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Influence of a boundary perforation on the Dirichlet energy

by

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Abstract: We consider some singular perturbations of the boundary of a smooth domain. Such domain variations are not differentiable within the classical theory of shape calculus. We mimic the *topological asymptotic* and we derive an asymptotic expansion of the shape function in terms of a size parameter. The two-dimensional case of the Dirichlet energy is treated in detail. We give a full theoretical proof as well as a numerical confirmation of the results.

Keywords: singular shape perturbation, shape derivatives, topological derivatives, multi-scale asymptotic expansion.

1. Introduction

The classical shape calculus presented, for example, in Murat, Simon (1977), Sokołowski, Żochowski (1999), Delfour (2001), is based on a perturbation approach in functional space of diffeomorphisms. This requires some regularity on the class of domains to be considered: for example $\mathcal{C}^1$-deformations of the boundary of small $\mathcal{C}^1$-norm. A deformation of small $\mathcal{L}^\infty$-norm cannot be seen as perturbation in that framework even if the Hausdorff distance between the two domains is by definition small. Another limitation of the classical shape calculus is the impossibility to deal with changes of topology. The so-called "topological asymptotic" (Masmoudi, 2002; Garreau, 2001; Lewiński, Sokołowski, 1999; Nazarov, Sokołowski, 1994) has been introduced to deal with the possibility of nucleations. The question this method addresses is the following: How does a shaping function behave when a hole of radius ε is dug at a fixed point M inside a body? One should notice that the small parameter ε is then a physical size parameter and not a pseudo-time (or a distance in spaces of diffeomorphisms) like in the classical methods.

In this paper, we consider the same question on a model shaping function, except that the point M lies on the boundary of the domain. Hence, the problem we consider in this paper is a singular boundary perturbation. Such problems have been studied by Maz'ya and Nazarov (1988) in the situation where the material is removed at a corner point. Our geometry is a limit case of the latter; we present here an alternative method to solve the problem with the tools of classical shape calculus. Our way to deal with it is directly inspired of the work of Sokołowski (Lewiński, Sokołowski, 1999; Sokołowski, Żochowski, 1999). A similar problem where angles are rounded was considered in Samet (2003) with a different approach. Our work has two main motivations: on the one hand, to generalize the topological asymptotic to the boundary case and, on the other hand to consider a singular case where the classical shape calculus is not directly operational.

More precisely, let f be a $\mathcal{C}^\infty$-function with compact support in $\mathbb{R}^2$. As the shaping function, we consider the Dirichlet energy J on the bounded domains Ω of class $\mathcal{C}^\infty$ in $\mathbb{R}^2$ such that $\mathrm{Supp} f \subset \Omega$. The Dirichlet energy is defined as

$$J(\Omega) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega f u = -\frac{1}{2}\int_\Omega |\nabla u|^2, \tag{1}$$

where u is the solution in $\mathcal{H}^1_0(\Omega)$ of the Poisson's equation $-\Delta u = f$ in Ω.

The main originality of the deformations we consider is their scale: let Ω_0 be an admissible domain, we introduce a scale parameter $\varepsilon \in (0, \varepsilon_0)$ and a reference smooth domain denoted by ω. For convenience, we assume that ω is star-shaped with respect to its gravity center O chosen as a point in the boundary $\partial\Omega_0$ of Ω_0. We denote by $\varepsilon\omega$ the image of ω by the homothety of center O and ratio ε. The perturbed domain Ω_ε is defined as

$$\Omega_\varepsilon = \Omega_0 \setminus \varepsilon\omega. \tag{2}$$

Fig. 1 makes explicit the geometrical setting.

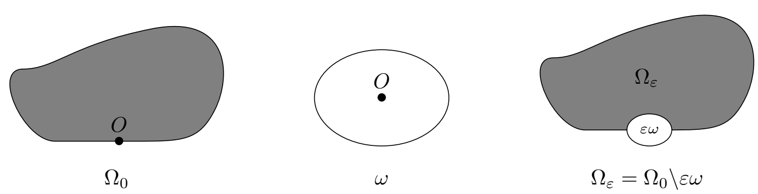

Figure 1. The geometrical setting

For a fixed ε, this deformation is not smooth as angular points appear at the intersection of $\partial\omega$ and $\partial\Omega_0$. In the Hausdorff sense, this is, however, of order ε. Hence it is a perturbation of the identity in this weak sense but not in any smooth sense. This means that the classical differential shape calculus can not provide Taylor-like formula in order to describe the behavior of $\varepsilon \mapsto J(\Omega_\varepsilon)$ for small ε. Our goal in this work is to obtain an asymptotical expansion of $J(\Omega_\varepsilon)$ starting from $J(\Omega_0)$.

The leading term of the asymptotics depends of the shaping function J. Let us consider two simple cases: the area and the perimeter. It is clear that the leading term is of order one for the perimeter and of order two for the area. This fact shows that the parameter ε is not appropriate to the classical shape calculus since both the area and the perimeter are differentiable with respect to the shape.

This paper is organized in the following way. first we establish an asymptotic expansion of the Dirichlet energy with respect to ε. This is done in two steps: first we derive an asymptotic expansion of the solution u_ε of Poisson equation inside Ω_ε; then we apply this result to obtain the behavior of the cost function. The complete, the proof of the expansion is presented in the third section. In the last part of this work, we present some numerical work to illustrate the results of Section 2.

2. The asymptotic expansion

2.1. Asymptotic expansion of the state function

This section is devoted to the asymptotic expansion in powers of ε of the solution u_ε to the Poisson's equation $-\Delta u_\varepsilon = f$ in $\mathcal{H}_0^1(\Omega_\varepsilon)$, starting from u_0, solution of the same equation in Ω_0. This is a now classical question (see Lewiński, Sokołowski, 1999) and we follow the method introduced in that paper; the complete expansion is written in Theorem 2.1 and justified in the proof of that result (see Section 3).

For convenience and in order to simplify the computations, we assume for a while that the boundary $\partial\Omega_0$ is flat around O. The general smooth case is much more complicated. The origin is chosen as O, the axis are taken as the tangent and normal to Ω_0 at O oriented so that Ω_0 locally lies in the upper half plane around O.

Even a localized perturbation of the domain induces a variation of the solution of Poisson's equation in the whole domain. This variation is not supported locally around O, but nevertheless mainly concentrated around it. Hence, the first step is to consider a blow-up around O – that is the center of the hole dug in the domain. We introduce the scaled (or *fast*) variable $y = x/\varepsilon$ (here $x = (x_1, x_2)$ and $y = (y_1, y_2)$ belong to $\mathbb{R}^2$). This canonical change of variables maps the ball $B(0, \varepsilon)$ into the unit ball and introduces the right scale to study our equations independently of ε.

Since ω is star-shaped with respect to O, its boundary has a parametrization $(\rho(\theta), \theta)$, $\theta \in [0, 2\pi]$ in polar coordinates. The function ρ is non-negative, and smooth because of the regularity assumption on Ω_0. Let $\partial\omega^+$ denote $\partial\omega \cap \{x_2 > 0\}$. To fix the scale we assume that $\rho(0) = \rho(2\pi) = 1$. With passing to the limit as $\varepsilon \to 0$, Ω_ε tends in this blow-up to the limit domain

$$\Omega_\infty = \big\{(r,\theta), r > \rho(\theta) \text{ and } \theta \in (0,\pi)\big\}. \tag{3}$$

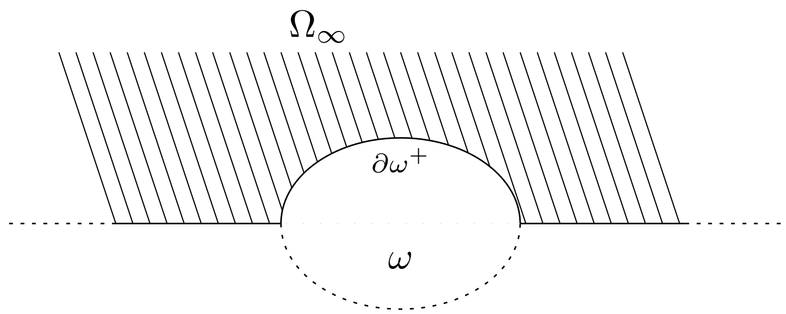

Figure 2. The limit domain

We consider the new problem satisfied by the difference $u = u_\varepsilon - u_0$:

$$\begin{aligned} -\Delta u &= 0 \text{ in } \Omega_\varepsilon, \\ u &= 0 \text{ on } \partial\Omega_\varepsilon \setminus \varepsilon\partial\omega^+, \\ u &= -u_0(\varepsilon y) \text{ on } \varepsilon\partial\omega^+. \end{aligned}$$

From the regularity assumptions on both Ω_0 and f, u_0 is known to belong to $\mathcal{C}^\infty(\overline{\Omega_0})$. Therefore, we can write a Taylor formula for u_0: for $x = \varepsilon y \in \partial\Omega_\varepsilon$, we get:

$$\begin{aligned} [u_\varepsilon - u_0](\varepsilon y) = -u_0(\varepsilon y) &= -\left[u_0(O) + \sum_{i=1}^{n} \frac{1}{i!} D^{(i)} u_0(O)[\varepsilon y, \dots, \varepsilon y]\right] + o(\varepsilon^n) \\ &= -\sum_{i=1}^{n} \varepsilon^i w_i(y) + o(\varepsilon^n), \end{aligned}$$

with an obvious definition of w_i. Since $-\Delta(u_\varepsilon - u_0) = 0$ in Ω_ε, the difference $u_\varepsilon - u_0$ is the harmonic extension of its trace on $\partial\Omega_\varepsilon$. The idea is to approximate this harmonic function by harmonic extensions of the approximated boundary conditions. Precisely, we have the following lemma. It can be proven by symmetry with respect to the line $x_2 = 0$ and the image method (see Tychonoff, Samarski, 1963, or Nédélec, 2000, for details).

Lemma 2.1 *There exists a unique function V_i defined on Ω_∞ by*

$$-\Delta V_i = 0 \textit{ in } \Omega_\infty, \tag{4}$$

and the boundary conditions:

$$
\begin{aligned}
V_i(y) &= 0 \text{ on } \partial\Omega_\infty \setminus \partial\omega^+, &(5)\\
V_i(y) &= w_i \text{ for } y \in \partial\omega^+, &(6)
\end{aligned}
$$

with the following expansion as an asymptotic series at infinity (with smooth functions ψ_n^i and $\mathbf{\Psi}_n^i$)

$$
V_i(y) \sim \sum_{n\geq 1} \psi_n^i(\theta)|y|^{-n} \quad \text{and} \quad \nabla V_i(y) \sim \sum_{n\geq 2} \mathbf{\Psi}_n^i(\theta)|y|^{-n} \quad \text{as } |y| \to +\infty. \quad (7)
$$

The functions (V_i), called *profiles*, describe the behavior of u_ε in the neighborhood of O. Since they are defined in the infinite domain Ω_∞, we need to truncate them: let $\chi : \mathbb{R}^2 \to \mathbb{R}$ be a $\mathcal{C}^\infty$-cut-off function such that

$$
|x| \leq 1/2 \Rightarrow \chi(x) = 1 \quad \text{and} \quad |x| \geq 1 \Rightarrow \chi(x) = 0. \quad (8)
$$

We now state the main result of this section: it is a two-scale asymptotic expansion of u_ε at every order. In fact, we need only the order two version but for its proof we use a bootstrapping method that requires the complete expansion.

Theorem 2.1 ***(Complete expansion of the state function)*** *Let Ω_0 be a $\mathcal{C}^\infty$ admissible domain with $O \in \partial\Omega_0$. For any admissible reference domain ω and any $\varepsilon \in (0, \varepsilon_0)$, we define Ω_ε by (2) and u_ε as the solution in $\mathcal{H}_0^1(\Omega_\varepsilon)$ of the Poisson's equation $-\Delta u_\varepsilon = f$.*

Then, for all $n \in \mathbb{N}$, there exists a function z_n^ε defined on Ω_ε such that

$$
u_\varepsilon(x) = u_0(x) + \chi(x)\left[\sum_{i=1}^{n} \varepsilon^i V_i\left(\frac{x}{\varepsilon}\right)\right] + \sum_{i=1}^{n-1} \varepsilon^{i+1} u^i(x) + z_n^\varepsilon(x), \quad (9)
$$

where the profile V_i solves the Dirichlet problem (4)-(6). The functions u^i are solutions of

$$
\begin{aligned}
-\Delta u^i &= \varphi_i \text{ in } \Omega_\varepsilon, &(10)\\
u^i(x) &= 0 \text{ on } \partial\Omega_\varepsilon, &(11)
\end{aligned}
$$

where φ^i arises from derivatives of the cut-off function, see (31) and Remark 2.1.

Moreover, if $\Phi_{\varepsilon,\varepsilon_0}$ denotes a given diffeomorphism mapping Ω_ε into Ω_{ε_0}, there exists a constant C, independent of ε such that

$$
\|z_n^\varepsilon \circ \Phi_{\varepsilon,\varepsilon_0}^{-1}\|_{\mathcal{H}^1(\Omega_{\varepsilon_0})} \leq C\,\varepsilon^{n+1}. \quad (12)
$$

We first give some remarks and comments on this result.

REMARK 2.1 *The remainder z_n^ε and the functions u^i depend of the choice of the cut-off function χ. The terms u^i are corrector terms that compensate for the cut-off effect away of the origin point, they are not of the same nature as the singular profiles which are intrinsic and not χ dependant. Let us make precise the construction of the first one: by definition, we have*

$$\Delta z_1^\varepsilon = \varepsilon \Delta\chi(x) V_1(\frac{x}{\varepsilon}) + 2\langle \nabla\chi(x) \nabla V_1(\frac{x}{\varepsilon})\rangle. \tag{13}$$

Thanks to the expansion (7) at infinity of $V_1(y)$, we obtain $\Delta z_i^\varepsilon = \varepsilon^2\varphi_1 + o(\varepsilon^2)$. The term φ_1 is corrected by u^1 while the leading terms will be handled at the next steps.

REMARK 2.2 *The fact that these functions are controlled in $\mathcal{H}_0^1(\Omega_{\varepsilon_0})$ independently of ε is crucial for the applications to shape calculus we have in mind. Therefore, we have to transport the functions on a domain independent of the parameter ε in order to remove all dependency with respect to ε of our upper bounds even hidden in the functional spaces.*

REMARK 2.3 *Another important implication of this result for the rest of this work is the following constatation. Whereas the state function is continuous with respect to the parameter ε, its gradient is not continuous: The main-order discontinuity is completely described by the first singular profile V_1. Hence this first singular profile will appear for shape function involving the gradient of the state.*

For the application to the shape functional we consider in this paper, we only need the second order expansion. We can be more explicit for the functions w_1 and w_2 involved in the problems defining the profiles: For $y \in \partial\omega^+$, we have

$$w_1(y) = -\langle \nabla u_0(O), y\rangle = -\varepsilon\partial_n u_0(O) y_2,$$
$$w_2(y) = -\frac{\varepsilon^2}{2} D^2 u_0(O)(y,y).$$

Sketch of the proof of Theorem 2.1. The proof itself is postponed to Section 3. First we rewrite everything on the fixed domain Ω_{ε_0}. Let $\Phi_{\varepsilon,\varepsilon_0}$ be a diffeomorphism mapping Ω_ε into Ω_{ε_0}. We set

$$z_n^\varepsilon(x) = u_\varepsilon(x) - u_0(x) - \chi(x)\sum_{i=1}^n \varepsilon^i V_i\left(\frac{x}{\varepsilon}\right) - \sum_{i=1}^{n-1} \varepsilon^{i+1} u^i(x). \tag{14}$$

Our task is to find an estimation of $\tilde{z}_n^\varepsilon = z_n^\varepsilon \circ \Phi_{\varepsilon,\varepsilon_0}^{-1}$ in $\mathcal{H}^1(\Omega_{\varepsilon_0})$. This estimate must be uniform with respect to ε. This will be done through the use of the classical estimates for elliptic equations in Sobolev spaces. If we apply the Laplace operator to the rest z_n^ε we get

$$\begin{aligned} \Delta z_n^\varepsilon &= g_{n,\chi} \text{ in } \Omega_\varepsilon, \\ z_n^\varepsilon &= 0 \text{ on } \partial\Omega_\varepsilon \setminus \varepsilon\partial\omega^+, \\ z_n^\varepsilon &= w_n \text{ on } \varepsilon\partial\omega^+. \end{aligned}$$

The problem solved by $\tilde{z}_n^\varepsilon$ writes

$$
\begin{aligned}
\mathcal{L}_\varepsilon v &= g_{n,\chi} \circ \Phi_{\varepsilon,\varepsilon_0}^{-1} \text{ in } \Omega_{\varepsilon_0}, \\
v &= 0 \text{ on } \partial\Omega_{\varepsilon_0} \setminus \varepsilon_0 \partial\omega^+, \\
v &= w_n \circ \Phi_{\varepsilon,\varepsilon_0}^{-1} \text{ on } \varepsilon_0 \partial\omega^+,
\end{aligned} \tag{15}
$$

where $\mathcal{L}_\varepsilon$ is the elliptic operator obtained from $-\Delta$ after transport (see Section 3 for its expression) and w_n is the rest in the boundary condition after the n-th order Taylor approximation

$$
w_n(x) = -u_0(x) - \left[\sum_{i=1}^{n} \varepsilon^i V_i \left(\frac{x}{\varepsilon} \right) \right] = \varepsilon^n r_n(x, \varepsilon), \tag{16}
$$

$$
\nabla w_n(x) = -\nabla u_0(x) - \left[\sum_{i=1}^{n} \varepsilon^{i-1} \nabla V_i \left(\frac{x}{\varepsilon} \right) \right] = \varepsilon^{n-1} R_n(x, \varepsilon), \tag{17}
$$

where r_n and R_n are smooth bounded functions with limit 0 when $x \to 0$. The right-hand side in (15) arises from the cut-off function; it is supported in the annulus where the derivatives of the truncation function χ are supported and satisfies (see Section 3 for details)

$$
\|g_{n,\chi}\|_{\mathcal{L}^2(\Omega_\varepsilon)} = O(\varepsilon^{n+1}).
$$

To use classical estimates, we precise $\Phi_{\varepsilon,\varepsilon_0}$ to obtain uniform constants of ellipticity and continuity for $\mathcal{L}_\varepsilon$. Then we need to obtain uniform estimates for the $\mathcal{H}^{1/2}$-norm of the trace and for the $\mathcal{H}^1$-norm of the right hand side. The estimates obtained in this way are sub-optimal and we use a bootstrap method to recover the desired estimates.

2.2. Asymptotic expansion of the shaping function

We consider the Dirichlet energy of this problem that is the functional J defined on the class of open subsets Ω of $\mathbb{R}^2$ by (1). Considering the perturbations Ω_ε defined formerly, we seek an asymptotic expansion of the real-valuated function

$$
j(\varepsilon) = J(\Omega_\varepsilon)
$$

around 0. Obviously, the classical differential shape calculus cannot be applied directly. However, if we fix for a while $\delta > 0$, the Taylor expansion of $j(\delta + \varepsilon)$ with respect to ε can be computed. Then a continuity argument allows to pass to the limit $\varepsilon \to 0$. In the following lines, we will use indifferently cartesian or polar coordinates.

The deformation field. Let R be the maximal size of acceptable perturbations. Let ξ be a cut-off function distinct from χ and depending of ε, such that

$$|x| < \varepsilon/3 \text{ or } |x| > 2R/3 \implies \xi(x) = 0 \text{ and } \varepsilon/2 < |x| < R/2 \implies \xi(x) = 1.$$

We define the deformation field in polar coordinates as $\mathbf{V} = \delta\rho(\theta)\mathbf{u}_r$. Here, $\mathbf{u}_r$ is the unit radial vector of the polar coordinates. The cut-off function ξ is needed to first avoid the singularity at the origin and to leave invariant the boundary $\partial\Omega_0$ away from the point O. In the annulus $\varepsilon/2 < r < R/2$ where the deformations take place, the vector field is constant along the radial lines. The family of deformed domains is then $T_t[\Omega(\varepsilon)] = \Omega_{\varepsilon+t}$ where T_t stands for the flow of the vector field $\mathbf{V}$. Hence for $t = \delta$ we have $\Omega(\delta) = \Omega_{\varepsilon+\delta}$.

The starting point. We have:

$$j(\varepsilon + \delta) = j(\varepsilon) + \int_0^\delta DJ(\Omega(t); \mathbf{V})dt, \tag{18}$$

where $DJ(\Omega(t); \mathbf{V})$ is the classical shape derivative (see Dambrine, Sokołowski, Żochowski, 2003, for more details on the justification of the derivation in the smooth case). An additional difficulty is caused here by the presence of two singular points at the intersection of $(\varepsilon + t)\partial\omega^+$ and $\partial\Omega_0$. The angles in the domain $\Omega(t)$ are of opening less than π. Therefore the solutions $u_{\varepsilon+t}$ are $\mathcal{H}^2(\Omega_{\varepsilon+t})$ (see Nazarov, Plamenevsky, 1994, Grisvard, 1985, Castabel, Douge, 1994, for details on equations in domain with corners). Hence $\langle\nabla u_{\varepsilon+t}, \mathbf{V}\rangle \in \mathcal{H}^1(\Omega_{\varepsilon+t})$ and $\langle\nabla u_{\varepsilon+t}, \mathbf{V}\rangle \in \mathcal{H}^{1/2}(\partial\Omega_{\varepsilon+t})$ and that is enough to allow this differentiation.

$$DJ(\Omega(t); \mathbf{V}) = -\frac{1}{2}\int_{\partial\Omega_{\varepsilon+t}} |\nabla u_{\varepsilon+t}|^2 \, \langle\mathbf{V}, \mathbf{n}(t)\rangle \; d\sigma_{\partial\Omega(t)}.$$

Notice that, in fact, the integrand vanishes outside of the ball $B(O, \varepsilon + t)$. Let us make explicit this integral. The normal component of $\mathbf{V}(t)$ writes simply

$$\langle\mathbf{V}, \mathbf{n}(t)\rangle = \frac{1}{\sqrt{\rho^2 + (\rho')^2}} \langle\delta\mathbf{u}_r, -\rho\mathbf{u}_r + \rho'\mathbf{u}_\theta\rangle = -\frac{\delta\rho^2}{\sqrt{\rho^2 + (\rho')^2}}.$$

We now turn to the term in gradient of $u_{\varepsilon+t}$. We know from Theorem 2.1 that for $y \in \partial\omega^+$

$$\nabla u_{\varepsilon+t}((\varepsilon + t)y) = [\nabla u_0(O) + o(\varepsilon + t)] + [\nabla V_1(y) + o(\varepsilon + t)] \,.$$

Since the problem solved by u_0 has homogeneous Dirichlet boundary conditions, the gradient is normal to the boundary: $\nabla u_0(O) = \partial_n u_0(O)\mathbf{n}$ and we get for $x \in (\varepsilon + t)\partial\omega^+$

$$\nabla u_{\varepsilon+t}(x) = \begin{pmatrix} \partial_1 V_1\left(\dfrac{x}{\varepsilon+t}\right) \\ \\ \partial_n u_0(0) + \partial_2 V_1\left(\dfrac{x}{\varepsilon+t}\right) \end{pmatrix} + o(\varepsilon+t).$$

In order to regroup all the dependency in $\partial_n u_0$, we introduce the normalized profile $\mathcal{V}_1$ defined as $V_1/\partial_n u_0(O)$; it solves

$$-\Delta\mathcal{V}_1 = 0 \text{ in } \Omega_\infty, \tag{19}$$

$$\mathcal{V}_1(y) = 0 \text{ on } \partial\Omega_\infty \setminus \partial\omega^+; \tag{20}$$

$$\mathcal{V}_1(y) = y_2 \text{ for } y \in \partial\omega^+ \text{ i.e. } \mathcal{V}_1(\rho(\theta),\theta) = \rho(\theta)\sin\theta \text{ for } \theta \in (0,\pi). \tag{21}$$

Then a straightforward computation leads to

$$|\nabla u_{\varepsilon+t}(x)|^2 = |\partial_n u_0(O)|^2 \left[\left(1+\partial_2\mathcal{V}_1\left(\frac{x}{\varepsilon+t}\right)\right)^2 + \left(\partial_1\mathcal{V}_1\left(\frac{x}{\varepsilon+t}\right)\right)^2\right] + o(\varepsilon+t).$$

To simplify the notations in the following lines, we confound $\mathcal{V}_1(\theta)$ for $\mathcal{V}_1(y)$ if $y = (\rho(\theta),\theta) \in \partial\omega^+$. For convenience, we rewrite the shape derivative as an integral on a fixed (with respect to the pseudo-time t) boundary $\partial\omega^+$. First, we notice that the dilatation of ratio $1/(\varepsilon+t)$ maps $(\varepsilon+t)\partial\omega^+$ onto $\partial\omega^+$. The arc-length $d\sigma_{\partial\omega^+}$ is given by $d\sigma = \left(\rho^2(\theta) + (\rho')^2(\theta)\right)^{1/2} d\theta$. We get:

$$\begin{aligned} DJ(\Omega_{\varepsilon+t};\mathbf{V}) &= -\frac{1}{2}\int_{(\varepsilon+t)\partial\omega^+} |\nabla u_{\varepsilon+t}|^2 \, \langle \mathbf{V},\mathbf{n}(t)\rangle \, d\sigma_{(\varepsilon+t)\partial\omega^+}; \\ &= -\frac{|\partial_n u_0(O)|^2}{2}\int_{\partial\omega^+} \Big[(1+\partial_2\mathcal{V}_1(y))^2 + (\partial_1\mathcal{V}_1(y))^2 \\ &\qquad\qquad + o(\varepsilon+t)\Big] \langle \mathbf{V},\mathbf{n}(t)\rangle \, (\varepsilon+t) d\sigma_{\partial\omega^+}, \\ &= \frac{(\varepsilon+t)|\partial_n u_0(O)|^2}{2}\int_0^\pi \Big[(1+\partial_2\mathcal{V}_1(\theta))^2 + (\partial_1\mathcal{V}_1(\theta))^2 + o(\varepsilon+t)\Big]\rho^2(\theta)d\theta, \\ &= \frac{(\varepsilon+t)|\partial_n u_0(O)|^2}{2}\left[A(\rho) + o(\varepsilon+t)\right], \end{aligned}$$

Here $A(\rho)$ is a shape-dependent number, an important point is that it does not depend on the pseudo-time t. This is caused by the particular choice of the deformation field that forces all the deformed boundaries to be dilation of the same original one. If this property of self-similarity is not fulfilled because of the assumption of flatness of $\partial\Omega_0$ around O, the first singular profile $\mathcal{V}_1$ would change during the deformations from ε to $\varepsilon+\delta$ and the computation would be much more delicate to perform. Hence we have:

$$\begin{aligned} j(\varepsilon+\delta) &= j(\varepsilon) + \frac{|\partial_n u_0(O)|^2}{2} \int_0^\delta \left(A(\rho) + o(\varepsilon+t)\right)(\varepsilon+t)dt \\ &= j(\varepsilon) + \delta^2 |\partial_n u_0(O)|^2 \left[\frac{A(\rho)}{4} + o(\varepsilon+\delta)\right]. \end{aligned}$$

Now we pass to the limit $\varepsilon \to 0$ in this formula. Using the well-known continuity of this functional with respect to the shape (see Delfour, Zolésio, 2001, for example) to see that $j(\varepsilon) \to j(0)$ when $\varepsilon \to 0$, we get the wanted expansion.

THEOREM 2.2 ***(Asymptotic behavior of the shape function)*** *The shaping function J behaves like*

$$J(\Omega_\delta) = J(\Omega_0) + \delta^2 |\partial_n u_0(O)|^2 \mathcal{A}(\omega) + o(\delta^2) \tag{22}$$

with

$$\mathcal{A}(\omega) = \frac{1}{4} \int_0^\pi \left[\left(1 + \partial_2 \mathcal{V}_1(\theta)\right)^2 + \left(\partial_1 \mathcal{V}_1(\theta)\right)^2\right] \rho^2(\theta) d\theta. \tag{23}$$

REMARK 2.4 *The quantity $\mathcal{A}(\omega)$ depends only on the geometry of the hole we dug and not at all on the position on the center of the hole and of the state function. It plays exactly the same role as the so-called polarization matrix. By analogy, we call it the polarization number.*

REMARK 2.5 *Formulæ* (22) *and* (23) *correspond to the results stated in Theorem 4.1 in Maz'ya, Nazarov (1988), though the singular profiles used by these authors are not written in the same way. Our approach underlines the links between the shape gradient – which is not defined for the present singular perturbation of the domain – and the leading term in the asymptotic of the functional. In Maz'ya, Nazarov (1988), the term $\partial_n u_0(O)$ derives from the expansion into singular functions. It turns out that the Taylor expansion of u_0 at point O coincides with the singular expansion at a corner point in the limit case where the opening equals π; hence $\partial_n u_0(O)$ is nothing but the first singular coefficient of u_0 at O.*

3. Complete proof of Theorem 2.1

As the leading line of that proof has been explained in Section 2, we just provide the complete technical arguments in this section. For convenience C will denote any non negative constant (independent of ε).

Construction of the diffeomorphism $\Phi_{\varepsilon,\varepsilon_0}$ and geometrical preliminaries. We take advantage of the geometry and hence we use the polar coordinates. We search this diffeomorphism $\Phi_{\varepsilon,\varepsilon_0}$ under the form

$$\Phi_{\varepsilon,\varepsilon_0}(re^{i\theta}) = P(r,\theta)e^{i\theta}.$$

For r big enough, we search $P(r)=r$ and we require $P(\varepsilon\rho(\theta),\theta)=\varepsilon_0\rho(\theta)$. The idea is to use an interpolation polynomial for the small r with conditions at $r=\varepsilon$ and a smooth connection up to order 2 to $P(r)=r$ at some point to be determined.

First consider the following fact of calculus. Let a,b,c be three real numbers such that $0<a<b<c$. The polynomial $P_{[a,b,c]}$ defined by

$$P_{[a,b,c]}(X)=\frac{b-a}{(a-c)^3}(X-c)^3+X.$$

satisfies the interpolation conditions:

$$P_{[a,b,c]}(a)=b,\quad P_{[a,b,c]}(c)=c,\quad P'_{[a,b,c]}(c)=1,\quad P''_{[a,b,c]}(c)=0.$$

Moreover, if $3b-2a<c$ then $\forall x\in[a,c]$,

$$1=P'_{[a,b,c]}(c)\geq P'_{[a,b,c]}(x)\geq P'_{[a,b,c]}(a)=\frac{2a+c-3b}{c-a}>0, \tag{24}$$

and $P_{[a,b,c]}$ is a bijection from $[a,c]$ into $[b,c]$. For any $\theta\in(0,\pi)$, we can choose $a=\varepsilon\rho(\theta)$, $b=\varepsilon_0\rho(\theta)$ and $c=3\varepsilon_0\|\rho\|_\infty:=R_0$ and satisfy to the condition $3b-2a<c$. Let $P(r,\theta)$ be $P_{[\varepsilon\rho(\theta),\varepsilon_0\rho(\theta),R_0]}(r)$. It writes:

$$P(r,\theta)=\frac{(\varepsilon_0-\varepsilon)\rho(\theta)}{\left(R_0-\varepsilon\rho\left(\theta\right)\right)^3}\left(r-R_0\right)^3+r. \tag{25}$$

We define a increasing function ϕ_ε on $[\varepsilon,+\infty)$ by

$$\phi_\varepsilon(r,\theta)=\begin{cases} P(r,\theta) & \text{if } r\in(\varepsilon,R_0],\\ r & \text{if } r\geq 2\varepsilon_0.\end{cases} \tag{26}$$

Let $\Phi_{\varepsilon,\varepsilon_0}$ be the diffeomorphism of $\mathbb{R}^2\setminus B(0,\varepsilon)$ into $\mathbb{R}^2\setminus B(0,\varepsilon_0)$ defined in polar coordinates by

$$\Phi_{\varepsilon,\varepsilon_0}(r,\theta)=\left(\phi_\varepsilon\left(r,\theta\right),\theta\right). \tag{27}$$

Far away from 0 (*i.e.* for $r>2\varepsilon_0$), $\Phi_{\varepsilon,\varepsilon_0}$ is nothing but the identity and therefore we get $\Phi_{\varepsilon,\varepsilon_0}(\Omega_\varepsilon)=\Omega_{\varepsilon_0}$. Moreover, we have $\Phi_{\varepsilon,\varepsilon_0}(\Omega_\varepsilon\cap B(0,R_0))=\Omega_{\varepsilon_0}\cap B(0,R_0)$.

To obtain bounds on the coefficients of $\mathcal{L}_\varepsilon$ we need the derivatives of $\Phi_{\varepsilon,\varepsilon_0}$. The non-trivial case is $|x|<R_0$. Let $x\in\Omega_\varepsilon\cap B(0,2\varepsilon_0)$ and let Ψ be the change of coordinates application that is $\Psi(r,\theta)=(r\cos\theta,r\sin\theta)$. We consider the diffeomorphism $\Phi_{\varepsilon,\varepsilon_0}=\Psi\circ\Phi_{\varepsilon,\varepsilon_0}\circ\Psi^{-1}$ to deal with Cartesian coordinates and we get at the point $x=(r\cos\theta,r\sin\theta)$:

$$\begin{aligned} D\Phi_{\varepsilon,\varepsilon_0}(x) &= D\Psi\left[\Phi_{\varepsilon,\varepsilon_0}\circ\Psi^{-1}(x)\right].D\Phi_{\varepsilon,\varepsilon_0}\left[\Psi^{-1}(x)\right].D\Psi^{-1}\left[x\right],\\ &= \begin{pmatrix}\cos\theta & -\phi_\varepsilon(r,\theta)\sin\theta\\ \sin\theta & \phi_\varepsilon(r,\theta)\cos\theta\end{pmatrix}\begin{pmatrix}\partial_r\phi_\varepsilon(r,\theta) & 0\\ \partial_\theta\phi_\varepsilon(r,\theta) & 1\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\dfrac{\sin\theta}{r} & \dfrac{\cos\theta}{r}\end{pmatrix}.\end{aligned}$$

Hence, we get

$$\det D\Phi_{\varepsilon,\varepsilon_0} = \frac{\phi_\varepsilon(r,\theta)\partial_r\phi_\varepsilon(r,\theta)}{r} \text{ and } \det D\Phi^{-1}_{\varepsilon,\varepsilon_0} = \frac{r}{\phi_\varepsilon(r,\theta)\partial_r\phi_\varepsilon(r,\theta)}.$$

By construction of ϕ_ε, we have both

$$\begin{aligned}&\frac{\varepsilon_0\rho(\theta)}{R_0} \leq \frac{\phi_\varepsilon(r,\theta)}{r} \leq \frac{R_0}{\varepsilon},\\ &\frac{(2\varepsilon-3\varepsilon_0)\rho(\theta)+R_0}{R_0-\varepsilon\rho(\theta)} \leq \partial_r\phi_\varepsilon \leq 1.\end{aligned}$$

Hence, we have the upper bounds,

$$|\det D\Phi_{\varepsilon,\varepsilon_0}| \leq \frac{R_0}{\varepsilon} \text{ and } |\det D\Phi^{-1}_{\varepsilon,\varepsilon_0}(\Phi_{\varepsilon,\varepsilon_0}(x))| \leq \frac{\varepsilon_0\rho(\theta)}{R_0-3\varepsilon_0\rho(\theta)}. \tag{28}$$

Moreover, in this proof, we use of the surfacic Jacobian $\Phi_{\varepsilon,\varepsilon_0} = \det(D\Phi_{\varepsilon,\varepsilon_0}\|^t (D\Phi_{\varepsilon,\varepsilon_0})^{-1}\mathbf{n}_\varepsilon\|)$ on the boundary of the holes. Here the boundaries are homothetic hence $\Phi_{\varepsilon,\varepsilon_0} = \varepsilon_0/\varepsilon$ the ratio of the dilatation.

Uniform ellipticity of $\mathcal{L}(\varepsilon)$. Taking advantage of the geometrical configuration, we write the problem (15) solved by $\tilde{z} = z\circ\Phi^{-1}_{\varepsilon,\varepsilon_0}$ in polar coordinates (we are only interested in the case $r < R_0$ where the operator $\mathcal{L}_\varepsilon$ is not the Laplace operator):

$$\left[(P')^2+\frac{(\rho')^2}{r^2}\right]\partial^2_{rr}\tilde{z}+\frac{1}{r^2}\partial^2_{tt}\tilde{z}+\left[P''+\frac{P'}{r}+\frac{\rho''}{r^2}\right]\partial_r\tilde{v} = g_\chi\circ\Phi^{-1}_{\varepsilon,\varepsilon_0} \text{ in } \Omega_{\varepsilon_0}\cap B(0,R_0). \tag{29}$$

By construction, P and its derivatives are uniformly bounded and there exist two constants λ and Λ such that for all $\varepsilon < \varepsilon_0$ we have

$$a_{i,j}(\varepsilon,x)\xi_i\xi_j \geq \lambda|\xi|^2 \text{ and } \sum |a_{i,j}(\varepsilon,x)|^2 \leq \Lambda^2$$

for all $x \in \Omega_{\varepsilon_0}$ and all $\xi \in \mathbb{R}^2$. This very classical result (ellipticity is preserved by transport) is simply caused by the continuity of the eigenvalues of the matrix $(a_{i,j})$ with respect to ε. Moreover, there exists also a third constant which dominates the coefficient in the order one derivatives of $\tilde{z}$.

Estimate of the boundary condition $\|\tilde{z}^\varepsilon_n\|_{\mathcal{H}^{1/2}(\Omega_{\varepsilon_0})}$. The natural way to control a norm in an $\mathcal{H}^{1/2}$ space on a boundary is to compute the $\mathcal{H}^1$-norm of well-chosen extension. This is not appropriate for this problem since the $\mathcal{H}^{1/2}$-norm is non-local and hence we can not take advantage of the support of $\tilde{z}_n$. The boundary term $\tilde{z}_n^\varepsilon = w \circ \Phi_{\varepsilon,\varepsilon_0}^{-1}$ of the problem (15) is a piecewise $\mathcal{C}^\infty$ continuous function and belongs to $\mathcal{H}^1(\partial\Omega_{\varepsilon_0})$. Since this trace vanishes outside $B(O,\varepsilon_0)$, the $\mathcal{H}^1$-norm is an integral over $\partial\Omega_{\varepsilon_0} \cap \partial B(0,\varepsilon_0)$ and we have $\|\tilde{z}_n^\varepsilon\|_{\mathcal{H}^{1/2}(\Omega_{\varepsilon_0})} \leq \|\tilde{z}_n^\varepsilon\|_{\mathcal{H}^1(\Omega_{\varepsilon_0})}$. We will estimate it to derive a bound on $\|\tilde{z}_n^\varepsilon\|_{\mathcal{H}}^{1/2}$. Inside $B(0,\varepsilon_0)$, that function is $\mathcal{C}^\infty$ and vanishes outside this ball, hence we get:

$$\begin{aligned}\|\tilde{z}_n^\varepsilon\|^2_{\mathcal{H}^1(\partial\Omega_{\varepsilon_0})} &= \int_{\partial\Omega_{\varepsilon_0}\cap\ \partial B(0,\varepsilon_0)} |\tilde{z}_n^\varepsilon|^2 + |\nabla \tilde{z}_n^\varepsilon|^2 \\ &= \int_{\partial\Omega_{\varepsilon_0}\cap\partial B(0,\varepsilon_0)} |w_n \circ \Phi_{\varepsilon,\varepsilon_0}^{-1}|^2 + |\nabla(w_n \circ \Phi_{\varepsilon,\varepsilon_0}^{-1})|^2.\end{aligned}$$

We transport this on the boundary $\partial\Omega_\varepsilon$ in order to use our assumptions. The expression of the transport of a tangential derivative can be found for example in the appendix to Dambrine, Sokołowski, Żochowski (2003).

$$\begin{aligned}&\|\tilde{z}_n^\varepsilon\|^2_{\mathcal{H}^1(\partial\Omega_{\varepsilon_0})} = \int_{\varepsilon\partial\omega^+} |w_n|^2 \Phi_{\varepsilon,\varepsilon_0} + I \text{ with} \\ &I = \int_{\varepsilon\partial\omega^+} |D\Phi_{\varepsilon,\varepsilon_0}^{-1}[\nabla w_n - \langle \nabla w_n, \mathbf{n}\rangle \mathbf{n} \\ &\qquad\qquad - \frac{1}{\|D\Phi_{\varepsilon,\varepsilon_0}^{-1} \neq \|^2} \langle D\Phi_{\varepsilon,\varepsilon_0}^{-1} \nabla w_n, D\Phi_{\varepsilon,\varepsilon_0}^{-1} \neq\rangle \neq]|^2 \Phi_{\varepsilon,\varepsilon_0}.\end{aligned}$$

Remember that $\Phi_{\varepsilon,\varepsilon_0} = \det\big(D\Phi_{\varepsilon,\varepsilon_0}\|\ {}^t(D\Phi_{\varepsilon,\varepsilon_0})^{-1} \neq \|\big) = \varepsilon_0/\varepsilon$ is the surfacic Jacobian and that $\neq$ denotes the unit normal vector to $\partial\Omega_\varepsilon$ pointing to the exterior. The first term of the sum is

$$\int_{\partial\Omega_\varepsilon\cap\partial B(0,\varepsilon)} |w|^2 \Phi_{\varepsilon,\varepsilon_0} = \int_{\varepsilon\partial\omega^+} \varepsilon^{2n} |r_n(x)|^2 \frac{\varepsilon_0}{\varepsilon} d\sigma = \varepsilon_0 \varepsilon^{2n} \|r_n\|^2_{L^\infty}.$$

By (17) and Cauchy-Schwarz inequality, we have

$$|I| \leq \int_{\varepsilon\partial\omega^+} C\varepsilon^{2(n-1)} \|R_n\|^2_\infty \frac{\varepsilon_0}{\varepsilon} \leq C\varepsilon_0 \|R_n\|^2_\infty \varepsilon^{2(n-1)}.$$

Hence we get

$$\|\tilde{z}_n^\varepsilon\|_{\mathcal{H}^{1/2}(\Omega_{\varepsilon_0})} \leq C\varepsilon^{n-1}. \tag{30}$$

Estimation of the norms of $g_{n,\chi}$. We give here more details on the definition of the correctors u^i. We will show, by induction, that $g_{n,\chi} = \Delta z_n^\varepsilon$ has an expansion in integer powers of ε. This is clear for $n = 1$ thanks to (13) and (7). Let us assume the following expansion for $\Delta z_{n-1}^\varepsilon$:

$$\Delta z_{n-1}^\varepsilon = \varepsilon^n \varphi_{n-1} + \varepsilon^{n+1} \varphi_{n-1}^{[1]} + \varepsilon^{n+2} \varphi_{n-1}^{[2]} + \cdots, \tag{31}$$

with functions $\varphi_{n-1}^{[i]}$ independent of ε. Let us now consider z_n^ε: By construction, we obviously have

$$\begin{aligned}\Delta z_n^\varepsilon &= \Delta z_{n-1}^\varepsilon - \varepsilon^n \Delta u^{n-1} - \varepsilon^{n-1}\Delta(\chi\, V_i(\frac{x}{\varepsilon}))\\ &= \sum_{i\geq 1} \varepsilon^{n+i}\varphi_{n-1}^{[i]} - \varepsilon^n[\Delta\chi V_n(\frac{x}{\varepsilon}) + 2\varepsilon^{-1}\nabla\chi\cdot\nabla V_n(\frac{x}{\varepsilon})].\end{aligned}$$

Since $\Delta\chi$ and $\nabla\chi$ are supported in an annulus $R_1 < |x| < R_2$, the second term of the above right-hand side is governed by the behavior of the profiles at infinity. Indeed, thanks to relations (7) we obtain the following expansion:

$$\Delta z_n^\varepsilon = \sum_{i\geq 1}\varepsilon^{n+i}\varphi_{n-1}^{[i]} - \sum_{i\geq 1}\varepsilon^{n+i}\zeta_n^{[i]},$$

which yields to (31) at rank n, with $\varphi_n = \varphi_{n-1}^{[1]} + \zeta_n^{[1]}$. We can deduce the estimate for $g_{n,\chi}$:

$$\|g_{n,\chi}\|_{\mathcal{L}^2(\Omega_\varepsilon)} \leq C(\chi)\varepsilon^{n+1}. \tag{32}$$

Now the real right-hand side $g_\chi \circ \Phi_{\varepsilon,\varepsilon_0}$ has the same L^2 norm as g_χ since $\Phi_{\varepsilon,\varepsilon_0}$ is nothing but the identity on the support of g_χ. Hence the jacobian is just 1 and

$$\|g_{n,\chi}\circ\Phi_{\varepsilon,\varepsilon_0}\|_{\mathcal{L}^2(\Omega_{\varepsilon_0})} \leq C(\chi)\varepsilon^{n+1}. \tag{33}$$

The bootstrap. Applying classical elliptic *a priori* estimates in Ω_{ε_0} to the solution of (15), we obtain from (30) and (33) the first estimate

$$\forall n\in\mathbb{N},\quad \|\tilde{z}_n^\varepsilon\|_{\mathcal{H}^1(\Omega_{\varepsilon_0})} \leq C(\Omega_{\varepsilon_0})\varepsilon^{n-1}.$$

We also have

$$\begin{aligned}z_n^\varepsilon(x) = z_{n+2}^\varepsilon(x) - \varepsilon^{n+1}\chi(x)V_{n+1}\left(\frac{x}{\varepsilon}\right) - \varepsilon^{n+2}\chi(x)V_{n+2}\left(\frac{x}{\varepsilon}\right) - \varepsilon^{n+1}u^n(x)\\ +\varepsilon^{n+2}u^{n+1}(x).\end{aligned}$$

Using uniform estimates on both the profiles and the corrector, we obtain

$$\|\tilde{z}_n^\varepsilon\|_{\mathcal{H}^1(\Omega_{\varepsilon_0})} \leq C(\Omega_{\varepsilon_0})\varepsilon^{n+1} + C\varepsilon^{n+1} + C\varepsilon^{n+2} + C\varepsilon^{n+1} + C\varepsilon^{n+2} \leq C\varepsilon^{n+1}.$$

This is the expected upper bound (12) and the proof is completed. ■

4. The particular case of circular holes

The general case developed in Section 2 applies to the particular case where ω is a ball. However, in this specific case, the computations can be carried out completely: The singular profiles and the polarization number $A(\rho)$ can be computed explicitly. We think these results can have practical use hence we present them in this section. This explicit computation will be used for the numerical validation as well.

The singular profiles. We introduce $\Omega_\infty = \{y = (y_1, y_2), y_2 > 0, |y| > 1\}$, the limit domain. From the regularity assumptions on both Ω_0 and f, u_0 is known to be $\mathcal{C}^\infty(\overline{\Omega_0})$. Therefore, we get that for all $n \geq 1$:

$$u_0(\varepsilon y) = \sum_{i=1}^{n} \varepsilon^i w_i(y) + \varepsilon^n z_n^\varepsilon(\varepsilon, y). \tag{34}$$

The functions w_k are defined by the derivatives of u_0. Namely, one has for the fist orders:

$$w_1(y) = \langle \nabla u_0(0), y \rangle \text{ and } w_2(y) = \frac{1}{2} D^2 u_0(0).[y, y].$$

Using the polar coordinates in Ω_∞, we get directly:

$$w_1(\theta) = |\partial_n u_0| \sin\theta.$$

The second term w_2 can also be described. The matrix $D^2 u_0$ is a hessian and therefore is symmetric. Because of the state equation, the matrix $D^2 u_0(0)$ writes $D^2 u_0(0) = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$. Then, we get

$$w_2(\theta) = \frac{1}{2} \begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = b + 2a \sin 2\theta.$$

To respect the boundary conditions (5), the w_i satisfy

$$w_i(0) = 0 \text{ and } w_i(\pi) = 0. \tag{35}$$

Therefore, we get that $b = 0$ and that $\det D^2 u_0(0) = -a^2$ and

$$w_2(\theta) = \frac{1}{2}\sqrt{-\det D^2 u_0(0)} \sin 2\theta.$$

Taking advantage of the specific form of the w_i, namely $w_i(\theta) = c_i \sin i\theta$, and of the geometry, we use an inversion to pose the problem in the unit ball, then the Poisson kernel to solve

$$\begin{aligned} -\Delta u &= 0 \text{ in } \Omega_\infty, \\ u &= 0 \text{ on } y_2 = 0,\ |y| \geq 1, \\ u &= w_i(\theta) \text{ on } \theta \in [0, \pi]. \end{aligned} \tag{36}$$

The obtained singular profiles are

$$V_i(r, \theta) = c_i \frac{\sin i\theta}{r^i}. \tag{37}$$

This is in particular the case for $i = 1, 2$. We get

$$V_1(r,\theta) = \partial_u u_0(O)\frac{\sin\theta}{r} \text{ and } \mathcal{V}_1(x_1,x_2) = \frac{x_2}{x_1^2+x_2^2} \tag{38}$$

$$V_2(r,\theta) = \frac{1}{2}\sqrt{-\det D^2 u_0(0)}\,\frac{\sin 2\theta}{r^2}. \tag{39}$$

Note that, obviously, these functions satisfy the announced behavior at infinity.

The polarization number. We apply formula (23) in the particular case $\rho(\theta) = 1$. We use (38) and get

$$\begin{aligned}\mathcal{A}(\omega) &= \frac{1}{4}\int_0^\pi \left[(1+\partial_2\mathcal{V}_1(\theta))^2 + (\partial_1\mathcal{V}_1(\theta))^2\right] d\theta,\\ &= \frac{1}{4}\int_0^\pi (1+\cos 2\theta)^2 + (\sin 2\theta)^2\, d\theta = \frac{1}{2}\int_0^\pi (1+\cos 2\theta)\ d\theta.\end{aligned}$$

Hence, we have

$$\mathcal{A}(\omega) = \frac{\pi}{2}. \tag{40}$$

An example of geometry with complete explicit quantities. We consider the case of the upper half-disk: $\Omega_0 = \{(r,\theta),\ \theta \in [0,\pi] \text{ and } r < 1\}$. For this particular domain, we consider Poisson's equation with the right hand side $f(r,\theta) = -\sin\theta$. This right hand side has not a compact support in Ω_0. However, this assumption is not necessary and was made for convenience and we still have the expected behavior. We can carry out the computations explicitly:

$$\begin{aligned}u_0(r,\theta) &= \frac{1}{3}\sin\theta(r^2-r),\\ |\partial_n u_0(O)|^2 &= \frac{1}{9},\\ J(\Omega_0) &= -\frac{\pi}{144},\\ u_\varepsilon(r,\theta) &= \frac{1}{3}\sin\theta\frac{\varepsilon^2(1-r^2)+(1+\varepsilon)(r^3-r^2)}{(1+\varepsilon)r}\\ J(\Omega_\varepsilon) &= -\frac{\pi}{144}+\frac{\pi}{18}\varepsilon^2-\frac{\pi}{9}\varepsilon^3+O(\varepsilon^4).\end{aligned}$$

We recover the expression (40) and the expansion (22).

5. The numerical validation

In this section, we present some numerical experiments, which illustrate the expansion (22). We consider the square $\Omega_0 = (-\frac{1}{2}, \frac{1}{2}) \times (0,1)$, on the boundary of which we dig a semi-circular hole: The domain Ω_ε is the defined as

$$\Omega_\varepsilon = \{x \in \Omega_0 \; ; \; |x| > \varepsilon\}.$$

As before, we denote by u_ε resp. u_0 the solution in $\mathcal{H}_0^1(\Omega_\varepsilon)$ resp. $\mathcal{H}_0^1(\Omega_0)$ of $-\Delta u = f$ with

$$f(x_1, x_2) = \begin{cases} 1 & \text{if } |x_1| < \frac{1}{4} \text{ and } |x_2 - \frac{1}{2}| < \frac{1}{4}, \\ 0 & \text{otherwise.} \end{cases}$$

The data f has a compact support in Ω_ε (for $\varepsilon < \frac{3}{4}$), but it is not smooth as required in the previous sections. Actually, it is sufficient for f to be smooth near $x = 0$ since we only use the regularity of u_0 near this point.

We used the finite element library Mélina (see Martin, 2004) to compute an approximation of both u_ε and u_0 for $\varepsilon = 2^{-i}$ with $i = 2, \ldots, 10$. Fig. 3 shows the high order (isoparametric $\mathbb{Q}_8$-type) meshes used for the values $\varepsilon = 1/4$ (8 elements, 561 degrees of freedom) and $\varepsilon = 1/8$ (12 elements, 825 degrees of freedom). We emphasize the fact that the geometry has to be approximated in a precise way, since the asymptotic phenomenon we want to observe is fine (see the error of order 10^{-10} in Fig. 5). The use of high-order elements is particularly adapted in the case of domain with curved boundaries.

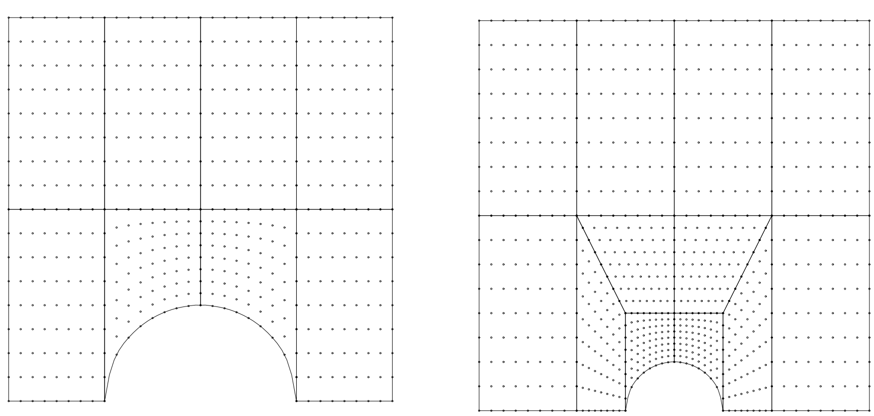

Figure 3. The $\mathbb{Q}_8$-mesh of the domain Ω_ε for $\varepsilon = 0.25$ and $\varepsilon = 0.125$

Fig. 5 presents the results of the computations (done on a calculator at the École Normale Supérieure de Cachan Bretagne, IBM Risc6000). In the table on the left, the values of $J(\Omega_\varepsilon)$ for $\varepsilon = 2^{-i}$ $(i = 2, \ldots, 10)$ are given, and can be compared with $J(\Omega_0)$. The graph on the right shows – in logarithmic axes – the evolution of $J(\Omega_\varepsilon)$ with respect of ε. Since it is a straight line of slope -2, the numerical results validate the dependency in ε^2 of the expansion (22).

ε	$J(\Omega_\varepsilon)$
0.25	$-5.4441191.10^{-4}$
0.125	$-5.4897622.10^{-4}$
0.0625	$-5.4997119.10^{-4}$
0.03125	$-5.5021303.10^{-4}$
0.015625	$-5.5027309.10^{-4}$
0.0078125	$-5.5028808.10^{-4}$
0.00390625	$-5.5029183.10^{-4}$
0.001953125	$-5.5029277.10^{-4}$
0.0009765625	$-5.5029300.10^{-4}$
0	$-5.5029307.10^{-4}$

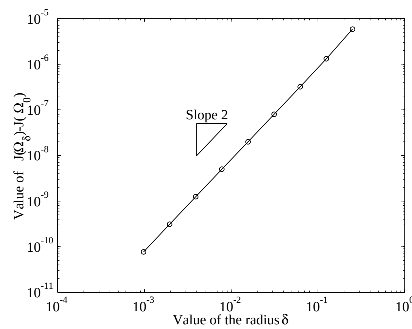

Figure 4. Comparison between $J(\Omega_\varepsilon)$ and $J(\Omega_0)$ with respect to ε (logarithmic axes)

In order to highlight the factor $A(\omega) = \frac{\pi}{2}$, we have done a computation with the following right-hand side:

$$f(x_1, x_2) = 2\pi^2 \cos(\pi x_1) \sin(\pi x_2),$$

for which we know the exact solution for $\varepsilon = 0$: $u_0(x_1, x_2) = \cos(\pi x_1) \sin(\pi x_2)$ (however, we do not have any explicit expression for the solution u_ε). In this case,

$$\partial_n u_0(0,0) = -\pi \quad \text{and} \quad J(\Omega_0) = -\frac{\pi^2}{4}.$$

In Table 1, we show the evolution of the quantity

$$\mathcal{E}(\varepsilon) = \frac{J(\Omega_\varepsilon) - J(\Omega_0)}{\varepsilon^2 \partial_n u_0(0,0)^2}$$

with respect to ε. We clearly see the convergence to $\frac{\pi}{2}$, predicted by formula (22).

Table 1. Quantities $J(\Omega_\varepsilon)$ and $\mathcal{E}(\varepsilon) = \frac{J(\Omega_\varepsilon)-J(\Omega_0)}{\varepsilon^2 \partial_n u_0(0,0)^2}$

ε	$J(\Omega_\varepsilon)$	$\mathcal{E}(\varepsilon)$
0.25	−1.664808	1.301116
0.125	−2.236814	1.495258
0.0625	−2.407592	1.551334
0.03125	−2.452308	1.565902
0.015625	−2.463619	1.569580
0.0078125	−2.466455	1.570502
0.00390625	−2.467165	1.570730
0.001953125	−2.467342	1.570783
0.0009765625	−2.467386	1.570778
0	−2.467401	1.570796

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