjective framework were first studied in Libura and Nikulin (2003). In this paper we present an extension of results Libura and Nikulin (2003) for the case when the considered optimization criteria have more general forms called Σ-MINMAX and Σ-MINMIN.

The paper is organized as follows. In Section 2 we consider vector combinatorial optimization problem with Σ-MINMAX and Σ-MINMIN partial criteria which consists in finding the set of Pareto optimal solutions. In analogy to Libura and Nikulin (2003), for a given Pareto optimal solution we introduce the relative error as a function of the norm of data perturbations. This leads us to natural extension of the stability function and the accuracy functions for the type of criteria we consider to the multiobjective case. We give formulae to calculate values of both functions. Afterward, we define the so called stability (respectively – accuracy) radius as extreme norm of perturbations of problem parameters for which stability (accuracy) function is equal to zero. In Section 3 analogous results are stated for the case of lexicographic optimality. In this section both functions are redefined in order to reflect specific of lexicographic efficiency.

2. Stability and accuracy functions of Pareto optimal solution

Let $E = \{e_1, e_2, ..., e_n\}$, $n > 1$, be a given set, and let $T \subseteq 2^E$, $|T| > 1$, be a family of non-empty subsets of E. Denote $\mathbf{R}_+ = \{u \in \mathbf{R} : u > 0\}$. For $e \in E$ and $m \geq 1$, we define

$$c(e) = (c_1(e), c_2(e), ..., c_m(e)) \in \mathbf{R}_+^m$$

and a matrix $C = \{c_i(e_j)\} \in \mathbf{R}_+^{m \times n}$. Put for $k \in \mathbf{N}$, $N_k = \{1, 2, ..., k\}$ and let for $t \in T$, $N(t) = \{j : e_j \in t\}$.

We will consider so-called Σ-MINMAX or Σ-MINMIN multiobjective optimization problem (see e.g. Girlich et al., 1999). Namely, we want to minimize over $t \in T$ the following vector objective function:

$$f(C,t) = (f_1(C,t), f_2(C,t), ..., f_m(C,t)),$$

where for $i \in N_m$,

$$f_i(C,t) = \max \left\{ \sum_{e \in q} c_i(e) : \ q \subseteq t, \ |q| = \min\{|t|, k_i\} \right\}, \tag{1}$$

or

$$f_i(C,t) = \min \left\{ \sum_{e \in q} c_i(e) : \ q \subseteq t, \ |q| = \min\{|t|, k_i\} \right\}. \tag{2}$$

Here k_i, $i \in N_m$, are given a priori numbers such that for $i \in N_m$ and $K = \max\{|t| : \ t \in T\}$, $1 \le k_i \le K$.

When $k_i = K$, $i \in N_m$, then both functions (1) and (2) transform into a linear objective function:

$$f_i(C,t) = \sum_{e \in t} c_i(e),$$

which leads to the MINSUM criterion. When $k_i = 1$, then function (1) converts into the function

$$f_i(C,t) = \max\{c_i(e) : \ e \in t\}$$

and we have a bottleneck criterion (MINMAX). Similarly, for $k_i = 1$ function (2) turns into

$$f_i(C,t) = \min\{c_i(e) : \ e \in t\},$$

which leads to the MINMIN criterion.

Let for a matrix $C \in \mathbf{R}_+^{m \times n}$ and a feasible solution $t \in T$,

$$\pi(C,t) = \{t' \in T : \ f(C,t') \le f(C,t), \ f(C,t') \ne f(C,t)\}.$$

The Pareto set $P^m(C)$ is defined in a traditional way, namely:

$$P^m(C) = \{t \in T : \ \pi(C,t) = \emptyset\}.$$

In other words, a feasible solution t is Pareto optimal if and only if there is no solution $t' \in T$ such that $f_i(C,t') \le f_i(C,t)$ for all $i \in N_m$ and at least one strict inequality holds. If the sets E and T are fixed, then an instance of $m-$criteria combinatorial optimization problem is uniquely determined by the matrix $C \in \mathbf{R}_+^{m \times n}$. Therefore, we will denote it by $Z_P^m(C)$.

It is assumed that the set T is fixed, but the matrix of weights C may vary or is estimated with errors. Moreover, it is assumed that for some originally specified matrix $C^0 = \{c_i^0(e_j)\} \in \mathbf{R}_+^{m\times n}$ we know one Pareto optimal solution $t^0 \in T$.

When coefficients of objective functions change, then an initially efficient solution may become no longer efficient. We will evaluate the quality of this solution from the point of view of its robustness with respect to data perturbations. Namely, we will calculate for a given norm of perturbations the maximal possible 'inefficiency' of this solution. The measure of this 'inefficiency' depends on the optimality principle for the multiobjective problem.

In case of Pareto optimality the 'inefficiency' of the solution $t^0 \in P^m(C^0)$ for a given matrix C may be measured by the value of the relative error of this solution:

$$\varepsilon_P(C,t^0) = \max_{t\in T}\min_{i\in N_m}\frac{f_i(C,t^0)-f_i(C,t)}{f_i(C,t)} \geq 0. \tag{3}$$

Observe, that if $t^0 \in P^m(C)$, then $\varepsilon_P(C,t^0)=0$. If t^0 loses Pareto optimality in an instance problem $Z_P^m(C)$, then the relative error $\varepsilon_P(C,t^0)>0$ characterizes the 'inefficiency' of t^0.

The considered measure (3) is a natural analogue of the suboptimality measure of feasible solution in the scalar case. Indeed, for m=1, the Pareto set transforms into the set of optimal solutions. Therefore relative error $\varepsilon_P(C,t^0)$ converts into (see Libura, 1999):

$$\varepsilon_P(C,t^0) = \frac{f_1(C,t^0) - \min\limits_{t\in T} f_1(C,t)}{\min\limits_{t\in T} f_1(C,t)}.$$

In fact, we are interested in the maximal value of the error $\varepsilon_P(C,t^0)$ when the matrix C belongs to some specified set. Two particular cases are considered in the following.

In the first case we are interested in absolute perturbations of the weights of elements and the quality of a given solution is described by the so-called *stability function*. For a given $p \geq 0$ the value of the stability function is equal to the maximal relative error of a given solution under the assumption that no weights of elements are increased or decreased by more than p.

In the second case we deal with relative perturbations of weights. This leads to the concept of *accuracy function*. The value of the accuracy function for a given $\delta \in [0,1)$ is equal to the maximum relative error of the solution t^0 under the assumption that the weights of the elements are perturbed by no more than $\delta \cdot 100\%$ of their original values.

Let $X \subseteq E$ be the set of non-stable elements, i.e. elements for which weights may change, and let

$$C^0(X) = \{C \in \mathbf{R}_+^{m\times n}:\ c_i(e_j) = c_i^0(e_j),\ e_j \in E\backslash X,\ i\in N_m\ j \in N_n\}.$$

For a given $p \in [0, q(C^0, X))$, where $q(C^0, X) = \min\{c_i^0(e_j) : \ e_j \in X, \ i \in N_m, \ j \in N_n\}$, we consider a set

$$\Omega_p(C^0, X) = \{C \in C^0(X) : \ |c_i(e_j) - c_i^0(e_j)| \leq p, \ i \in N_m, \ j \in N_n\}.$$

For a Pareto optimal solution $t^0 \in P^m(C^0)$, an arbitrary set of non-stable elements X, and $p \in [0, q(C^0, X))$, the value of the stability function is defined as follows:

$$S_P(t^0, X, p) = \max_{C \in \Omega_p(C^0, X)} \varepsilon_P(C, t^0).$$

In a similar way, for a given $\delta \in [0, 1)$, we consider a set

$$\Theta_\delta(C^0, X) = \{C \in C^0(X) : \ |c_i(e_j) - c_i^0(e_j)| \leq \delta c_i^0(e_j), i \in N_m, \ j \in N_n\}.$$

For a Pareto optimal solution $t^0 \in P^m(C^0)$, an arbitrary set of non-stable elements X and $\delta \in [0, 1)$, the value of the accuracy function is defined as follows:

$$A_P(t^0, X, \delta) = \max_{C \in \Theta_\delta(C^0, X)} \varepsilon_P(C, t^0).$$

Observe that $S_P(t^0, X, p) \geq 0$ for any $p \in [0, q(C^0, X))$ as well as $A_P(t^0, X, \delta) \geq 0$ for each $\delta \in [0, 1)$. Moreover, if we consider two initially efficient solutions $t', t'' \in P^m(C^0)$ such that $S_P(t', X, p) \leq S_P(t'', X, p)$ for $p \subseteq [0, q(C^0, X))$ or $A_P(t', X, \delta) \leq A_P(t'', X, \delta)$ for $\delta \subseteq [0, 1)$, then the solution t' may be regarded as 'at least as good' as the solution t'' from the stability (robustness) point of view, because it guarantees the same or smaller 'inefficiency' for the considered data perturbations.

For any $t, t' \in T$ let $t \otimes t' = (t \backslash t') \cup (t' \backslash t)$. Thus $|t \otimes t'| = |(t \backslash t') \cup (t' \backslash t)| = |t| + |t'| - 2|t \cap t'|$. Let for any $t \neq t'$,

$$\Delta(t, t', X) = \begin{cases} |(t \otimes t') \cap X| & \text{if } i \in I_{SUM}, \\ \min\{|t \cap X|, k_i\} + \min\{|t' \cap X|, k_i\} & \text{otherwise,} \end{cases}$$

and $\Delta(t, t', X)$=0 if $t = t'$. Here $I_{SUM} = \{i \in N_m : \ k_i = K\}$.

THEOREM 2.1 *For an optimal solution $t^0 \in P^m(C^0)$, an arbitrary set X of non-stable elements, and $p \in [0, q(C^0, X))$, the stability function can be expressed by the formula:*

$$S_P(t^0, X, p) = \max_{t \in T} \min_{i \in N_m} \frac{f_i(C^0, t^0) - f_i(C^0, t) + p\,\Delta(t, t', X)}{f_i(C^0, t) - p \ \min\{|t \cap X|, k_i\}}. \tag{4}$$

For an optimal solution $t^0 \in P^m(C^0)$, an arbitrary set X of non-stable elements, and $\delta \in [0, 1)$, the accuracy function can be described by the formula:

$$A_P(t^0, X, \delta) = \max_{t \in T} \min_{i \in N_m} \frac{f_i(C^0, t^0) - f_i(C^0, t) + \delta f_i(C^0, (t \otimes t^0) \cap X)}{f_i(C^0, t) - \delta f_i(C^0, t \cap X)}. \tag{5}$$

Proof. We will prove only (4). The proof of (5) is analogous.

$$S_P(t^0, X, p) = \max_{C\in\Omega_p(C^0,X)} \varepsilon_P(C, t^0) = \max_{C\in\Omega_p(C^0,X)} \max_{t\in T} \min_{i\in N_m} \frac{f_i(C,t^0) - f_i(C,t)}{f_i(C,t)} =$$

$$= \max_{t\in T} \max_{C\in\Omega_p(C^0,X)} \min_{i\in N_m} \frac{f_i(C,t^0) - f_i(C,t)}{f_i(C,t)} \leq$$

$$\leq \max_{t\in T} \min_{i\in N_m} \max_{C\in\Omega_p(C^0,X)} \frac{f_i(C,t^0) - f_i(C,t)}{f_i(C,t)}.$$

For any fixed $t \in T$ and $i \in N_m$ the maximum of the ratio $\frac{f_i(C,t^0)-f_i(C,t)}{f_i(C,t)}$ over $C \in \Omega_p(C^0, X)$ is attained when

$$c_i(e_j) = \begin{cases} c_i^0(e_j) + p & \text{if } j \in N(t^0 \cap X), \\ c_i^0(e_j) - p & \text{if } j \in N(t \cap X). \end{cases}$$

Thus, we get

$$S_P(t^0, X, p) \leq \max_{t\in T} \min_{i\in N_m} \frac{f_i(C^0,t^0) - f_i(C^0,t) + p\Delta(t,t^0,X)}{f_i(C^0,t) - p\min\{|t\cap X|, k_i\}}.$$

Now it remains to prove that

$$S_P(t^0, X, p) \geq \max_{t\in T} \min_{i\in N_m} \frac{f_i(C^0,t^0) - f_i(C^0,t) + p\Delta(t,t^0,X)}{f_i(C^0,t) - p\min\{|t\cap X|, k_i\}}.$$

Consider a matrix $C^* = \{c_i^*(e_j)\} \in \mathbf{R}^{m\times n}$ with elements defined for any index $i \in N_m$ as follows:

$$c_i^*(e_j) = \begin{cases} c_i^0(e_j) + p & \text{if } j \in N(t^0 \cap X), \\ c_i^0(e_j) - p & \text{otherwise.} \end{cases}$$

Then

$$\max_{t\in T} \min_{i\in N_m} \frac{f_i(C^*,t^0) - f_i(C^*,t)}{f_i(C^*,t)} = \max_{t\in T} \min_{i\in N_m} \frac{f_i(C^0,t^0) - f_i(C^0,t) + p\Delta(t,t^0,X)}{f_i(C^0,t) - p\min\{|t\cap X|, k_i\}}.$$

So, we have that

$$S_P(t^0, X, p) \geq \max_{t\in T} \min_{i\in N_m} \frac{f_i(C^0,t^0) - f_i(C^0,t) + p\Delta(t,t^0,X)}{f_i(C^0,t) - p\min\{|t\cap X|, k_i\}}.$$

■

Observe that t^0 is a Pareto optimal solution of $Z_P^m(C^0)$ if and only if $S_P(t^0, X, p) = A_P(t^0, X, \delta) = 0$. So, it is of special interest to know the largest values of p and δ, for which $S_P(t^0, X, p) = 0$ and $A_P(t^0, X, \delta) = 0$, respectively. Therefore, for any arbitrary set of non-stable elements X we will introduce the stability radius $R_P^S(t^0, X)$ and the accuracy radius $R_P^A(t^0, X)$ in the following way:

$$R_P^S(t^0, X) = \sup\{p \in [0, q(C^0, X)) : \ S_P(t^0, X, p) = 0\},$$

$$R_P^A(t^0, X) = \sup\{\delta \in [0, 1) : \ A_P(t^0, X, \delta) = 0\}.$$

THEOREM 2.2 *For an optimal solution $t^0 \in P^m(C^0)$ and an arbitrary set X of non-stable elements,*

$$R_P^S(t^0, X) = \min\left\{q(C^0, X), \min_{t \in T \setminus \{t^0\}} \max_{i \in N_m} \frac{f_i(C^0, t) - f_i(C^0, t^0)}{\Delta(t, t^0, X)}\right\}, \tag{6}$$

and

$$R_P^A(t^0, X) = \min\left\{1, \min_{t \in T_\alpha} \max_{i \in N_m} \frac{f_i(C^0, t) - f_i(C^0, t^0)}{f_i(C^0, (t \otimes t^0) \cap X)}\right\}, \tag{7}$$

where $T_\alpha = \{t \in T : \ f_i(C^0, (t \otimes t^0) \cap X) \neq 0$ for all $i \in N_m\}$.

Proof. We will prove only (6). The proof of (7) is analogous. If $p = 0$, then $S_P(t^0, X, 0) = 0$. Let $S_P(t^0, X, p) > 0$. This inequality holds if and only if

$$\max_{t \in T} \min_{i \in N_m} \frac{f_i(C^0, t^0) - f_i(C^0, t) + p\Delta(t, t^0, X)}{f_i(C^0, t) - p \min\{|t \cap X|, k_i\}} > 0.$$

But the latter means that

$$p > \bar{p} = \min_{t \in T \setminus \{t^0\}} \max_{i \in N_m} \frac{f_i(C^0, t) - f_i(C^0, t^0)}{\Delta(t, t^0, X)}.$$

Thus, if $\bar{p} \leq q(C^0, X)$, then we get that $S_P(t^0, X, p) = 0$ on interval $[0, \bar{p})$. Otherwise, stability function is equal to zero on $[0, q(C^0, X))$. ■

3. Stability and accuracy functions of lexicographically optimal solution

The lexicographic optimality principle is widely spread in optimization (see e.g. Ehrgott, 1997, Ehrgott and Gandiebleux, 2000). This principle is used, for example, for solving stochastic programming problems and to define a priority in complex systems which consist of different sublevels. Observe that any scalar constrained optimization problem may be transformed to unconstrained bicriteria lexicographic problem by using as first criterion some exact penalty function for problem constrains, and an original objective function as second criterion.

In this section we will consider a variant of lexicographic optimization with respect to all permutations of partial criteria.

Let S_m be the set of all permutations of N_m. For $s = (s_1, s_2, ..., s_m) \in S_m$, the binary relation $\prec_s$ of a lexicographic order is defined as follows: $t \prec_s t'$ if and only if $f(C,t) = f(C,t')$ or there exists an index $j \in N_m$ such that for all $k \in N_{j-1}$ we have $f_{s_j}(C,t) < f_{s_j}(C,t')$ and $f_{s_k}(C,t) = f_{s_k}(C,t')$. Here $N_0 = \emptyset$ for $j = 1$.

Under the vector (m-criteria) combinatorial optimization problem $Z_L^m(C)$ we understand the problem of finding the lexicographic set $L^m(C)$ defined in the following way:

$$L^m(C) = \bigcup_{s \in S_m} L^m(C,s),$$

where

$$L^m(C,s) = \{t \in T : \ t \prec_s t' \ \ \forall\, t' \in T\}.$$

The elements of the set $L^m(C)$ are called lexicographic optima of the problem $Z_L^m(C)$. It is easy to see that any lexicographic optimum belongs to the Pareto set.

For a given matrix C, we will measure the quality of $t^0 \in L^m(C^0)$ by the value of the relative error $\varepsilon_L(C,t^0)$, which is introduced as follows:

$$\varepsilon_L(C,t^0) = \min_{i \in N_m} \max_{t \in T} \frac{f_i(C,t^0) - f_i(C,t)}{f_i(C,t)}.$$

If $t^0 \in L^m(C)$ for any instance of problem $Z_L^m(C)$, then $\varepsilon_L(C,t^0) = 0$. If t^0 looses lexicographic optimality in an $Z_L^m(C)$, then the relative error $\varepsilon_L(C,t^0) > 0$ characterizes the quality of t^0.

For a lexicographical optimal solution $t^0 \in L^m(C^0)$, an arbitrary set of non-stable elements X and $p \in [0, q(C^0,X))$, the value of the stability function is defined as follows:

$$S_L(t^0,X,p) = \max_{C \in \Omega_p(C^0,X)} \varepsilon_L(C,t^0).$$

Similarly, for a lexicographical optimal solution $t^0 \in P^m(C^0)$, an arbitrary set of non-stable elements X and $\delta \in [0,1)$, the value of the accuracy function is defined as follows:

$$A_L(t^0,X,\delta) = \max_{C \in \Theta_\delta(C^0,X)} \varepsilon_L(C,t^0).$$

The two subsequent theorems provide the formulae for calculating the values of stability and accuracy functions and corresponding radii in lexicographic case. We will omit their proofs because they are similar to the Pareto case.

THEOREM 3.1 *For a lexicographical optimal solution $t^0 \in L^m(C^0)$, an arbitrary set X of non-stable elements, and $p \in [0, q(C^0,X))$,*

$$S_L(t^0,X,p) = \min_{i \in N_m} \max_{t \in T} \frac{f_i(C^0,t^0) - f_i(C^0,t) + p\Delta(t,t',X)}{f_i(C^0,t) - p\ \min\{|t \cap X|, k_i\}}.$$

For a lexicographical optimal solution $t^0 \in L^m(C^0)$, an arbitrary set X of non-stable elements, and $\delta \in [0,1)$,

$$A_L(t^0, X, \delta) = \min_{i \in N_m} \max_{t \in T} \frac{f_i(C^0, t^0) - f_i(C^0, t) + \delta f_i(C^0, (t \otimes t^0) \cap X)}{f_i(C^0, t) - \delta f_i(C^0, t \cap X)}.$$

By analogy, for an arbitrary set of non-stable elements X, we define the stability radius and the accuracy radius as follows:

$$R_L^S(t^0, X) = \sup \left\{ p \in [0, q(C^0, X)) : \ S_L(t^0, X, p) = 0 \right\},$$

$$R_L^A(t^0, X) = \sup \left\{ \delta \in [0,1) : \ A_L(t^0, X, \delta) = 0 \right\}.$$

THEOREM 3.2 *For a lexicographical optimal solution $t^0 \in L^m(C^0)$ and an arbitrary set of non-stable elements X*

$$R_L^S(t^0, X) = \min \left\{ q(C^0, X), \max_{i \in N_m} \min_{t \in T \setminus \{t^0\}} \frac{f_i(C^0, t) - f_i(C^0, t^0)}{\Delta(t, t^0, X)} \right\},$$

$$R_L^A(t^0, X) = \min \left\{ 1, \max_{i \in N_m} \min_{t \in T_\alpha} \frac{f_i(C^0, t) - f_i(C^0, t^0)}{f_i(C^0, (t \otimes t^0) \cap X)} \right\}.$$

4. Examples

Consider the vector traveling salesman problem defined on graph $G = K_4$. Let the ground set E be equal to the set of all edges of G, i.e., $E = \{e_1, e_2, ..., e_6\}$. The set of feasible solutions T represents a family of all subsets of edges which form the Hamiltonian cycles in the graph G. There are only three such subsets (see Fig. 1), thus we have $T = \{t_1, t_2, t_3\}$, where $t_1 = \{e_1, e_2, e_5, e_6\}$, $t_2 = \{e_1, e_3, e_4, e_6\}$, $t_3 = \{e_2, e_3, e_4, e_5\}$.

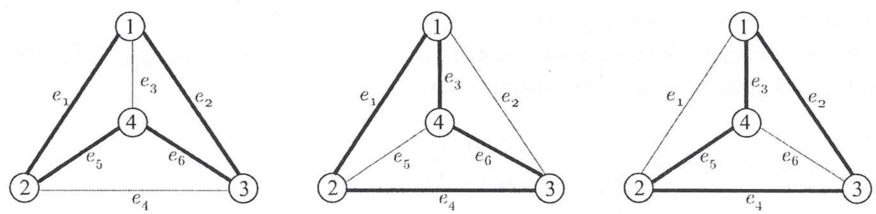

Figure 1. All Hamiltonian cycles in graph K_4

We will consider 2-criteria optimization problem with the initial matrix of weights

$$C^0 = \begin{bmatrix} 2 & 1 & 2 & 3 & 1 & 2 \\ 1 & 3 & 1 & 1 & 2 & 2 \end{bmatrix}.$$

Assume parameters $k_i = 3,\ i = 1, 2$. It means that we calculate the value of objective function with respect to all the possible Hamiltonian paths in a given Hamiltonian cycle. Let our partial criteria, which we want to minimize, have the form:

$$f_1(C,t) = \max\left\{\sum_{e\in q} c_1(e) : q \subseteq t,\ |q| = 3\right\}.$$

$$f_2(C,t) = \min\left\{\sum_{e\in q} c_2(e) : q \subseteq t,\ |q| = 3\right\}.$$

Then $f(C^0, t_1) = (5, 5),\ f(C^0, t_2) = (7, 3),\ f(C^0, t_3) = (6, 4),\ P^2(C^0) = \{t_1, t_2, t_3\}$. Let all elements of E be non-stable, i.e. $X = E$. By Theorem 2.1, we calculate that

$$S_P(t_1, E, p) = \max\{0, \frac{6p-1}{6-3p}, \frac{6p-2}{7-3p}\} = \begin{cases} 0 & \text{if } p \in [0, \frac{1}{6}], \\ \frac{6p-1}{6-3p} & \text{if } p \in (\frac{1}{6}, 1), \end{cases}$$

$$S_P(t_2, E, p) = \max\{0, \frac{6p-2}{5-3p}, \min\{\frac{6p+1}{6-3p}, \frac{6p-1}{4-3p}\}\} = \begin{cases} 0 & \text{if } p \in [0, \frac{1}{6}], \\ \frac{6p+1}{6-3p} & \text{if } p \in (\frac{1}{6}, \frac{5}{9}], \\ \frac{6p-1}{4-3p} & \text{if } p \in (\frac{5}{9}, 1), \end{cases}$$

$$S_P(t_3, E, p) = \max\{0, \frac{6p-1}{5-3p}, \frac{6p-1}{7-3p}\} = \begin{cases} 0 & \text{if } p \in [0, \frac{1}{6}], \\ \frac{6p-1}{5-3p} & \text{if } p \in (\frac{1}{6}, 1). \end{cases}$$

Observe that for any solution t_1, t_2, t_3, the stability radius is equal to $\frac{1}{6}$. But for instance, t_1 is 'better' than t_2 and t_3, since $S_P(t_1, E, p) \leq S_P(t_2, E, p)$ and $S_P(t_1, E, p) \leq S_P(t_3, E, p)$ for all $p \in [0, 1)$, with strict inequalities on some subinterval of $[0, 1)$ (see Fig. 2).

If we consider lexicographic optimality principle, then we get $L^2(C^0) = \{t_1, t_2\}$. By Theorem 3.1, we obtain that

$$S_L(t_1, E, p) = \begin{cases} 0 & \text{if } p \in [0, \frac{1}{6}], \\ \frac{6p-1}{6-3p} & \text{if } p \in (\frac{1}{6}, 1), \end{cases}$$

$$S_L(t_2, E, p) = \begin{cases} 0 & \text{if } p \in [0, \frac{1}{6}], \\ \frac{6p-1}{4-3p} & \text{if } p \in [\frac{1}{6}, \frac{13}{15}], \\ \frac{6p+2}{5-3p} & \text{if } p \in (\frac{13}{15}, 1). \end{cases}$$

We can see now that $R_L^S(t_1, E) = R_L^S(t_2, E) = 1/6$, whereas for $p \in (1/6, 1)$, we get $S_L(t_1, E, p) < S_L(t_2, E, p)$ (see Fig. 3). Altogether, this implies that t_1 is 'better' than t_2.

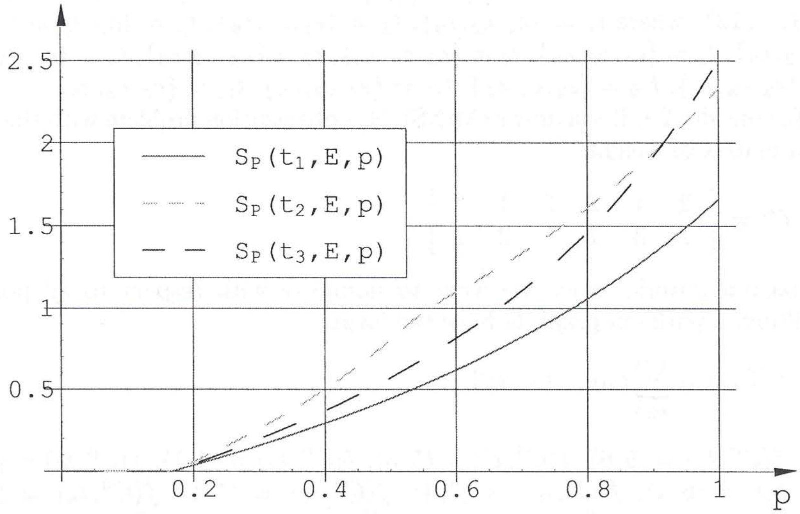

Figure 2. Stability functions $S_P(t_1, E, p)$, $S_P(t_2, E, p)$, $S_P(t_3, E, p)$

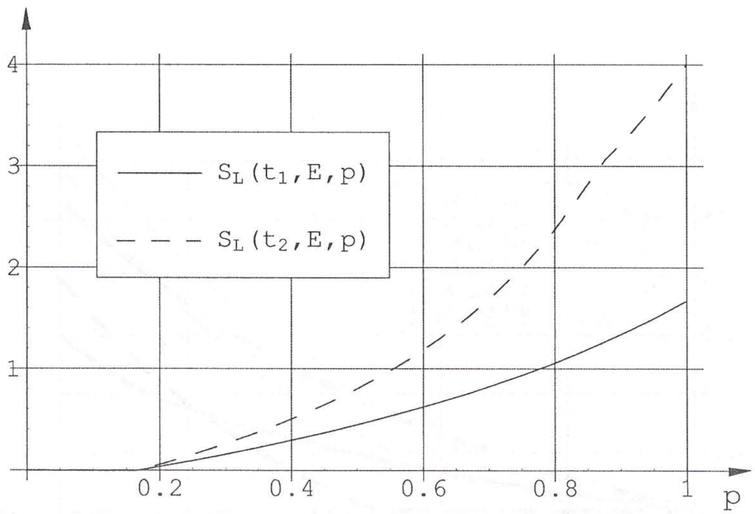

Figure 3. Stability functions $S_L(t_1, E, p)$, $S_L(t_2, E, p)$

Now assume that the set of feasible solutions T represents a family of all subsets of edges which form the Hamiltonian paths in the graph G. Thus $T = \{t_j : j = 1, ..., 12\}$, where $t_1 = \{e_1, e_2, e_6\}$, $t_2 = \{e_1, e_5, e_6\}$, $t_3 = \{e_2, e_6, e_5\}$, $t_4 = \{e_1, e_2, e_5\}$, $t_5 = \{e_1, e_3, e_6\}$, $t_6 = \{e_2, e_4, e_6\}$, $t_7 = \{e_3, e_4, e_6\}$, $t_8 = \{e_1, e_3, e_4\}$, $t_9 = \{e_2, e_3, e_4\}$, $t_{10} = \{e_2, e_4, e_5\}$, $t_{11} = \{e_3, e_4, e_5\}$, $t_{12} = \{e_2, e_3, e_5\}$.

We consider 2-criteria linear (MINSUM) optimization problem with the same initial matrix of weights

$$C^0 = \begin{bmatrix} 2 & 1 & 2 & 3 & 1 & 2 \\ 1 & 3 & 1 & 1 & 2 & 2 \end{bmatrix}.$$

The partial criteria which we want to minimize with respect to all possible Hamiltonian paths in graph G have the form:

$$f_i(C, t) = \sum_{e \in t} c_i(e), \ i = 1, 2.$$

Then $f(C^0, t_1) = (5, 6)$, $f(C^0, t_2) = (5, 5)$, $f(C^0, t_3) = (4, 7)$, $f(C^0, t_4) = (4, 6)$, $f(C^0, t_5) = (6, 4)$, $f(C^0, t_6) = (7, 4)$, $f(C^0, t_7) = (7, 4)$, $f(C^0, t_8) = (7, 3)$, $f(C^0, t_9) = (6, 5)$, $f(C^0, t_{10}) = (5, 6)$, $f(C^0, t_{11}) = (6, 4)$, $f(C^0, t_{12}) = (4, 6)$. In this case we have

$$P^2(C^0) = \{t_2, t_4, t_5, t_8, t_{11}, t_{12}\}.$$

Let all elements of E be non-stable, i.e. $X = E$. Using Theorem 2.1, we constructed the plots for stability functions of Pareto optimal solutions (see Fig. 4).

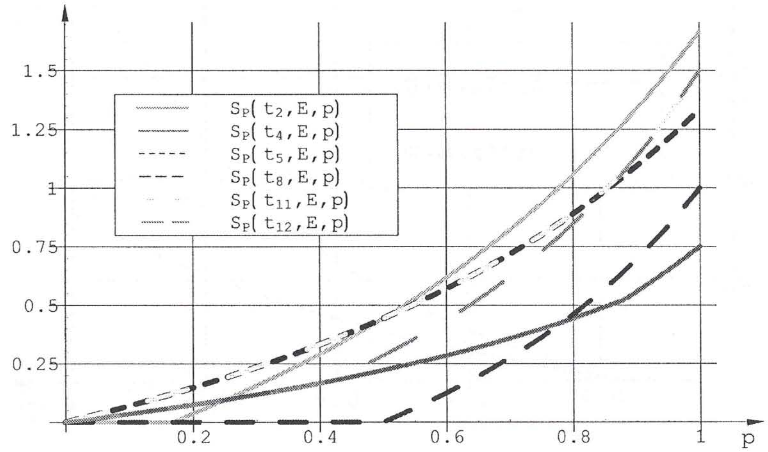

Figure 4. Stability functions $S_P(t_i, E, p)$ *for* $i = 2, 4, 5, 8, 11, 12$

It is easy to see that solution t_8 is 'the most preferable' because it has the largest stability radius equal to $\frac{1}{2}$ and the stability function which dominates almost all other stability functions. Observe that t_8 does not belong to the 'best' Hamiltonian cycle considered earlier. If we continue our analysis we can definitely say that solutions t_4, t_5, t_{11}, t_{12} are non-stable, i.e. they have stability radius equal to 0. But stability radius of t_2 is equal to $\frac{1}{6}$ and it means that t_2 is 'more preferable' (at least inside its stability area) than t_4, t_5, t_{11}, t_{12}.

5. Conclusions

The accuracy and stability functions describe the quality of efficient solution in the situation when coefficients in criteria are subject to uncertainty. The definitions of these functions are directly related to a specific optimality principle. The stability and accuracy radii give us the maximum values of independent perturbations which preserve the efficiency of a given solution.

Examples in previous section suggest that changes or inaccuracies in estimating objective function coefficients may influence significantly the set of efficient solutions of multicriteria combinatorial optimization problem. Moreover, some initially efficient solutions cannot be considered 'robust', because very small changes of data destroy their efficiency. Therefore, a possibility of ranking initially efficient solutions from the 'robustness' point of view is of special importance for a decision maker.

The simplest measure of the 'robustness' of the efficient solution is its stability radius or the accuracy radius. But frequently these radii are not sufficient to rank the efficient solutions and it is necessary to calculate complementary more general characteristics of solutions like stability and accuracy functions.

The formulae proved in the paper do not lead directly to efficient methods of calculating the values of defined functions and radii. Nevertheless, we see some possibility of extending to multicriteria case results of Libura (1999, 2000) and Libura et al. (1998), which are based on subsets of so-called k-best solutions (Hamacher and Queyranne, 1985/6).

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