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Analysis of undamped second order systems with dynamic feedback

by

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Abstract: In this paper the stabilization problem of undamped second order system is considered. The stabilization by first order dynamic feedback is studied. The global asymptotic stability of the respectively closed-loop system is proved by LaSalle's theorem. As an example of application of the proposed method an electric ladder network L and lc type is presented. Numerical calculations were made using the Matlab/Simulink program.

Keywords: asymptotic stability, feedback stabilization, ladder network.

1. Introduction

The problem of feedback stabilization of second order systems has been considered recently by Kobayashi (2001) and Kobayashi & Oya (2004). Kobayashi studied stabilization by adaptive nonlinear feedback for finite dimensional systems (Kobayashi, 2001) and for infinite dimensional systems (Kobayashi & Oya, 2004). The state matrix of the second order system has eigenvalues only on the imaginary axis. Is evident that a second order system with static feedback is not asymptotically stable. If a system is controllable and observable, then it can be stabilized by a dynamic feedback (Luenberger observer with linear regulator). The order of the dynamic feedback is equal to the order of the system. In this paper we will consider stabilization by first order linear dynamic feedback. Our dynamic feedback is optimal from the point of view of its order.

2. Undamped second order system

Let $A^T = A = [a_{ij}] \in R^{n\times n}$ be a real symmetric matrix and let A be a positive definite matrix ($A > 0$). Now we consider the following system

$$\ddot{x}(t) + Ax(t) = Bu(t), \quad y(t) = Cx(t), \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R$ is the control input, $y(t) \in R$ is the output, $B \in R^{n\times 1}$ and $C \in R^{1\times n}$.

Let $\widetilde{x}(t) = [x(t)^T \; \dot{x}(t)^T]^T$ and let

$$\widetilde{A} = \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} 0 \\ B \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} C & 0 \end{bmatrix}. \tag{2}$$

Then from (1) we have

$$\dot{\widetilde{x}}(t) = \widetilde{A}\widetilde{x}(t) + \widetilde{B}u(t), \quad y(t) = \widetilde{C}\widetilde{x}(t). \tag{3}$$

REMARK 2.1 *The real symmetric matrix $A^T = A$ has only real eigenvalues λ_i and $\lambda_i > 0$, because $A > 0$. The eigenvalues of the state matrix $\widetilde{A}$ of the system (3) are given by following formulas: $\widetilde{s}_i = \pm j\sqrt{\lambda_i}$, $j^2 = -1$, $i = 1, 2, \ldots, n$. The state matrix $\widetilde{A}$ has eigenvalues on the imaginary axis. In this case the system (3) is called undamped second order system.*

REMARK 2.2 *The pair $(\widetilde{A}; \widetilde{B})$ is controllable if and only if the pair $(A; B)$ is controllable. The pair $(A; B)$ is controllable if and only if $rank[s_iI - A \vdots B] = n$ for $s_i \in \lambda(A)$, $i = 1, 2, \ldots, n$, where $\lambda(A)$ is the spectrum of $A \in R^{n\times n}$ (see for example Klamka, 1990, p. 21, 149, and Klamka, 1991). Similarly, the pair $(A; B)$ is controllable if and only if $rank[B \vdots AB \vdots A^2B \vdots \ldots \vdots A^{n-1}B] = n$.*

REMARK 2.3 *It is obvious that the pair $(C; A)$ is observable if and only if the pair $(A^T; C^T)$ is controllable (see, for example, Klamka, 1990, p. 63, and Klamka, 1991), i.e.*

$$rank \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = n, \tag{4}$$

where $A \in R^{n\times n}$. Similarly (criterion of Hautus 1969) the pair $(C; A)$ is observable if and only if the $rank M(s; C, A) = n$ for any complex number s (in particular, for any $s = s_i \in \lambda(A)$), where

$$M(s; C, A) = \begin{bmatrix} sI - A \\ C \end{bmatrix}. \tag{5}$$

Let $\dot{x}(t) = Ax(t)$, $y(t) = Cx(t)$. The following relation holds:

$$\{(C; A) \text{ is observable}\} \Leftrightarrow \{y(t) = 0,\ t \geq 0 \ \Rightarrow\ x(t) = 0,\ t \geq 0\}. \tag{6}$$

Remark 2.4 *We notice that (see (2))*

$$rank \begin{bmatrix} \widetilde{C} \\ \widetilde{C}\widetilde{A} \\ \widetilde{C}\widetilde{A}^2 \\ \vdots \\ \widetilde{C}\widetilde{A}^{n-1} \end{bmatrix} = rank \begin{bmatrix} C & 0 \\ 0 & C \\ CA & 0 \\ 0 & CA \\ CA^2 & 0 \\ 0 & CA^2 \\ \vdots & \vdots \\ 0 & CA^{n-1} \end{bmatrix} = 2n \tag{7}$$

if and only if the condition (4) is satisfied. Thus, if and only if the pair $(C; A)$ *is observable, then the pair* $(\widetilde{C}; \widetilde{A})$ *is observable.*

3. Feedback stabilization

The system (3) is stable, but not asymptotically stable. It is evident that the system (3) with static feedback $u(t) = -Ky(t)$ is not asymptotically stable. If $(\widetilde{A}; \widetilde{B})$ is controllable and $(\widetilde{C}; \widetilde{A})$ observable, then from the classical result it can be shown that the dynamic feedback (Luenberger observer with linear regulator) asymptotically stabilizes the system (3). The order of the dynamic feedback is $2n$ or $2n-1$ (reduced order Luenberger observer). Now we will consider stabilization by first order linear dynamic feedback.

We consider the linear dynamic feedback given in the following form:

$$\begin{aligned} u(t) &= -K(y(t) + w(t)), \quad K > 0, \\ \dot{w}(t) &= -aw(t) + bu(t), \quad a > 0,\ b > 0, \end{aligned} \tag{8}$$

where $dim\, w(t) = 1$. The closed-loop system (3), (8) is described by the following equation in the state space $X = R^n \times R^n \times R$:

$$\dot{z}(t) = Fz(t), \quad F \in R^{(2n+1)\times(2n+1)}, \quad z(t) \in X = R^{2n+1}, \tag{9}$$

$$F = \begin{bmatrix} 0 & I & 0 \\ -[A + BKC] & 0 & -BK \\ -bKC & 0 & -[a + bK] \end{bmatrix}, \quad z(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \\ w(t) \end{bmatrix} \tag{10}$$

or differently as the block-diagonal system (in this case with $\alpha = 1$ and $\beta = 1$ in output $s(t)$)

$$\begin{aligned} \begin{bmatrix} \dot{\widetilde{x}}(t) \\ \dot{w}(t) \end{bmatrix} &= \begin{bmatrix} \widetilde{A} & 0 \\ 0 & -a \end{bmatrix} \begin{bmatrix} \widetilde{x}(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} \widetilde{B} \\ b \end{bmatrix} u(t), \\ s(t) &= \begin{bmatrix} \alpha\widetilde{C} & \beta \end{bmatrix} \begin{bmatrix} \widetilde{x}(t) \\ w(t) \end{bmatrix} \end{aligned} \tag{11}$$

with static feedback

$$u(t) = -Ks(t), \quad K > 0. \tag{12}$$

Remark 3.1 *From Remark 2.1 for $a > 0$ (see (8)) $\lambda(\widetilde{A}) \cap \lambda(-a) = \emptyset$. Let the pair $(\widetilde{C}; \widetilde{A})$ be observable (see Remark 2.3 and 2.4). Then for $\alpha \neq 0$ and $\beta \neq 0$*

$$rank \begin{bmatrix} sI - \widetilde{A} & 0 \\ 0 & s + a \\ \alpha\widetilde{C} & \beta \end{bmatrix} = 2n + 1, \quad \forall s \tag{13}$$

and consequently the block-diagonal system (11) with output $s(t)$ is observable.

Now we consider the following functional

$$\begin{aligned} V(x(t), \dot{x}(t), w(t)) = \tfrac{1}{2} x(t)^T A x(t) + \tfrac{1}{2} \dot{x}(t)^T \dot{x}(t) \\ + \tfrac{1}{2} \tfrac{a}{b} w(t)^2 + \tfrac{1}{2} K [C x(t) + w(t)]^2. \end{aligned} \tag{14}$$

We notice that if $A^T = A > 0$, $a > 0$ and $b > 0$, then

$$V(x, \dot{x}, w) \geq \frac{1}{2} x^T A x + \frac{1}{2} \dot{x}^T \dot{x} + \frac{1}{2} \frac{a}{b} w^2 > 0 \tag{15}$$

for $x \neq 0$, $\dot{x} \neq 0$, $w \neq 0$ and

$$V(x, \dot{x}, w) \to \infty, \quad \text{if} \quad x, \dot{x}, w \to \infty. \tag{16}$$

From (14), (11) with $\alpha = 1$ and $\beta = 1$, (12) and from elementary calculations we obtain

$$\dot{V}(x(t), \dot{x}(t), w(t)) = u(t)[B^T - C]\dot{x}(t) - b[\frac{a}{b} w(t) - u(t)]^2, \tag{17}$$

where $u(t)$ is given by (12).

Remark 3.2 *If output matrix $C = B^T$, then*

$$\dot{V}(x(t), \dot{x}(t), w(t)) = -b[\frac{a}{b} w(t) - u(t)]^2 \leq 0. \tag{18}$$

In this case V given by (14) is the Lyapunov functional for system (9).

Substituting $u(t)$ from (12) into (18) we finally obtain

$$\dot{V}(x(t), \dot{x}(t), w(t)) = -b[K B^T x(t) + (\frac{a}{b} + K) w(t)]^2 = -b s(t)^2 \leq 0, \tag{19}$$

where $s(t) = \alpha B^T x(t) + \beta w(t)$ is the output of the system (11) with $C = B^T$ (see (2)), $\alpha = K$ and $\beta = \frac{a}{b} + K$.

By LaSalle's theorem (LaSalle & Lefschetz, 1966, p. 196) the solution of (9) asymptotically tends to the maximal invariant subset of E, where

$$E = \{(x, \dot{x}, w) : \dot{V}(x, \dot{x}, w) = 0\}. \tag{20}$$

From $\dot{V}(x, \dot{x}, w) = 0$ (see (19)) we have $s = 0$ and $u = 0$ (see (12)). The system (11) is observable (see Remark 3.1) and thus from $s = 0$ (see Remark 2.3, relation (6)) we obtain $x = 0$, $\dot{x} = 0$ and $w = 0$. Consequently, it is easy to see that the largest invariant subset contained in $E = \{0\}$ is the set $S = \{0\}$. Thus, the following theorem has been proved:

THEOREM 3.1 *Let* $A^T = A \in R^{n\times n}$, $A > 0$ *(positive definite matrix). Let the pair* $(A; B)$ *be controllable. If matrix* $C = B^T$ *and* $K > 0$, $a > 0$ *and* $b > 0$, *then the closed-loop system (3), (8) described by equation (9) is globally asymptotically stable, i.e.* $Re\,\lambda(F) < 0$, *where* F *is the state matrix of system (9).*

4. Properties of the matrix e^{Ft}

The matrices F and e^{Ft} are respectively the state matrix and the fundamental matrix of the closed-loop system (9). The norm of fundamental matrix depends on the inner product in Hilbert space X.

Consider the Hilbert space $X = R^n \times R^n \times R$ with the following inner product:

$$(d|e) = d^T e, \quad d \in X, \quad e \in X. \tag{21}$$

Every inner product on Hilbert space X induces the norm on X:

$$\|d\| = \sqrt{(d|d)}. \tag{22}$$

Let $S : X \to X$ be a linear operator. The natural norm of S is given by

$$\|S\| = \sup\{\|Sd\| : \|d\| \leq 1\}. \tag{23}$$

If inner product (21) induces the norm (22), then

$$\|S\| = \max_k \sqrt{\lambda_k(S^T S)}, \tag{24}$$

where $\lambda_k(S^T S)$ is eigenvalue of matrix $S^T S$, $k = 1, 2, 3, \ldots, dimX$.

Let F be the matrix (10). The family of matrices e^{Ft}, $0 \leq t \leq \infty$, is a strongly continuous semigroup (C_0 semigroup) of bounded linear operators on X (Pazy, 1983, p. 4). The matrix F is the infinitesimal generator of the semigroup e^{Ft}. If $Re\,\lambda(F) < 0$, then there exist constants $\gamma < 0$ and $M \geq 1$ such that

$$\|e^{Ft}\| \leq M e^{\gamma t} \quad \text{for} \quad 0 \leq t \leq \infty, \tag{25}$$

where $\|.\|$ is given by (24). In this case the semigroup e^{Ft} is exponentially stable with the growth constant γ and consequently the generator F is exponentially stable.

Now we consider the space $X = R^n \times R^n \times R$ with the inner product given in the following form (see, for example, Kobayashi & Oya, 2004, p. 76):

$$(d|e)_c = (Ad_1|e_1) + (d_2|e_2) + \frac{a}{b}(d_3|e_3) + K(Cd_1 + d_3|Ce_1 + e_3), \qquad (26)$$

where $a > 0$, $b > 0$, $K > 0$. We notice that for $d \in X$ and F given by (10) with $C = B^T$

$$(Fd|d)_c + (d|Fd)_c = -2b\|KB^T d_1 + [\frac{a}{b} + K]d_3\|_c^2 \leq 0. \qquad (27)$$

Thus, the linear operator F (in this case matrix F) with $C = B^T$ is dissipative. From the Lumer Phillips theorem (Pazy, 1983, p. 14, Beckenbach, 1968, p. 150) matrix F is the infinitesimal generator of C_0 semigroup of contractions on X, i. e.

$$\|e^{Ft}\|_c \leq 1 \quad \text{for} \quad t \geq 0. \qquad (28)$$

From Theorem 3.1, $Re\,\lambda(F) < 0$, where F is the state matrix of the system (9). Thus, it is evident that $\|e^{Ft}\|_c \to 0$ if $t \to \infty$.

These results will be illustrated in the following example.

5. Electric ladder network

Consider an electric ladder network of the L and lc-type shown in Fig. 1. The parameters of the network $L > 0$, $l > 0$ and $c > 0$ are known. Let $y(t) = x_1(t)/Lc$.

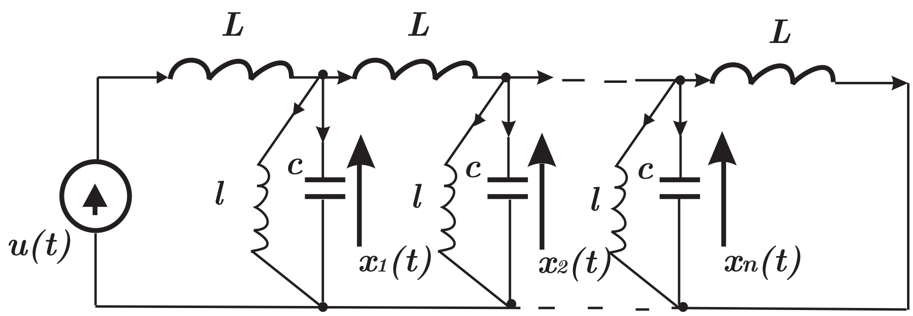

Figure 1. Electric ladder network of the L and lc-type

The electric system shown in Fig. 1 is described by following equations:

$$\begin{aligned} &Lc\,\ddot{x}_i(t) - x_{i-1}(t) + (2 + \tfrac{L}{l})\,x_i(t) - x_{i+1}(t) = 0, \quad i = 1, 2, \ldots, n \\ &x_0(t) = u(t), \quad x_{n+1}(t) = 0, \quad y(t) = x_1(t)/Lc, \end{aligned} \qquad (29)$$

or by equation (1) with $C = B^T$ and

$$A = \frac{1}{Lc}\begin{bmatrix} 2+\frac{L}{l} & -1 & 0 & \cdots & 0 \\ -1 & 2+\frac{L}{l} & -1 & \cdots & 0 \\ 0 & -1 & 2+\frac{L}{l} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2+\frac{L}{l} \end{bmatrix}, \quad B = \frac{1}{Lc}\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (30)$$

The eigenvalues of symmetric Jacobi matrix A given in (30) are represented by the following equations (see, for example, Bellman, 1960, p. 215 or Lancaster, 1969, p. 104):

$$\lambda_i(A) = \frac{1}{Lc}\,[4\sin^2(\frac{\varphi_i}{2}) + \frac{L}{l}], \quad \varphi_i = \frac{i\,\pi}{n+1}, \quad i = 1, 2, \ldots, n. \quad (31)$$

We notice (see (31)), that $\lambda_i(A) > 0$. Thus the matrix A is a positive definite matrix. Similarly to Mitkowski (2004), we can prove that the pair $(A; B)$ is controllable. From Theorem 3.1 it is obvious that the closed-loop system (29), (8)) is globally asymptotically stable.

6. Simulation results

Let us consider the undamped electric system (29) with feedback (8), where $n = 5$, $Lc = 1$, $L/l = 1$ and $a = 1$, $b = 1$. Let $x_1(0) = 0.2$, $x_i(0) = 0$, $i = 2, 3, \ldots, n$, $\dot{x}_i(0) = 0$, $i = 1, 2, 3, \ldots, n$ and $w(0) = 1$.

In Fig. 2 output trajectory $y(t) = x_1(t)/Lc$ for $K = 0$ is shown (see feedback (8)).

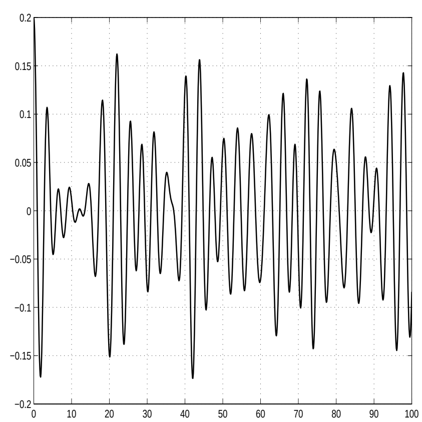

Figure 2. Trajectory $y(t)$ of undamped system (29)

Now we consider the closed-loop system (29), (8) with the cost functional

$$J(K) = \int_0^{100} [y(t)^2 + 0.01u(t)^2]dt. \tag{32}$$

In Fig. 3 the cost functional $J(K)$ of closed-loop system (29), (8) is shown. In Fig. 4 the output trajectory $y(t) = x_1(t)/Lc$ for $K = 5$ is shown (see feedback (8)).

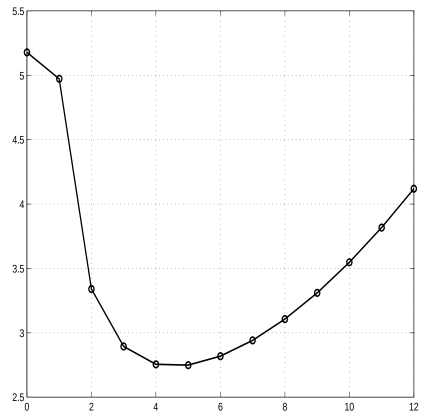

Figure 3. Cost functional $J(K)$

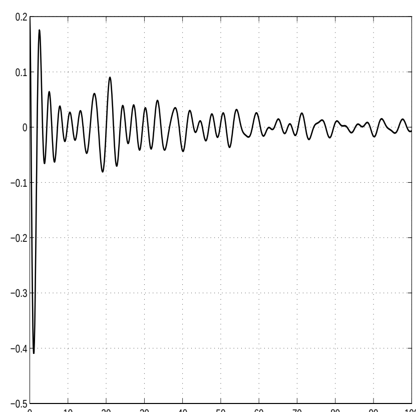

Figure 4. Trajectory $y(t)$ with feedback (8) for $K = 5$

7. Conclusion

In this paper, we considered the stabilization problem of undamped second order system (1). If the output matrix $C = B^T$, then the system (1) can be stabilized by the first order dynamic feedback (8). To prove that the closed-loop system (1), (8) is globally asymptotically stable (see Theorem 3.1), we have used LaSalle's invarince principle (LaSalle & Lefschetz, 1966, p. 196). The observability of the system (1) plays an essential role in the proof. Example (the electric ladder network of L and lc -type, see (29)) shows the quality of stabilization depends on K (see Figs. 2, 3 and 4).

Results can be generalized. The approximately observable (similarly to Klamka, 1990, p. 135, 148) system (1) in appropriate Hilbert space with bounded operator $C = B^*$ and with self-adjoint operator A with compact resolvent can be stabilized by one dimensional dynamic feedback (8). In this case we have to use a generalized version of LaSalle's invariance principle (see for example Slemrod, 1976, p. 406, and Kobayashi & Oya, 2004, p. 77) for C_0 semigroup of contractions on X (see (28)).

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