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Pole assignment by feedback control of the second order coupled singular distributed parameter systems[1]

by

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Abstract: Pole assignment by feedback control of the second order coupled singular distributed parameter systems is discussed via functional analysis and operator theory in Hilbert space. The solutions of the problem and the constructive expression of the solutions are given by the generalized inverse one of bounded linear operator. This research is theoretically important for studying the stabilization and asymptotical stability of the second order coupled singular distributed parameter systems.

Keywords: feedback control, pole assignment, coupled singular distributed parameter systems, generalized inverse one of operator.

1. Introduction

Singular distributed parameter systems are systems which are much more often encountered than the distributed parameter systems. They appear in the study of the temperature distribution in a composite heat conductor, voltage distribution in electromagnetically coupled superconductive circuits (Ge, 1993a; Joder, 1991; Trzaska, Marszałek, 1993; Yang, Liu, 2000; Yue, Liu, 1996). There is an essential distinction between them and the ordinary distributed parameter systems. When under disturbance, they not only lose stability, but also great changes take place in their structure, such as leading to impulsive behavior etc.

One of the most important research problems is the study of the pole assignment of the singular distributed parameter systems (Ge, 1999, 2000; Ge, Ma, 2000). There have been some papers discussing pole assignment of the first order coupled singular distributed parameter systems (Ge, 2000; Ge, Ma, 2000).

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In mathematical and engineering control systems it is of great importance that the control object is described by the second order singular distributed parameter system while the controller is governed by the singular lumped parameter system. The physical measurement values of the second order singular distributed parameter system are fed to the controller, in which the control signal is produced and transmitted to the actuator. The latter realizes the feedback control for the system. Since the controller is usually described by the singular ordinary differential equation, we must study the pole assignment for the second order singular distributed parameter system coupled with the singular lumped parameter one. When the placement of controller for the second order singular distributed parameter system is known, we choose appropriate placement of the observation for the second order singular distributed parameter system, such that the closed loop system, which is the second order singular distributed parameter system coupled with the singular lumped parameter one, possesses assignable poles. This note deals with the pole assignment by feedback control of the second order singular distributed parameter system coupled with the singular lumped parameter system. The solutions of the problem and the constructive expression of the solutions are given by the generalized inverse one of bounded linear operator.

Let H denote the complex separable Hilbert space, E_0 and A_0 be linear operators in H, A_0 be an invertible and closed densely defined linear operator, E_0 be a bounded one, and g_i, b_0, $y \in H(i = 0,1,2)$, $b_0 \neq 0$. There exists $A_0^{1/2}$. Let R^n denote the n-dimensional Euclidean space, $R^{n\times n}$ denote the set of real matrices. Further, z, g, $k_i \in R^n(i = 0,1,2)$ and $k_j \neq 0(i = 0,1,2)$, E_2, $F \in R^{n\times n}$ and $\det E_2 = 0$. For the systems

$$E_0\ddot{y} = A_0y + ub_0 \quad y(0) = y_0, \quad \dot{y} = y_1 \tag{1}$$

$$E_2\dot{z} = Fz + w, \quad z(0) = z_0 \tag{2}$$

if $u =< z, g >$ and $w =< E_0\ddot{y}, g_2 > k_2+ < E_0\dot{y}, g_1 > k_1+ < E_0y, g_0 > k_0$ are the feedback controls, where $< \cdot,\cdot >$ denotes the inner product, then (1) and (2) become

$$E_0\ddot{y} = A_0y+ < z, g > b_0 \quad y(0) = y_0, \quad \dot{y}(0) = y_1 \tag{3}$$

$$E_2\dot{z} = Fz+ < E_0\ddot{y}, g_2 > k_2+ < E_0\dot{y}, g_1 > k_1+ < E_0y, g_0 > k_0, \quad z(0) = z_0 \tag{4}$$

Let $G_0z =< z, g > b_0$, $G_{0i}y =< E_0y, g_i > k_i(i = 0,1,2)$, then the expressions of (3) and (4) become

$$\begin{cases} E_0\ddot{y} = A_0y + G_0z, \ y(0) = y_0, \ \dot{y}(0) = y_1 \\ E_2\dot{z} = Fz + G_{02}\ddot{y} + G_{01}\dot{y} + G_{00}y, \quad z(0) = z_0 \end{cases} \tag{5}$$

The problem of pole assignment for (1) and (2) is whether there exist $g_i \in H(i = 0,1,2)$ for an arbitrary set $\{\alpha_i\}_1^N$ of N complex numbers such that the closed-loop second order coupled singular distributed parameter system (5) possesses the poles $\{\alpha_i\}_1^N$.

Let $E_1 = \begin{bmatrix} I & 0 \\ 0 & E_0 \end{bmatrix}$, $A = \begin{bmatrix} 0 & A^{1/2} \\ A_0^{1/2} & 0 \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, where $v_1 = A_0^{1/2}y$, $v_2 = \dot{y}$. From (5) we obtain

$$\begin{cases} E_1\dot{v} = Av + Gz, \\ E_2\dot{z} = Fz + G_1\dot{v} + G_2 v \end{cases} \tag{6}$$

where $Gz =< z, g > \begin{bmatrix} 0 \\ b_0 \end{bmatrix} =< z, g > b$, $G_2 v = G_{00}A_0^{-1/2}v_1 + G_{01}v_2$, $G_1 v = G_{02}v_2$, and $b = \begin{bmatrix} 0 \\ b_0 \end{bmatrix}$.

Let $B_0 = \begin{bmatrix} E_1 & 0 \\ -G_1 & E_2 \end{bmatrix}$, $T_0 = \begin{bmatrix} A & G \\ G_2 & F \end{bmatrix}$, and $\omega = \begin{bmatrix} v \\ z \end{bmatrix}$. From (6) we have

$$B_0\dot{\omega} = T_0\omega, \quad \omega(0) = \omega_0 \tag{7}$$

The generalized eigenvalue problems of (5) and (7) can be written, respectively, as follows:

$$\begin{cases} \lambda^2 E_0 y = A_0 y + G_0 z \\ \lambda E_2 z = Fz + \lambda^2 G_{02}y + \lambda G_{01}y + G_{00}y \end{cases} \tag{8}$$

and

$$\lambda B_0\omega = T_0\omega \tag{9}$$

The following result can be proved directly:

Lemma 1.1 *Let A_0 be an invertible linear operator and let E_0 be a bounded linear operator. If (λ, y, z) is a solution of (8), $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} A^{1/2}y \\ \lambda y \end{bmatrix}$, $\omega = \begin{bmatrix} v \\ z \end{bmatrix}$, then (λ, ω) is a solution of (9). Conversely, if (λ_0, ω_0) is a solution of (9), where $\omega_0 = \begin{bmatrix} v_0 \\ z_0 \end{bmatrix}$, $v_0 = \begin{bmatrix} v_{01} \\ v_{02} \end{bmatrix}$, then $(\lambda_0, A_0^{-1/2}v_{01}, z_0)$ is a solution of (8).*

According to Lemma 1.1, the problem of pole assignment for (1) and (2) becomes whether there exist $g_i \in H(i = 0, 1, 2)$ for an arbitrary set $\{\alpha_i\}_1^N$ of N complex numbers such that the closed-loop singular system (7) possesses the poles $\{\alpha_i\}_1^N$.

In this note, the constructive expressions of $g_i(i = 1, 2)$ are given via the generalized inverse one of bounded linear operator in Hilbert space.

2. Preliminaries

In the following, E_0^* denotes the adjoint operator of E_0, $\sigma_p(E_0, A_0) = \{\lambda : \lambda$ is a generalized eigenvalue of E_0 and $A_0\}$ denotes the finite generalized point spectrum of E_0 and A_0, i.e. the finite poles of system (1); $\rho(E_1, A) = \{\alpha : (\alpha E_1 - A)$ is a regular operator$\}$; $R(\alpha E_1, A) = (\alpha E_1 - A)^{-1}$ denotes the inverse operator of $(\alpha E_1 - A)$ for $\alpha \in \rho(E_1, A)$; I denotes the identity operator.

DEFINITION 2.1 *(Ge, 1993b) Let $B(H)$ denote the Banach algebra of all bounded linear operators on H and $B \in B(H)$. If there exists $B^+ \in B(H)$ such that $BB^+B = B$, $B^+BB^+ = B^+$, $(B^+B)^* = B^+B$, and $(BB^+)^* = BB^+$, then B^+ is called the generalized inverse one of B.*

LEMMA 2.1 *If there exists B^+, then (i) B^+ is unique; (ii) there exists $(B^*)^+$, and $(B^*)^+ = (B^+)^*$.*

Proof. (i) If G_1 and G_2 are two operators satisfying

$$BG_1B = B,\ G_1BG_1 = G_1,\ (G_1B)^* = G_1B,\ (BG_1)^* = BG_1$$

and

$$BG_2B = B,\ G_2BG_2 = G_2,\ (G_2B)^* = G_2B,\ (BG_2)^* = BG_2,$$

then

$$\begin{aligned} G_1 &= G_1BG_1 = G_1(BG_1)^* = G_1G_1^*B^* = G_1G_1^*(BG_2B)^* \\ &= G_1G_1^*B^*G_2^*B^* = G_1(BG_1)^*(BG_2)^* = G_1BG_1BG_2 \\ &= G_1BG_2 = (G_1B)^*(G_2BG_2) = (G_1B)^*(G_2B)^*G_2 \\ &= (G_2BG_1B)^*G_2 = (G_2B)^*G_2 = G_2BG_2 = G_2. \end{aligned}$$

Therefore $G_1 = G_2 = B^+$, i.e. (i) holds.

(ii) Since $BB^+B = B$, $B^+BB^+ = B^+$, $(B^+B)^* = B^+B$, and $(BB^+)^* = BB^+$, by taking the adjoint of both sides of each equation we obtain

$$\begin{aligned} B^*(B^+)^*B^* = B^*, (B^+)^*B^*(B^+)^* = (B^+)^*, [(B^+)^*B^*]^* \\ = (B^+)^*B^*,\ [B^*(B^+)^*]^* = B^*(B^+)^*. \end{aligned}$$

Using the Definition 2.1 and (i) of Lemma 2.1, we obtain that (ii) holds. ■

LEMMA 2.2 *Let A_0 be an invertible linear operator, E_0 be bounded, E_0 and A_0 only have the finite generalized point spectrum, and $E_0A_0^{1/2} = A_0^{1/2}E_0$. If $B_F = \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix}$, $R = R(\lambda^2 E_0, A_0) = (\lambda^2 E_0 - A_0)^{-1}$, and $T = \begin{bmatrix} 0 & G \\ G_2 & 0 \end{bmatrix}$, then*

(i) $\lambda \in \rho(E_1, A)$ *if and only if* $\lambda^2 \in \rho(E_0, A_0)$ *and*

$$R(\lambda E_1, A) = (\lambda E_1 - A)^{-1} = \begin{bmatrix} \lambda E_0 R & A_0^{1/2} R \\ A_0^{1/2} R & \lambda R \end{bmatrix};$$

(ii) $\lambda \in \rho(B_0, B_F)$ *if and only if* $\lambda \in \rho(E_1, A) \cap \rho(E_2, F)$.

Lemma 2.2 can be proved directly by the reference to Halmos (1982).

For $\lambda \in \rho(B_0, B_F)$, let

$$\begin{aligned}\alpha &= \lambda^2 < E_0 R b_0, g_2 > R(\lambda E_2, F) k_2 + \lambda < E_0 R b_0, g_1 > R(\lambda E_2, F) k_1 \\ &\quad + < E_0 R b_0, g_0 > R(\lambda E_2, F) k_0\end{aligned}$$

and $\omega(\lambda) =< \alpha, g >$. Then we have the following lemma:

LEMMA 2.3 *Let A_0 be an invertible and closed densely defined linear operator, E_0 and A_0 only have the finite generalized point spectrum, and $A_0^{1/2} E_0 = E_0 A_0^{1/2}$. If $\lambda \in \rho(B_0, B_F)$, then $\lambda \in \sigma_p(B_0, B_F)$ if and only if*

$$\omega(\lambda) = 1 \tag{10}$$

and $w_1 = \begin{bmatrix} R(\lambda E_1, A) b \\ \alpha \end{bmatrix}$ is an associated generalized eigenvector.

Proof. Let $\lambda \in \rho(B_0, B_F)$. If (10) is false, then $\lambda \in \rho(B_0, T_0)$. In fact, since

$$(\lambda B_0 - T_0)^* = \begin{bmatrix} \bar{\lambda} E_1^* - A^* & -G_2^* - \bar{\lambda} G_1^* \\ -G^* & \bar{\lambda} E_2^* - F^* \end{bmatrix} \tag{11}$$

for any element $\psi_1 \in H \times H$, $\psi_2 \in R^n$, $\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$, and $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, let

$$(\lambda B_0 - T_0)^* y = \psi. \tag{12}$$

From (11) and (12) we obtain

$$\left\{\begin{array}{ll} (\bar{\lambda} E_1^* - A^*) y_1 - (\bar{\lambda} G_0^* + G_2^*) y_2 = \psi_1, & (13) \\ -G^* y_1 + (\bar{\lambda} E_2^* - F^*) y_2 = \psi_2. & (14) \end{array}\right.$$

Since $\lambda \in \rho(B_0, B_F)$, from (14) we have

$$y_2 = R^*(\lambda E_2, F) \psi_2 + < y_1, b > R^*(\lambda E_2, F) g. \tag{15}$$

Using (13) and (15), we deduce

$$y_1 = h + < y_1, b > h_1 \tag{16}$$

where

$$\begin{aligned}h &= R^*(\lambda E_1, A) \psi_1 + R^*(\lambda E_1, A)(\bar{\lambda} G_1^* + G_2^*) R^*(\lambda E_2, F) \psi_2, \\ h_1 &= R^*(\lambda E_1, A)(\bar{\lambda} G_1^* + G_2^*) R^*(\lambda E_2, F) g.\end{aligned}$$

Thus

$$< y_1, b >=< h, b > + < y_1, b >< h_1, b > \tag{17}$$

and

$$\begin{aligned} < h_1, b > &= \bar{\lambda} < R^*(\lambda E_1, A) G_1^* R^*(\lambda E_2, F) g, b > \\ &\quad + < R^*(\lambda E_1, A) G_2^* R^*(\lambda E_2, F) g, b > \\ &= \bar{\lambda} < R^*(\lambda E_2, F) g, G_1 R(\lambda E_1, A) b > \\ &\quad + < R^*(\lambda E_2, F) g, G_2 R(\lambda E_1, A) b > . \end{aligned}$$

From Lemma 2.2 we have

$$G_1 R(\lambda E_1, A) b = G_1 \begin{bmatrix} \lambda E_0 R & A_0^{1/2} R \\ A^{1/2} R & \lambda R \end{bmatrix} \begin{bmatrix} 0 \\ b_0 \end{bmatrix} = \lambda < E_0 R b_0, g_2 > k_2,$$

$$G_2 R(\lambda E_1, A) b = < E_0 R b_0, g_0 > k_0 + \lambda < E_0 R b_0, g_1 > k_1.$$

Therefore

$$\begin{aligned} < h_1, b > &= (\bar{\lambda})^2 \overline{< E_0 R b_0, g_0 >} \cdot \overline{< R(\lambda E_2, F) k_2, g >} \\ &\quad + \bar{\lambda} \overline{< E_0 R b_0, g_1 >< R(\lambda E_2, F) k_1, g >} \\ &\quad + \overline{< E_0 R b_0, g_0 >< R(\lambda E_2, F) k_0, g >} = \overline{\omega(\lambda)}. \end{aligned}$$

Hence, (17) can be written as follows

$$< y_1, b > = \frac{< h, b >}{1 - \overline{\omega(\lambda)}}. \tag{18}$$

Using (15), (16) and (18) we obtain

$$y_1 = h + \frac{< h, b >}{1 - \overline{\omega(\lambda)}} h_1 \tag{19}$$

$$y_2 = R^*(\lambda E_2, F) \psi_2 + \frac{< h, b >}{1 - \overline{\omega(\lambda)}} R^*(\lambda E_2, F) g \tag{20}$$

From (19), (20) and the representation formulae of h and h_1, it is obvious that y_1 and y_2 are continuous at any element ψ. Therefore, the operator $[(\lambda B_0 - T_0)^*]^{-1}$ satisfying $y = [(\lambda B_0 - T_0)^*]^{-1} \psi$ is a bounded linear operator. Thus $(\lambda B_0 - T_0)^{-1}$ is a regular operator.

If $\lambda \in \rho(B_0, B_F)$ and $\omega(\lambda) = 1$, we need to prove that λ is a generalized eigenvalue of B_0 and T_0, and the associated generalized eigenvector is $w_1 = \begin{bmatrix} R(\lambda E_1, A) b \\ \alpha \end{bmatrix}$. In fact, let w_0 satisfy

$$(\lambda B_0 - T_0) w_0 = (\lambda B_0 - B_F - T) w_0 = (\lambda B_0 - B_F)[I - R(\lambda B_0, B_F) T] w_0 = 0.$$

From $\lambda \in \rho(B_0, B_F)$, it is obvious that $\lambda \in \sigma_p(B_0, T_0)$ if and only if

$$R(\lambda B_0, B_F) T w_0 = w_0 = \begin{bmatrix} w_{01} \\ w_{02} \end{bmatrix}.$$

Since

$$R(\lambda B_0, B_F)Tw_0 = \left[\begin{array}{c} R(\lambda E_1, A)Gw_{02} \\ \lambda R(\lambda E_2, F)G_1 R(\lambda E_1, A)Gw_{02} + R(\lambda E_2, F)G_2 w_{01} \end{array} \right],$$

$$R(\lambda E_1, A)Gw_{02} =< w_{02}, g > R(\lambda E_1, A)b = w_{01},$$

and

$$\begin{aligned} &\lambda R(\lambda E_2, F)G_1 R(\lambda E_1, A)Gw_{02} + R(\lambda E_2, F)G_2 w_{01} \\ &=< w_{02}, g > [\lambda R(\lambda E_2, F)G_1 R(\lambda E_1, A)b + R(\lambda E_2, F)G_2 R(\lambda E_1, A)b] \\ &=< w_{02}, g > [\lambda^2 < E_0 Rb_0, g_2 > R(\lambda E_2, F)k_2 \\ &\quad + \lambda < E_0 Rb_0, g_1 > R(\lambda E_2, F)k_1 + < E_0 Rb_0, g_0 > R(\lambda E_2, F)k_0] \\ &=< w_{02}, g > \alpha = w_{02}, \end{aligned}$$

we have $\lambda \in \sigma_p(B_0, T_0)$, and the associated generalized eigenvector is w_1. ■

LEMMA 2.4 *(Wang, 1982) Let $x_i \in H$ and $x_i \neq 0$ $(i = 1, 2, \cdots, N)$, $y_{N+k} \in H$ and $y_{N+k} \neq 0$, $k = 1, 2, \cdots$, and H_i^{N-1} denote the closed linear subspace generated by $\{x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N,\ y_{N+1}, y_{N+2}, \cdots\}$. If $x_i \notin H_i^{N-1}$, then there exists $g_1 \in H$ such that $< x_i, g_1 >= 1 (i = 1, 2, \cdots, N)$, and*

$$< y_k, g_1 >= 0 \quad (k = N + 1, N + 2, \cdots).$$

ASSUMPTION $(\mathbf{G_1})$ Let A_0 be an invertible and closed densely defined linear operator, and let E_0 be a bounded linear operator. There exists $A_0^{1/2}$ and $E_0 A_0^{1/2} = A_0^{1/2} E_0$. E_0 and A_0 have only the finite generalized point spectrum. Let $\{\lambda_k\}_1^\infty$ be the set of all finite generalized points of the spectrum of E_0 and A_0, and any λ_k be a single, and φ_k be the associated generalized eigenvector, i.e.

$$A_0 \varphi_k = \lambda_k E_0 \varphi_k \quad (k = 1, 2, \cdots).$$

Let $\{\psi_k\}_1^\infty$ denote the set of all generalized eigenvectors of E_0^* and A_0^* satisfying

$$A_0^* \psi_k = \bar{\lambda}_k E_0^* \psi_k \quad (k = 1, 2, \cdots),$$

and there exist the following relations between $\{E\varphi_k\}_1^\infty$ and $\{\psi_k\}_1^\infty$:

$$< E_0 \varphi_k, \psi_l >= \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases} \quad (k, l = 1, 2, \cdots).$$

ASSUMPTION $(\mathbf{G_2})$ Let $E_2, F \in R^{n \times n}$, F_2 and F have only the finite generalized point spectrum $\{r_k\}_1^{n_0} (n_0 < n)$, every r_k be a single, and u_k be the associated generalized eigenvector, i.e. $Fu_k = r_k E_2 u_k (k = 1, 2, \cdots, n_0)$. For the generalized eigenvalue $\bar{r}_k$ of E_2^* and F^*, the associated generalized eigenvector is v_k, i.e. $F^* v_k = \bar{r}_k E_2^* v_k (k = 1, 2, \cdots, n_0)$, and there exist the following relations between $\{E_2 u_k\}_1^{n_0}$ and $\{v_k\}_1^{n_0}$:

$$< E_2 u_k, v_l >= \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases} \qquad (k = 1, 2, \cdots, n_0).$$

3. Main result and proof

THEOREM 3.1 *Suppose E_0 and A_0 satisfy the assumption (G_1), E_2 and F satisfy the assumption (G_2), and there exists E_0^+. Let $\{\alpha_i\}_1^N$ be an arbitrary set of N complex numbers satisfying $\alpha_i \neq \alpha_j (i \neq j;\ i, j = 1, 2, \cdots, N)$, and $\alpha_i \notin \sigma_p(E_1, A) \cup \sigma_p(E_2, F) (i = 1, 2, \cdots, N)$. If*

$$d_{j\mu} = \alpha_j^\mu < R(\alpha_j E_2, F) k_\mu, g >\neq 0 \quad (j = 1, 2, \cdots, N; \mu = 0, 1, 2),$$

then there exist $g_\mu \in H (\mu = 0, 1, 2)$ such that $\{\alpha_i\}_1^N \cup \{\sqrt{\lambda_k}\}_{N+1}^\infty \subset \sigma_p(B_0, T_0)$, and

$$g_\mu = \sum_{j=1}^N g_j^{(\mu)} \psi_j + [I - (E_0^*)^+ E_0^*] a, \quad (\mu = 0, 1, 2,),$$

where

$$\overline{g_j^{(\mu)}} = \frac{\alpha_j^2 - \lambda_j}{3 b_j} \prod_{\substack{k=1 \\ k \neq j}}^N \Big(\frac{\alpha_k^2 - \lambda_j}{\lambda_k - \lambda_j} \Big) \cdot \sum_{k=1}^N \frac{\alpha_k^2 - \lambda_k}{d_{k\mu} (\alpha_k^2 - \lambda_j)} \cdot \prod_{\substack{i=1 \\ i \neq k}}^N \Big(\frac{\alpha_k^2 - \lambda_i}{\alpha_k^2 - \alpha_i^2} \Big),$$
$$j = 1, 2, \cdots, N; \mu = 0, 1, 2,$$

and a is any element in H.

Proof. Let $x_{\mu i} = \alpha_i^\mu < R(\alpha_i E_2, F) k_0, g > E_0 R(\alpha_i^2 E_0, A_0) b_0$, $i = 1, 2, \cdots, N$; $\mu = 0, 1, 2$;

$$y_{k+N} = E_0 \varphi_{k+N} \quad (k = 1, 2, \cdots)$$

and $H_{i\mu}^{N-1}$ denote the closed linear subspace generated by

$$\{x_{\mu 1}, x_{\mu 2}, \cdots, x_{\mu i+1}, x_{\mu i+2}, \cdots, x_N, y_{N+1}, y_{N+2}, \cdots\}.$$

Then $x_{\mu i} \notin H_{i\mu}^{N-1} (i = 1, 2, \cdots, N)$.

In fact, if $x_{\mu i} \in H_{i\mu}^{N-1} (i = 1, 2, \cdots, N)$, then there exist

$$\beta_{\mu 1}, \beta_{\mu 2}, \cdots, \beta_{\mu i-1}, \beta_{\mu i+1}, \cdots, \beta_{\mu N}, \beta_{\mu N+1}, \beta_{\mu N+2}, \cdots$$

such that

$$d_{i\mu} E_0 R(\alpha_i^2 E_0, A_0) b_0 = \sum_{\substack{j=1 \\ j \neq i}}^N \beta_{\mu j} d_{j\mu} E_0 R(\alpha_j^2 E_0, A_0) b_0 + \sum_{k=1}^\infty \beta_{\mu k+N} E_0 \varphi_{k+N}.$$

Thus

$$d_{i\mu} < E_0R(\alpha_i^2 E_0, A_0)b_0, \psi_l >= \sum_{\substack{j=1\\ j\neq i}}^{N} \beta_{\mu j} d_{j\mu} < E_0R(\alpha_j^2 E_0, A_0)b_0, \psi_l >, \quad l = 1, 2, \cdots, N; \mu = 0, 1, 2. \tag{21}$$

Since

$$(\overline{\alpha_j^2} - \overline{\lambda_l})E_0^*\psi_l = \overline{\alpha_j^2}E_0^*\psi_l - \bar{\lambda}_l E_0^*\psi_l = (\overline{\alpha_j^2}E_0^* - A_0^*)\psi_l (l = 1, 2, \cdots, N),$$

$$E_0^*\psi_l = \frac{1}{\overline{\alpha_j^2} - \overline{\lambda}_l}(\overline{\alpha_j^2}E_0^* - A_0^*)\psi_l (l = 1, 2, \cdots, N),$$

from (21), we obtain

$$\frac{d_{i\mu}b_l}{\alpha_i^2 - \lambda_l} = \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{\beta_{\mu j} d_{j\mu} b_l}{\alpha_i^2 - \lambda_l} (l = 1, 2, \cdots, N; \mu = 0, 1, 2). \tag{22}$$

Let $b_{lj} = \dfrac{1}{\alpha_j^2 - \lambda_l}(j, l = 1, 2, \cdots, N)$ and $D_N = \lfloor b_{lj} \rfloor_{N\times N}$. Then

$$\det D_N = (-1)^{\frac{N(N-1)}{2}} \frac{\prod\limits_{1\le i<j\le N} (\alpha_i^2 - \alpha_j^2) \prod\limits_{1\le i<j\le N} (\lambda_i - \lambda_j)}{\prod\limits_{i=1}^{N}\prod\limits_{j=1}^{N} (\alpha_i^2 - \lambda_j)}.$$

Thus, $\det D_N \neq 0$ by the given assumptions. Therefore, (22) has no solution. Hence

$$x_{\mu i} \notin H_{i\mu}^{N-1} \quad (i = 1, 2, \cdots, N; \mu = 0, 1, 2).$$

Using Lemma 2.4, we obtain that there exist $g_\mu \in H(\mu = 0, 1, 2)$ such that

$$d_{i\mu} < E_0R(\alpha_i^2 E_0, A_0)b_0, g_\mu >= 1/3 \quad (i = 1, 2, \cdots, N; \mu = 0, 1, 2) \tag{23}$$

$$< y_k, g_\mu >= 0 \quad (k = N + 1, N + 2, \cdots; \mu = 0, 1, 2). \tag{24}$$

From (23) and (24), it is easy to prove $\omega(\alpha_i) = 1$ and

$$g_\mu = \sum_{k=1}^{N} g_k^{(\mu)}\psi_k + [I - (E_0^*)^+ E_0^*]a \quad (\mu = 0, 1, 2)$$

where a is any element in H, and

$$1/3 = d_{i\mu} < E_0R(\alpha_i^2 E_0, A_0)b_0, g_\mu >= d_{i\mu} \sum_{j=1}^{N} \overline{g_j^{(\mu)}} < E_0Rb_0, \psi_j > \quad (\mu = 0, 1, 2)$$

i.e.
$$\sum_{j=1}^{N}\overline{g_j^{(\mu)}}\frac{b_i}{\alpha_i^2-\lambda_j}=\frac{1}{3d_{i\mu}}\quad(i=1,2,\cdots,N;\mu=0,1,2)\tag{25}$$

The solution of (25) is
$$\overline{g_k^{(\mu)}}=\frac{\alpha_k^2-\lambda_k}{3b_k}\prod_{\substack{j=1\\ j\neq k}}^{N}\Big(\frac{\alpha_j^2-\lambda_k}{\lambda_j-\lambda_k}\Big)\sum_{j=1}^{N}\frac{\alpha_j^2-\lambda_j}{d_{j\mu}(\alpha_j^2-\lambda_k)}\cdot\prod_{\substack{i=1\\ i\neq j}}^{N}\Big(\frac{\alpha_j^2-\lambda_i}{\alpha_j^2-\alpha_i^2}\Big),$$
$$(k=1,2,\cdots,N;\mu=0,1,2).$$

From Lemma 2.3, we have $\alpha_i\in\sigma_p(B_0,T_0)(i=1,2,\cdots,N)$.

For $\lambda_k\in\sigma_p(E_0,A_0)(i=N+1,N+2,\cdots)$, let $V_k=\begin{bmatrix}A^{1/2}\varphi_k\\ \sqrt{\lambda_k}\varphi_k\end{bmatrix}$. It is easy to prove that $G_2V_k=G_1V_k=0$, and

$$\begin{aligned}T_0\begin{bmatrix}V_k\\ 0\end{bmatrix}&=\begin{bmatrix}A & G\\ G_2 & F\end{bmatrix}\begin{bmatrix}V_k\\ 0\end{bmatrix}=\begin{bmatrix}AV_k\\ G_2V_k\end{bmatrix}=\begin{bmatrix}\sqrt{\lambda_k}A_0^{1/2}\varphi_k\\ A_0\varphi_k\\ G_2V_k\end{bmatrix}\\ &=\sqrt{\lambda_k}\begin{bmatrix}I & 0 & 0\\ 0 & E_0 & \\ & -G_1 & E_2\end{bmatrix}\begin{bmatrix}A_0^{1/2}\varphi_k\\ \sqrt{\lambda_k}\varphi_k\\ 0\end{bmatrix}=\sqrt{\lambda_k}B_0\begin{bmatrix}V_k\\ 0\end{bmatrix}.\end{aligned}$$

Thus $\{\sqrt{\lambda_k}\}_{N+1}^{+\infty}\subset\sigma_p(B_0,T_0)$. Hence Theorem 2.1 holds. ∎

4. An illustrative example

Consider the following systems in $H\times H$ and $R^n\times R^n$, respectively:
$$\begin{cases}\begin{bmatrix}I & 0\\ I & 0\end{bmatrix}\begin{bmatrix}\ddot y_1\\ \ddot y_2\end{bmatrix}=\begin{bmatrix}A_{11} & 0\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}+\begin{bmatrix}b_1\\ 0\end{bmatrix}u\\ \begin{bmatrix}I_1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot z_1\\ \dot z_2\end{bmatrix}=\begin{bmatrix}F_{11} & 0\\ F_{21} & F_{22}\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}+\begin{bmatrix}v_1\\ 0\end{bmatrix}\end{cases}\tag{26}$$

where I_1 denotes the identity matrix in R^n, A_{11} is a discrete spectral operator, and there exists A_{11}^{-1}; A_{22} and F_{22} are invertible, there exists $A_{22}^{1/2}$, $A_{21}=A_{11}-A_{22}$. It is easy to prove that

$$\begin{bmatrix}A_{11} & 0\\ A_{21} & A_{22}\end{bmatrix}^{1/2}\begin{bmatrix}A_{11}^{1/2} & 0\\ A_{11}^{1/2}-A_{22}^{1/2} & A_{22}^{1/2}\end{bmatrix},$$
$$\begin{bmatrix}I & 0\\ I & 0\end{bmatrix}\begin{bmatrix}A_{11} & 0\\ A_{21} & A_{22}\end{bmatrix}^{1/2}=\begin{bmatrix}A_{11} & 0\\ A_{21} & A_{22}\end{bmatrix}^{1/2}\begin{bmatrix}I & 0\\ I & 0\end{bmatrix}.$$

Let the feedback controls be
$$u=<z_1,g>,\quad v_1=<\ddot y,g_2>k_2+<\dot y_1,g_1>k_1+<y_1,g_0>k_0.$$

Then (26) becomes

$$\left\{\begin{array}{l} \begin{bmatrix} I & 0 \\ I & 0 \end{bmatrix}\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + < z_1, g > \begin{bmatrix} b_1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ \quad + \begin{bmatrix} < \ddot{y}_1, g_2 > k_2 + < \dot{y}_1, g_1 > k_1 + < y_1, g_0 > k_0 \\ 0 \end{bmatrix}. \end{array}\right. \tag{27}$$

It is obvious that (27) is the second order coupled singular distributed parameter system. From (27) we obtain

$$\left\{\begin{array}{l} \ddot{y}_1 = A_{11}y_1 + < z_1, g > b_1 \\ \dot{z}_1 = F_{11}z_1 + < \ddot{y}_1, g_2 > k_2 + < \dot{y}_1, g_1 > k_1 + < y_1, g_0 > k_0 \end{array}\right. . \tag{28}$$

Hypothesis (H_1) Let A_{11} be a discrete spectral operator, $\{\lambda_k\}_1^\infty$ be the set of all point spectrum of A_{11}, and any λ_k be single, and φ_k be the associated eigenvector, i. e.

$$A_{11}\varphi_k = \lambda_k\varphi_k \quad (k = 1, 2, \cdots).$$

There exists A_{11}^{-1}. Let $\{\psi_k\}_1^\infty$ denote the set of all eigenvector of A_{11}^* satisfying

$$A_{11}^*\psi_k = \overline{\lambda_k}\psi_k \quad (k = 1, 2, \cdots),$$

there exist the following relations between $\{\varphi_k\}_1^\infty$ and $\{\psi_k\}_1^\infty$:

$$< \varphi_k, \psi_l > = \left\{\begin{array}{ll} 1 & k = l \\ 0 & k \neq l \end{array}\right. \quad (k, l = 1, 2, \cdots).$$

Hypothesis (H_2) Let $F_{11} \in R^{n\times n}$, $\{r_k\}_1^n$ be the set of all points of the spectrum of F_{11}, and every r_k be a single, and u_k be the associated eigenvector, i.e. $F_{11}u_k = r_k u_k (k = 1, 2, \cdots, n)$. For the eigenvalue $\overline{r_k}$ of F_{11}^*, the associated eigenvector is v_k, i. e. $F_{11}^* v_k = \overline{r_k} v_k (k = 1, 2, \cdots, n)$. There exist the following relations between $\{u_k\}_1^n$ and $\{v_k\}_1^n$:

$$< u_k, v_l > = \left\{\begin{array}{ll} 1 & k = l \\ 0 & k \neq l \end{array}\right. \quad (k, l = 1, 2, \cdots, n).$$

It is easy to prove that (27) satisfies the Assumptions (G_1) and (G_2) of this note if (28) satisfies the Hypotheses (H_1) and (H_2). Therefore, there exists the second order coupled singular distributed parameter system, which satisfies the hypothesis of this note.

Conclusion

In this paper, pole assignment by feedback control of the second order singular distributed parameter system coupled with the first order singular lumped parameter system is discussed via functional analysis and operator theory in Hilbert space. The solutions of the problem and the constructive expression of the solutions are given by the generalized inverse one of bounded linear operator. This research is theoretically important and convenient for studying the feedback control and pole assignment of the coupled singular distributed parameter systems. If (2) is the second order singular lumped parameter system, the results which are obtained in this paper need to be modified.

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