**Control and Cybernetics**

vol. **33** (2004) No. 4

# A necessary and sufficient condition for stability of the convex combination of polynomials

by

**Stanisław Białas**

Department of Mathematics
AGH University of Science and Technology
Al. Mickiewicza 30, 30–059 Cracow, Poland
e-mail: sbialas@uci.agh.edu.pl

**Abstract:** This paper gives a necessary and sufficient condition for the Hurwitz (Schur) stability of the convex combination of the complex polynomials $f_1(x), f_2(x), \ldots, f_m(x)$.

It provides a generalization of the Ackermann, Barmish (1988), Barlett, Hollot, Huang (1988) and Białas (1985).

**Keywords:** Hurwitz stability, Schur stability, convex combination of polynomials.

## 1. Introduction

The complex polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is called Hurwitz (Schur) stable if all its roots are in the open left-half plane (in the open unit circle).

Let $\tilde{F}_n(\tilde{S}_n)$ denote the set all Hurwitz (Schur) stable complex polynomials, whose degrees are less than or equal to $n$.

Consider the complex polynomials

$$f_i(x) = a_n^{(i)} x^n + a_{n-1}^{(i)} x^{n-1} + \cdots + a_1^{(i)} x + a_0^{(i)} \tag{1}$$

for $i = 1, 2, \ldots, m$, whose the degrees are equal $n \geq 1$.

We will use the notations:

$$V_m = \{(\alpha_1, \alpha_2, \ldots, \alpha_m) \in R^m : \\ \alpha_i \geq 0 \ (i = 1, 2, \ldots, m), \alpha_1 + \alpha_2 + \cdots + \alpha_m = 1\},$$

$$C(f_1, f_2, \cdots, f_m) = \\ \{\alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_m f_m(x) : (\alpha_1, \alpha_2, \cdots, \alpha_m) \in V_m\},$$

$$F_n = \{f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = \\ a_n(x - x_1)(x - x_2)\cdots(x - x_n) : \\ \mathrm{Re}(x_j) < 0 \ (j = 1, 2, \ldots, n), a_n \neq 0,\ a_j \in C\ (j = 0, 1, \ldots, n)\},$$

$$S_n = \{f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = \\ a_n(x - x_1)(x - x_2)\cdots(x - x_n) : \\ |x_j| < 1\ (j = 1, 2, \ldots, n),\ a_n \neq 0,\ a_j \in C\ (j = 0, 1, \ldots, n)\}.$$

We will assume that

$$\alpha_1 a_n^{(1)} + \alpha_2 a_n^{(2)} + \cdots + \alpha_m a_n^{(m)} \neq 0$$

for every $(\alpha_1, \alpha_2, \ldots, \alpha_m) \in V_m$.

Definition 1.1 *The convex combination of the polynomials $C(f_1, f_2, \ldots, f_m)$ is called Hurwitz (Schur) stable if and only if $C(f_1, f_2, \ldots, f_m) \subset F_n$ ($C(f_1, f_2, \ldots, f_m) \subset S_n$).*

The literature, provides the known criteria for stability of convex combination $C(f_1, f_2, \ldots, f_m)$ in the particular case, when the polynomials (1) are real and the degrees of the polynomials are equal: $\deg(f_1(x)) = \deg(f_2(x)) = \cdots = \deg(f_m(x)) = n$. In 1985 a necessary and sufficient condition for $C(f_1, f_2) \subset F_n$ has been proved in Białas (1985). In Bartlett, Hotlot, Huang (1988) and Ackermann, Barmish (1988) a necessary and sufficient condition for $C(f_1, f_2, \ldots, f_m) \subset F_n(C(f_1, f_2, \ldots, f_m) \subset S_n)$ has been given. The three papers cited concerned the real polynomials.

This paper is the generalization of these three papers to the complex polynomials.

At first, we will prove the lemma, which will be useful in the next part of this paper.

Let $P_i(t) :< t_0, \infty) \to R^2$ $(i = 1, 2, 3)$ be continuous functions for $t \in < t_0, \infty)$, $A \subset R^2$. Let $\partial(A)$ denote the bound of the set $A$ and let dist $(x_0, A)$ denote the distance of the point $x_0$ to the set $A$, where $x_0 \in R^2$.

We will use the notations:

$$\overline{A} = (R^2 \backslash A) \cup \partial(A),$$

$$T(t) = \{\alpha_1 P_1(t) + \alpha_2 P_2(t) + \alpha_3 P_3(t) :\ (\alpha_1, \alpha_2, \alpha_3) \in V_3\}$$

for $t \geq t_0$.

Lemma 1.1 *If the functions $P_i(t)$ $(i = 1, 2, 3)$ are continuous for $t \in < t_0, \infty)$ and*

$$(0,0) \in T(t_0) \quad \text{and} \quad (0,0) \notin T(t) \tag{2}$$

*for every $t > t_0$, then $(0,0) \in \partial(T(t_0))$.*

*Proof.* Consider the function $\phi(t) = dist((0,0), \overline{T(t)})$. With the assumption that the functions $P_i(t)$ $(i = 1,2,3)$ are continuous for $t \geq t_0$ and with (2) it follows that

$$\phi(t) = 0 \quad \text{for} \quad t > t_0 \quad \text{and} \quad \lim_{t \to t_0^+} \phi(t) = 0.$$

Hence it follows that $(0,0) \in \partial(T(t_0))$. ■

## 2. The necessary and sufficient condition for Hurwitz (Schur) stability of the convex combination of polynomials

At first, we will consider the case of $m = 3$.

Let $f_1(x), f_2(x), f_3(x)$ be the polynomials (1).

Let

$$R(C(f_1, f_2, f_3)) = \{z \in C : \bigvee_{(\alpha_1,\alpha_2,\alpha_3)\in V_3} \alpha_1 f_1(z) + \alpha_2 f_2(z) + \alpha_3 f_3(z) = 0\}.$$

Now, we prove

Lemma 2.1 *Assume that $f_1(x), f_2(x), f_3(x)$ are the polynomials (1).*

*1) If $\alpha_0 = \max_{z\in R(C(f_1,f_2,f_3))} \text{Re}\ (z)$, $z_0 = \alpha_0 + i\beta$,*
*$\alpha_1' f_1(z_0) + \alpha_2' f_2(z_0) + \alpha_3' f_3(z_0) = 0$, where $(\alpha_1', \alpha_2', \alpha_3') \in V_3$, then*

$$z_0 \in R(C(f_1, f_2)) \text{ or } z_0 \in R(C(f_1, f_3)) \text{ or } z_0 \in R(C(f_2, f_3)) \tag{3}$$

*2) If $r_0 = \max_{z\in R(C(f_1,f_2,f_3))} |z|$, $z_1 = r_0 e^{i\varphi_0}$, $\beta_1' f_1(z_1) + \beta_2' f_2(z_1) + \beta_3' f_3(z_1) = 0$ where $(\beta_1', \beta_2', \beta_3') \in V_3$, then*

$$z_1 \in R(C(f_1, f_2)) \text{ or } z_1 \in R(C(f_1, f_3)) \text{ or } z_1 \in R(C(f_2, f_3)). \tag{4}$$

*Proof.* We first prove relations (3). Consider the set

$$A(z_0 + \epsilon) = \{\alpha_1 f_1(z_0 + \epsilon) + \alpha_2 f_2(z_0 + \epsilon) + \alpha_3 f_3(z_0 + \epsilon) : (\alpha_1, \alpha_2, \alpha_3) \in V_3,\ \epsilon \in R,\ \epsilon \geq 0\}.$$

We see, that the set $A(z_0+\epsilon)$ is the convex combination of the points: $f_1(z_0+\epsilon)$, $f_2(z_0+\epsilon)$, $f_3(z_0+\epsilon)$, and $A(z_0+\epsilon) \subset C$, $(0,0) \in A(z_0)$ and $(0,0) \notin A(z_0+\epsilon)$ for every $\epsilon > 0$. Hence, by applying Lemma 1.1, we obtain $(0,0) \in \partial(A(z_0))$. Thus, relations (3) are true.

The proof for relations (4) is analogous.

Let

$$\begin{aligned} B((r_0 + \epsilon)e^{i\varphi_0}) &= \{\alpha_1 f_1((r_0 + \epsilon)e^{i\varphi_0}) + \alpha_2 f_2((r_0 + \epsilon)e^{i\varphi_0}) \\ &\quad + \alpha_3 f_3((r_0 + \epsilon)e^{i\varphi_0}) : (\alpha_1, \alpha_2, \alpha_3) \in V_3,\ \epsilon \in R,\ \epsilon \geq 0\}. \end{aligned}$$

As seen, the set $B((r_0+\epsilon)e^{i\varphi_0})$ is the convex combination of the points: $f_1((r_0+\epsilon)e^{i\varphi_0})$, $f_2((r_0+\epsilon)e^{i\varphi_0})$, $f_3((r_0+\epsilon)e^{i\varphi_0})$, and $B((r_0+\epsilon)e^{i\varphi_0}) \subset C$, $(0,0) \in B(r_0 e^{i\varphi_0})$, $(0,0) \notin B((r_0+\epsilon)e^{i\varphi_0})$ for every $\epsilon > 0$. Thus, by applying Lemma 1.1, we obtain that relations (4) are true. This completes the proof of Lemma 2.1 ■

Now, we will prove the theorem, which gives the necessary and sufficient condition for Hurwitz (Schur) stability of the convex combination of the polynomials $f_1(x)$, $f_2(x)$, $f_3(x)$.

THEOREM 2.1 *If $f_1(x)$, $f_2(x)$, $f_3(x)$ are the polynomials (1), then the convex combination*

$$C(f_1, f_2, f_3) = \{\alpha_1 f_1(x) + \alpha_2 f_2(x) + \alpha_3 f_3(x) : (\alpha_1, \alpha_2, \alpha_3) \in V_3\}$$

*is Hurwitz (Schur) stable if and only if the convex combinations $C(f_1, f_2)$, $C(f_1, f_3)$, $C(f_2, f_3)$ are Hurwitz (Schure) stable.*

*Proof.* The necessary condition is trivial because $C(f_1, f_2), C(f_1, f_3), C(f_2, f_3) \subset C(f_1, f_2, f_3)$.

Now, let us prove the sufficient condition for Hurwitz stability.

Assume that $C(f_1, f_2)$, $C(f_1, f_3)$, $C(f_2, f_3) \subset F_n$ and we will prove that $C(f_1, f_2, f_3) \subset F_n$.

We will prove the sufficient condition by *reductio ad absurdum*. Assume that there exist a complex number $z_0 = \alpha_0 + i\beta$ and a polynomial $f(z) \in C(f_1, f_2, f_3)$ such that

$$f(z_0) = 0 \quad \text{and} \quad Re(z_0) \geq 0.$$

We can assume, without loss of generality, that $\alpha_0 = \max_{z \in R(C(f_1,f_2,f_3))} Re(z)$. Hence, taking also into account Lemma 2.1 it follows, that

$$z_0 \in R(C(f_1, f_2)) \quad \text{or} \quad z_0 \in R(C(f_1, f_3)) \quad \text{or} \quad z_0 \in R(C(f_2, f_3)).$$

This is a contradiction to the assumption: $C(f_1, f_2)$, $C(f_1, f_3)$, $C(f_2, f_3) \subset F_n$. This finishes the proof for the sufficient condition for Hurwitz stability of $C(f_1, f_2, f_3)$.

The proof for the sufficient condition for Schur stability of $C(f_1, f_2, f_3)$.

Assume that $C(f_1, f_2)$, $C(f_1, f_3)$, $C(f_2, f_3) \subset S_n$. We will prove that $C(f_1, f_2, f_3) \subset S_n$. The proof is analogous as for Hurwitz stability. For the proof by *reductio ad absurdum*, we assume that there exist a complex number $z_0 = r_0 r^{i\varphi_0}$ and a polynomial $f(z) \in C(f_1, f_2, f_3)$ such that $f(z_0) = 0$ and $r_0 \geq 1$. We can assume that $r_0 = \max_{z \in R(C(f_1,f_2,f_3))} |z|$. Hence and with Lemma 2.1 we have

$$z_0 \in R(C(f_1, f_2)) \quad \text{or} \quad z_0 \in R(C(f_1, f_3)) \quad \text{or} \quad z_0 \in R(C(f_2, f_3)).$$

This is a contradiction to the assumption that the sets $C(f_1, f_2)$, $C(f_1, f_3)$ and $C(f_2, f_3)$ are Schur stable.

This completes the proof of Theorem 1.1. ■

Now, we will prove the necessary and sufficient condition for the Hurwitz (Schur) stability of the convex combination of polynomials $C(f_1, f_2, \ldots, f_m)$.

THEOREM 2.2 *If $f_1(x), f_2(x), \ldots, f_m(x)$ are the polynomials (1) then the convex combination*

$$C(f_1, f_2, \ldots, f_m) = \{\alpha_1 f_1(x)+\alpha_2 f_2(x)+\cdots+\alpha_m f_m(x): \ (\alpha_1, \alpha_2, \ldots, \alpha_m) \in V_m\}$$

*is Hurwitz (Schur) stable if and only if the convex combinations $C(f_j, f_k)$ are Hurwitz (Schur) stable for $j = 1, 2, \ldots, m$; $k = 1, 2, \ldots, m$; $j < k$.*

*Proof.* The necessary conditions follow from the assumption that $C(f_j, f_k) \subset C(f_1, f_2, \ldots, f_m)$.

The sufficient condition for Hurwitz stability.

Assume that $C(f_j, f_k) \subset F_n$ $(j, k = 1, 2, \ldots, m; j < k)$ and we will prove that $C(f_1, f_2, \ldots, f_m) \subset F_n$. We will prove the sufficient condition by *reductio ad absurdum.* Assume that there exists a polynomial $f(x) \in C(f_1, f_2, \ldots, f_m)$, which is not Hurwitz stable. Therefore, for each $\rho > 0$ the polynomial $\rho f(x) \notin F_n$ and there exists $\rho_0 > 0$ such that $\rho_0 f(x)$ is inside the bound of $C(f_1, f_2, \ldots, f_m)$, for example $\rho_0 f(x) \in C(f_{i_0}, f_{j_0}, f_{k_0}) \subset C(f_1, f_2, \ldots, f_m)$.

From the assumption $C(f_j, f_k) \subset F_n$ $(j, k = 1, 2, \ldots, m; j < k)$ and Theorem 1 it follows that the convex combination $C(f_{i_0}, f_{j_0}, f_{k_0})$ is Hurwitz stable. This is a contradiction to $\rho_0 f(x) \in C(f_{j_0}, f_{j_0}, f_{k_0})$ and $f(x) \notin F_n$. This finishes the proof for the sufficient condition for Hurwitz stability of $C(f_1, f_2, \ldots, f_m)$.

The proof of the sufficient condition for Schur stability is analogous.

From Theorem 2.1 and with the assumption $C(f_j, f_k) \subset F_n$ $(C(f_j, f_k) \subset S_n)$ for $j < k$ we see that $C(f_1, f_2, \ldots, f_m) \subset F_n$ $(C(f_1, f_2, \ldots, f_m) \subset S_n)$.

This completes the proof of Theorem 2.2. ■

Now, we will prove the necessary and sufficient condition for Hurwitz (Schur) stability of the convex combination of two real polynomials, whose degrees can be different.

It is the generalization of Białas (1985), where the degrees of the polynomials were assumed to be equal.

Consider two real polynomials

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0; \ g(x) = b_k x^k + b_{k-1} x^{k-1} + \cdots + b_1 x + b_0,$$

where $a_n > 0$, $b_k > 0$, $k \leq n$.

Let

$$\begin{aligned} C(f, g) &= \{\alpha_1 f(x) + \alpha_2 g(x): \ \alpha_1 \geq 0, \ \alpha_2 \geq 0, \ \alpha_1 + \alpha_2 = 1\}, \\ \tilde{g}(x) &= b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0, \end{aligned}$$

where $b_j = 0$ for $k < j \le n$.

We see that $C(f,g) = C(f,\tilde{g})$.

Let $H(h)$ denote the Hurwitz matrix associated with the polynomial $h(x) \in C(f,\tilde{g})$, i.e.

$$H(f) = \begin{bmatrix} a_{n-1} & a_n & 0 & 0 & 0 & \dots & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} & a_n & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots & 0 & a_0 \end{bmatrix},$$

$$H(\tilde{g}) = \begin{bmatrix} b_{n-1} & b_n & 0 & 0 & 0 & \dots & 0 \\ b_{n-3} & b_{n-2} & b_{n-1} & b_n & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots & 0 & b_0 \end{bmatrix}.$$

Moreover, let $S(f)$ denote the matrix associated with the polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = f(x) \in C(f,\tilde{g})$ where

$$S(f) = \begin{bmatrix} a_n & a_{n-1} & a_{n-2} & \dots & a_3 & a_2 - a_0 \\ 0 & a_n & a_{n-1} & \dots & a_4 - a_0 & a_3 - a_1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & -a_0 & -a_1 & \dots & a_n - a_{n-4} & a_{n-1} - a_{n-3} \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-3} & a_n - a_{n-2} \end{bmatrix}.$$

We see that $H(f) \in R^{n \times n}$, $S(f) \in R^{(n-1)\times(n-1)}$ and it is known that

$$\det(S(f)) = a_n^{n-1} \prod_{1 \le i < j \le n} (1 - x_i x_j) \tag{5}$$

where $x_i, x_j$ are the roots of the polynomial $f(x)$.

Hence, given that the polynomial $f(x)$ is Hurwitz (Schur) stable it follows that there exists the inverse matrix $H^{-1}(f)$, $(S^{-1}(f))$.

Consider the matrices:

$$W = H^{-1}(f)H(\tilde{g}), \quad M = S^{-1}(f)S(\tilde{g}).$$

Let $\lambda_i(W)$ $(i = 1, 2, \dots, n)$ denote the eigenvalues of the matrix $W$ and let $\lambda_i(M)$ $(i = 1, 2, \dots, n-1)$ denote the eigenvalues of the matrix $M$.

We will prove the following theorem.

Theorem 2.3 *Assume that $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, $g(x) = b_k x^k + b_{k-1}x^{k-1} + \cdots + b_1 x + b_0$ are the real polynomials, where $a_n > 0$, $b_k > 0$, $k \le n$.*

$1^0$ *If the polynomials $f(x)$, $g(x)$ are Hurwitz stable then the convex combination $C(f,g)$ is Hurwitz stable if and only if*

$$\lambda_i(W) = \lambda_i(H^{-1}(f)H(\tilde{g})) \notin (-\infty, 0) \quad (i = 1, 2, \dots, n).$$

$2^0$ *If the polynomials $f(x)$, $g(x)$ are Schur stable then the convex combination $C(f,g)$ is Schur stable if and only if*

$$\lambda_i(M) = \lambda_i(S^{-1}(f)S(\tilde{g})) \notin (-\infty, 0) \quad (i = 1, 2, \ldots, n-1).$$

*Proof.* From the assumption $f(x) \in F_n$ ($f(x) \in S_n$) it follows that there exists $H^{-1}(f)$ ($S^{-1}(f)$).

The necessary condition for Hurwitz stability.

Assume that $C(f, \tilde{g}) \subset \tilde{F}_n$ and we will prove that $\lambda_i(W) \notin (-\infty, 0)$ $(i = 1, 2, \ldots, n)$.

From the assumption $C(f, \tilde{g}) \subset \tilde{F}_n$ it follows that for every polynomial $h(x) = \alpha_1 f(x) + \alpha_2 \tilde{g}(x) \in C(f, \tilde{g})$ we have

$$\det(H(h)) = \det(\alpha_1 H(f) + \alpha_2 H(\tilde{g})) \neq 0$$

for every $(\alpha_1, \alpha_2) \in V_2$ and $\alpha_1 \neq 0$. Therefore

$$\det(\alpha_1 H(f) + (1-\alpha_1) H(\tilde{g})) \neq 0,$$

$$\det(\alpha_1 I + (1-\alpha_1) H^{-1}(f) H(\tilde{g})) \neq 0,$$

$$\det\Big(\frac{\alpha_1}{\alpha_1 - 1} I - H^{-1}(f) H(\tilde{g})\Big) \neq 0$$

for every $\alpha_1 \in (0,1)$. Hence it follows that

$$\lambda_i(H^{-1}(f) H(\tilde{g})) \notin (-\infty, 0) \quad (i = 1, 2, \ldots, n).$$

The sufficient condition for Hurwitz stability.

We will prove that if $\lambda_i(W) = \lambda_i(H^{-1}(f)H(\tilde{g})) \notin (-\infty, 0)$ $(i = 1, 2, \ldots, n)$ then $C(f, \tilde{g}) \subset \tilde{F}_n$. From the ssumption $\lambda_i(W) \notin (-\infty, 0)$ we have

$$\det\Big(\frac{\alpha_1}{\alpha_1 - 1} I - H^{-1}(f) H(\tilde{g})\Big) \neq 0,$$

$$\det(\alpha_1 H(f) + (1-\alpha_1) H(\tilde{g})) \neq 0 \quad \text{for every } \alpha_1 \in (0,1),$$

$$\det(H(h)) \neq 0 \tag{6}$$

for all $h(x) = \alpha_1 f(x) + (1-\alpha_1)\tilde{g}(x)$, $\alpha_1 \in (0,1)$.

Denote by $D_i(\alpha_1)$ $(i = 1, 2, \ldots, n)$ the leading minors of the matrix

$$H(h) = H(\alpha_1 f(x) + (1-\alpha_1)\tilde{g}) = \alpha_1 H(f(x)) + (1-\alpha_1) H(\tilde{g})$$

for $\alpha_1 \in (0,1)$. Hence, given that the polynomial $f(x)$ is Hurwitz stable it follows that $D_i(1) > 0$ $(i = 1, \ldots, n)$.

We will prove by *reductio ad absurdum* that $D_i(\alpha_1) > 0$ $(i = 1, 2, \ldots, n)$ for every $\alpha_1 \in (0,1)$. Assume that there exist $\alpha_1^0$ and $i_0$ such that $\alpha_1^0 \in (0,1)$, $1 \leq i_0 \leq n$, $D_{i_0}(\alpha_1^0) = 0$. Hence, with (6) and the formula of Orleando's it follows that there exists $\overline{\alpha}_1 \in (\alpha_1^0, 1)$ such that $D_n(\overline{\alpha}_1) = 0$. This is a contradiction to (6). So, the sufficient condition for Hurwitz stability is true.

The proof of the necessary and sufficient condition for Schur stability is analogous as for Hurwitz stability. We will prove only the sufficient condition. We will prove that if $f(x), g(x) \in \tilde{S}_n$ and $\lambda_i(M) \notin (-\infty, 0)$ $(i = 1, 2, \ldots, n-1)$ then $C(f, g) \subset \tilde{S}_n$. From the assumption $\lambda_i(M) \notin (-\infty, 0)$ $(i = 1, 2, \ldots, n-1)$ we have

$$\det\Big(\frac{\alpha_1}{\alpha_1 - 1} I - S^{-1}(f) S(\tilde{g})\Big) \neq 0,$$
$$\det(\alpha_1 S(f) + (1 - \alpha_1) S(\tilde{g})) \neq 0,$$
$$\det(S(\alpha_1 f + (1 - \alpha_1)\tilde{g})) \neq 0 \tag{7}$$

for every $\alpha_1 \in (0, 1)$. Therefore, (7) is true for $\alpha_1 = 1$.

For the proof by *reductio ad absurdum* we assume that there exists $\alpha_0 \in (0, 1)$ such that the polynomial $\alpha_0 f(x) + (1 - \alpha_0) g(x)$ is not Schur stable. Hence if follows that there exists $\alpha_1 \in < \alpha_0, 1 >$ such that

$$\alpha_1 f(1) + (1 - \alpha_1) g(1) \leq 0 \tag{8}$$

or

$$(-1)^n [\alpha_1 f(-1) + (1 - \alpha_1) g(-1)] \leq 0 \tag{9}$$

or

$$\alpha_1 f(\beta) + (1 - \alpha_1) g(\beta) = 0 \quad \text{and} \quad \alpha_1 f(\overline{\beta}) + (1 - \alpha_1) g(\overline{\beta}) = 0 \tag{10}$$

where $\beta\overline{\beta} = 1$, $Im(\beta) \neq 0$.

The inequalities (8), (9) are contradictions to the assumption $f(x), g(x) \in \tilde{S}_n$. From (10) and (5) it follows that $\det(S(\alpha_1 f + (1 - \alpha_1) g)) = 0$ and this is a contradiction to (7). Thus the sufficient condition for Schur stability is true. ■

Example 2.1 *For the polynomials*

$$f(x) = x^3 + 2x^2 + 3x + 4 \quad ; \quad g(x) = x^2 + 2x + 1$$

*we have:*

$$\tilde{g}(x) = 0x^3 + x^2 + 2x + 1,$$
$$H(f) = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & 2 \\ 0 & 0 & 4 \end{bmatrix}, \qquad H(\tilde{g}) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$
$$W = H^{-1}(f) H(\tilde{g}) = \begin{bmatrix} 1 & -1 & -\frac{1}{4} \\ -1 & 2 & \frac{1}{2} \\ 0 & 0 & \frac{1}{4} \end{bmatrix},$$
$$\lambda_1(W) = \frac{1}{4}, \quad \lambda_2(W) = \frac{1}{2}(3 - \sqrt{5}), \quad \lambda_3(W) = \frac{1}{2}(3 + \sqrt{5}).$$

Moreover, the polynomials $f(x)$, $g(x)$ are Hurwitz stable. Hence and with Theorem 2.3 it follows that the polynomial $\alpha f(x)+(1-\alpha)g(x)$ is Hurwitz stable for all $\alpha \in< 0,1 >$.

EXAMPLE 2.2 *For the polynomials*

$$f(x) = x^3 + 2x^2 + 3x + 4 \quad ; \quad g(x) = \frac{1}{4}x + 1$$

*we have:*

$$\tilde{g}(x) = 0x^3 + 0x^2 + \frac{1}{4}x + 1,$$

$$H(f) = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & 2 \\ 0 & 0 & 4 \end{bmatrix}, \qquad H(\tilde{g}) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$W = H^{-1}(f)H(\tilde{g}) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{8} & \frac{1}{4} \\ 1 & \frac{1}{4} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{4} \end{bmatrix},$$

$$\lambda_1(W) = \frac{1}{4}, \quad \lambda_2(W) = 0, \quad \lambda_3(W) = -\frac{1}{4}.$$

Because of $\lambda_3(W) = -\frac{1}{4}$ and from Theorem 2.3 it follows that the convex combination $C(f,g)$ is not Hurwitz stable. Indeed, the polynomial $\alpha f(x)+(1-\alpha)g(x)$ for $\alpha = 0.01$ has the root with positive real part.

## References

ACKERMANN, J. and BARMISH, B.R. (1988) Robust Schur stability of a polytope of polynomials. *IEEE Trans. Automat. Contr.* **33** (10), 984-986.

BARTLETT, A.C., HOLLOT, C.V. and HUANG, L. (1988) Root locations of an entire polytope of polynomials: It suffices to check the edges. *Mathematics of Control, Signals and Systems* **1**, 61-71.

BIAŁAS, S. (1985) A necessary and sufficient condition for the stability of convex combinations of stable polynomials or matrices. *Bull. Polish Acad. Sci., Tech. Sci.* **33** (9-10), 473-480.