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Shape identification via metrics constructed from the oriented distance function*

by

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Abstract: This paper studies the generic identification problem: to find the best non-parametrized object Ω which minimizes some weighted sum of distances to I a priori given objects Ω_i for metric distances constructed from the $W^{1,p}$–norm on the *oriented* (resp. *signed*) *distance function* which occurs in many different fields of applications. It discusses existence of solution to the generic identification problem and investigates the Eulerian shape semiderivatives with special consideration to the non-differentiable terms occurring in their expressions. A simple example for the new *cracked sets* recently introduced in Delfour and Zolésio (2004b) is also presented. It can be viewed as an approximation of a cracked set by sets whose boundary is made up of pieces of lines or Bézier curves that are not necessarily connected.

Keywords: shape identification, sensitivity analysis, variational problems, set-valued and variational analysis, image processing, image enhancing, metric, distance, oriented distance function, signed distance function, Sobolev domains, velocity method, cracked sets.

1. Introduction

In problems where a non-parametrized geometric object is the variable, special metrics are used to measure the distance between two objects and to induce topologies from which existence and characterization of optimal objects can be

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obtained for design, identification, or control purposes. The choice of the metric is obviously very much problem dependent and corresponds to pertinent technological, physical, or geometric entities associated with the problem at hand. For instance, distance functions have been used for theoretical and computational purposes in free boundary problems (Gilbarg and Trudinger, 1977; Ishii and Souganidis, 1995), image processing and computer vision (Matheron, 1998; Serra, 1984, 1998; Aubin, 1999; Osher and Sethian, 1988; Malladi, Sethian, Vemuri, 1995; Adalsteinsson and Sethian, 1999; Caselles, Kimmel, Sapiro, 1997; Gomes and Faugeras, 2000), and robotics (Hoffmann et al., 1992; Hoffmann, 1990, 1994; Stifter, 1992). When computations are envisioned, the choice of metrics and formulations is also influenced by the fact that they must lead to algorithms which are efficient, easily implementable, and capable of handling available experimental data or measurements.

This paper focuses on theoretical and practical issues associated with the following generic *shape identification problem*: given I objects or data sets Ω_i, to find the best object Ω which minimizes some combination of the distances from Ω to each Ω_i, $1 \leq i \leq I$. This basic problem occurs in many areas of applications: *biometric identification* or *image enhancement* such as the production of a sharp image from images produced by an array of very large telescopes (VLT). For instance, the *European Southern Observatory (ESO) VTL consists of four 8-m telescopes, which should one day work in unison and simulate the resolution of a huge single instrument through interferometry - a technique familiar to astronomers using radio telescopes (see http://www.eso.org/projects/vlt/).* Even in this simple form the problem is technically very delicate since singularities and non-differentiabilities naturally occur even for sets of class C^∞. Among the many metrics available for non-parametrized sets, the paper specializes to metrics and constructions based on the the $W^{1,p}$–norm[1] on the *oriented* (resp. *signed or algebraic*) *distance function* b_Ω. This choice is simultaneously motivated by the existence of efficient computer packages using distance functions and, from the purely theoretical viewpoint, by the fact that it is playing a central and natural role in the shape and geometric analysis (see, for instance Delfour and Zolésio, 2001, 2004, Aubin, 1999, for a comprehensive analysis, Delfour, Doyon and Zolésio, 2005a, b, c, for new compactness results in shape optimization for sets verifying a *uniform cusp or fat segment property*, and the new *cracked sets* used in the context of the image segmentation problem in Delfour and Zolésio, 2004b).

Section 2 discusses the generic shape identification problem for the four objective functions that have been selected to illustrate the fundamental issues at sake: $f_{0,p}$ and $f_{1,p}$ are defined over a fixed bounded hold-all D (see (9) and (10)) and $g_{0,p}$ and $g_{1,p}$ are defined on the boundary Γ of the set Ω (see (11) and (12)). Section 3 reviews the family of $W^{2,p}$-Sobolev domains. For $p > N$,

[1]This topology was introduced in Delfour and Zolésio, 1994, and further investigated in Delfour and Zolésio, 1998, 2001.

the boundary integral is shown to be continuous for special classes of functions. This material is used in Section 4 to discuss the existence of solution to the four objective functions.

The other sections are devoted to the computation of *Eulerian shape semiderivatives* of the objective functions. Section 5 reviews the *Velocity Method* which transforms an initial domain Ω into domains $\Omega_t(V)$ indexed by the real parameter t under the action of a velocity field V. We compute the partial derivative of the oriented distance function of $\Omega_t(V)$ with respect to t. Its expression gives a complete description of the non-differentiability involved and is used to study the Eulerian semiderivatives of the four generic objective functions. Section 6 gives the expression of the shape semiderivative of the objective functions $f_{0,2}$ and $f_{1,2}$ defined on the hold-all D (see (31) and (32)) for sets Ω with thin boundary. For $f_{1,2}$ the Ω_i's are also assumed to have thin boundaries, but the semiderivative necessitates more smoothness on the curvatures of Ω or of all the Ω_i's in the whole hold-all D. This can be restrictive. Since Ω is the free variable it can be assumed sufficiently smooth to make sense of curvature terms, but this is more delicate for the data sets Ω_i that may have some skeleton away from their boundary Γ_i even if they are very smooth. In Section 7 the semiderivatives of the objective functions $g_{0,2}$ and $g_{1,2}$ defined on the boundary Γ (see (34) and (37)) require more smoothness assumptions than their counterparts on D. It is assumed that Ω and the Ω_i's have thin boundary and that Γ has finite $(N-1)$-dimensional Hausdorff measure to make sense of integration on Γ. Since no gradients are involved in the definition (34) of $g_{0,2}$, its semiderivative makes sense by relaxing one of the terms to its non-differentiable expression. Gradients are present in the definition of $g_{1,2}$ and more assumptions have to be put on the data sets Ω_i in order to make sense of the trace of their curvature on Γ. We discuss this much more restrictive case and give a result for a special case in dimension 2. The function $g_{1,2}$ is not always semidifferentiable.

Section 8 gives a simple example for the objective function $g_{2,0}$ that does not require any semidifferentiability assumption. The unknown set is a *cracked set* (first introduced in Delfour and Zolésio, 2004b, for the segmentation functional of Mumford and Shah) whose boundary is made up of line segments or Bézier curves specified by a connectivity matrix. All the semiderivatives are explicitly computed in terms of the projections. The constructions and computations readily extend to sets in three dimensions whose boundary is made up of two-dimensional triangular facets or curved triangular surfaces. The special set Γ can also be viewed as an approximation of a cracked set by sets whose boundary is made up of pieces of lines or Bézier curves that are not necessarily connected. All the formulae for the semiderivatives can also be obtained by choosing a special velocity field associated with each node and each control node in the case of a Bézier curve. This approach was used in Zolésio (1984) to compute the derivative of an objective function that depends on the solution of a finite element problem with respect to the internal nodes of the triangulation of the underlying domain.

This theoretical analysis of the oriented distance function in the four generic objective functions considered indicates that metrics involving gradients necessitate more restrictive assumptions on the set Ω and/or the data sets Ω_i. There are many intriguing issues and problems which are still open and apparent restrictions might be overcome while preserving explicit expressions of the semiderivatives. The two metrics without gradient terms are easier to handle and the lack of control over the gradients can be fixed by minimizing over classes of domains with bounds on the curvatures over a tubular neighborhood of the boundary. In that case, control is exerted through both the thickness of the tubular neighborhood and the amplitude of the curvatures.

In this paper the words *set*, *image*, and *object* will be used equivalently. Given an integer $N \geq 1$, m_N and H_{N-1} denote the N-dimensional Lebesgue and $(N-1)$-dimensional Hausdorff measures. The inner product and the norm in $\mathbf{R}^{\mathrm{N}}$ will be written $x \cdot y$ and $|x|$. The *complement* $\{x \in \mathbf{R}^{\mathrm{N}} : x \notin \Omega\}$ and the boundary $\overline{\Omega} \cap \overline{\complement\Omega}$ of a subset Ω of $\mathbf{R}^{\mathrm{N}}$ will be respectively denoted by $\complement\Omega$ or $\mathbf{R}^{\mathrm{N}} \backslash \Omega$ and by $\partial\Omega$ or Γ. The *distance function* $d_A(x)$ from a point x to a subset $A \neq \varnothing$ of $\mathbf{R}^{\mathrm{N}}$ is defined as $\inf\{|y-x| : y \in A\}$.

2. A generic shape identification problem

Assume that I objects $\{\Omega_i : i \in I\}$ are given in the Euclidean space $\mathbf{R}^{\mathrm{N}}$, $N \geq 1$ an integer. In most applications N is equal to 2 or 3. Given a metric $\rho(\Omega', \Omega)$ defined on the objects, we want to find the *best object* Ω which minimizes objective functions of the form

$$f_p(\Omega) \stackrel{\text{def}}{=} \left\{ \sum_{i=1}^{I} \rho(\Omega, \Omega_i)^p \right\}^{1/p} \quad \text{or} \quad f(\Omega) \stackrel{\text{def}}{=} \max_{1 \leq i \leq I} \rho(\Omega, \Omega_i) \tag{1}$$

for some finite integer $p \geq 1$.

2.1. Metrics from the oriented distance function

Given a subset Ω of $\mathbf{R}^{\mathrm{N}}$, $\Gamma \neq \varnothing$, the *oriented distance function* is defined as

$$b_\Omega(x) \stackrel{\text{def}}{=} d_\Omega(x) - d_{\complement\Omega}(x). \tag{2}$$

The function b_Ω is Lipschitz continuous of constant 1, and ∇b_Ω exists and $|\nabla b_\Omega| \leq 1$ almost everywhere in $\mathbf{R}^{\mathrm{N}}$. Thus $b_\Omega \in W^{1,p}_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}})$ for all p, $1 \leq p \leq \infty$.

Definition 2.1 *(i) Given a nonempty subset D of $\mathbf{R}^{\mathrm{N}}$, define the families*

$$C_b(D) \stackrel{\text{def}}{=} \left\{ b_\Omega : \Omega \subset \overline{D} \text{ and } \Gamma \neq \varnothing \right\}, \quad C_b^0(D) \stackrel{\text{def}}{=} \left\{ b_\Omega \in C_b(D) : \mathrm{m}_N(\Gamma) = 0 \right\}. \tag{3}$$

(ii) The boundary Γ of a subset Ω of $\mathbf{R}^{\mathrm{N}}$ is said to be thin[2] *if* $\mathrm{m}_N(\Gamma) = 0$.

[2] This terminology is not to be confused with the one of thin set in *Capacity Theory.*

In this paper we specialize to the following complete metrics associated with b_Ω over the subsets of a bounded open *hold-all* D

$$\rho_{C(D)}([\Omega'],[\Omega]) \overset{\text{def}}{=} \max_{x\in\overline{D}} |b_{\Omega'}(x)-b_\Omega(x)| \tag{4}$$

$$\rho_{L^p(D)}([\Omega'],[\Omega]) \overset{\text{def}}{=} \left\{\int_D |b_{\Omega'}-b_\Omega|^p\,dx\right\}^{1/p} \tag{5}$$

$$\rho_{W^{1,p}(D)}([\Omega'],[\Omega]) \overset{\text{def}}{=} \left\{\int_D |b_{\Omega'}-b_\Omega|^p + |\nabla b_{\Omega'}-\nabla b_\Omega|^p\,dx\right\}^{1/p}. \tag{6}$$

The space $C_b(D)$ is a complete metric space for the metrics (4), (5), and (6), but the space $C_b^0(D)$ is complete only with respect to the metric (6) (e.g. Delfour and Zolésio, 2001, Chapter 5). The metrics (6) are all equivalent for $1 \le p < \infty$.

The points of $\mathbf{R}^{\mathrm{N}}$ where the gradient of b_Ω does not exist can be divided into two categories: the ones on the boundary Γ and the ones outside of Γ.

Definition 2.2 *The set of* projections *of a point* $x \in \mathbf{R}^{\mathrm{N}}$ *onto the boundary* Γ *of a set* Ω, $\Gamma \neq \varnothing$,

$$\Pi_\Gamma(x) \overset{\text{def}}{=} \left\{p \in \mathbf{R}^{\mathrm{N}} : |b_\Omega(x)| = |p-x|\right\}$$

since $|b_\Omega(x)| = d_\Gamma(x)$*; the* skeleton *of* Ω

$$\mathrm{Sk}(\Omega) \overset{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}} : \Pi_\Gamma(x) \text{ is not a singleton}\right\} \tag{7}$$

(by definition $\mathrm{Sk}(\Omega) \subset \mathbf{R}^{\mathrm{N}}\setminus\Gamma$*); the* set of cracks *of* Ω

$$\mathrm{C}(\Omega) \overset{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}} : \nabla b_\Omega^2(x) \textit{ exists but } \nabla b_\Omega(x) \textit{ does not exist}\right\}.$$

The *projection* $p_\Gamma(x)$ of a point $x \notin \mathrm{Sk}(\Omega)$ onto the boundary Γ of Ω is given by

$$p_\Gamma(x) = x - \frac{1}{2}\nabla b_\Omega^2(x) = x - b_\Omega\,\nabla b_\Omega(x). \tag{8}$$

The following families of sets with thin boundary will be used in the paper.

Definition 2.3 *(i) The boundary* Γ *of a subset* Ω *of* $\mathbf{R}^{\mathrm{N}}$ *is said to be* integrable *if* Γ *is nonempty, thin, and the* $(N-1)$*-dimensional Hausdorff measure* H_{N-1} *is locally finite on* Γ.

(ii) The boundary Γ *of a subset* Ω *of* $\mathbf{R}^{\mathrm{N}}$ *is said to be* integrable with normal[3] *if* Γ *is integrable and* $H_{N-1}(\mathrm{C}(\Omega)) = 0$*, that is,* ∇b_Ω *exists* H_{N-1}*-almost everywhere on* Γ.

[3]For instance this is true for locally Lipschitzian domains, but the existence of a unique normal is meaningless for submanifolds of $\mathbf{R}^{\mathrm{N}}$ of codimension strictly greater than one.

There are a number of important open questions. In particular how can the sets of Definition 2.3 be characterized from the properties of the Hessian matrix of b_Ω, d_Ω, or $d_{\partial\Omega}$ for sets Ω with a thin boundary. We shall see later how this is intimately related to the well-posedness and the semidifferentiability of objective functions defined on the thin boundary Γ of a set Ω.

2.2. Generic objective functions

Consider the following objective functions on a fixed bounded hold-all D

$$f_{0,p}([\Omega]) \stackrel{\text{def}}{=} \sum_{i=1}^{I} \int_D |b_\Omega - b_{\Omega_i}|^p \, dx \tag{9}$$

$$f_{1,p}([\Omega]) \stackrel{\text{def}}{=} \sum_{i=1}^{I} \int_D |b_\Omega - b_{\Omega_i}|^p + |\nabla b_\Omega - \nabla b_{\Omega_i}|^p \, dx. \tag{10}$$

Both functions are well-defined for arbitrary sets Ω and Ω_i with nonempty boundary. In view of the boundedness of D, we only consider sets Ω that are bounded with a compact boundary Γ.

In some applications it might be desirable to use smoothness properties in some neighborhood of the boundary Γ of the variable set Ω rather than in the whole hold-all D. This can be done by specifying the properties of Ω in the *open tubular neighborhood* $U_h(\Gamma)$ of thickness $h > 0$ of its boundary Γ.

The shape of an object is essentially determined by its boundary. So it is also natural to consider the following integral over Γ

$$g_{0,p}([\Omega]) \stackrel{\text{def}}{=} \sum_{i=1}^{I} \int_\Gamma |b_\Omega - b_{\Omega_i}|^p \, d\Gamma \tag{11}$$

$$g_{1,p}([\Omega]) \stackrel{\text{def}}{=} \sum_{i=1}^{I} \int_\Gamma |b_\Omega - b_{\Omega_i}|^p + |\nabla b_\Omega - \nabla b_{\Omega_i}|^p \, d\Gamma. \tag{12}$$

Since $b_\Omega(x) = 0$ on Γ, the above expressions can be slightly simplified. Here some restrictions have to be put on the families of subsets Ω and Ω_i of $\mathbf{R}^{\mathrm{N}}$ since the two boundary integrals over Γ must make sense and the gradients ∇b_Ω and ∇b_{Ω_i} must be well-defined on Γ. For $g_{0,p}$ it is sufficient that Γ be integrable in the sense of Definition 2.3, but, for the function $g_{1,p}$, Γ it must be integrable with normal and, in addition, some assumptions have to be put on the sets Ω_i. In general for each i the gradient $\nabla b_{\Omega_i}(x)$ (which only exists almost everywhere in $\mathbf{R}^{\mathrm{N}}$) may not be defined or have a trace on Γ. For sets Ω_i which are polygonal in dimension $N = 2$ or whose boundary is made up of triangular facets in dimension $N = 3$, the gradient of b_{Ω_i} will exist H_{N-1}-almost everywhere on Γ_i and the function $g_{1,p}$ will be well-defined, but the Hessian matrix will be a matrix of measures in the *corners* due to the jump in the normal related to the angle at the corner.

THEOREM 2.1 *Assume that the sets* Ω *and* Ω_i, $i \in I$, *have boundaries which are integrable with normal. Further assume that* $H_{N-1}(\Gamma \cap \mathrm{Sk}(\Omega_i)) = 0$. *Then*

$$g_{1,2}([\Omega]) = \sum_{i=1}^{I} \int_\Gamma |b_{\Omega_i}(x)|^2 \, d\Gamma + 2\, H_{N-1}(\Gamma) - 2 \sum_{i=1}^{I} \int_{\Gamma \setminus \mathrm{Sk}(\Omega_i)} \nabla b_{\Omega_i} \cdot \nabla b_\Omega \, d\Gamma. \tag{13}$$

Proof. For integrable sets Ω with normal, $|\nabla b_\Omega| = 1$ H_{N-1} a.e on Γ. Similarly $|\nabla b_{\Omega_i}| = 1$ H_{N-1} a.e on Γ_i. By assumption $H_{N-1}(\Gamma \cap \mathrm{Sk}(\Omega_i)) = 0$ and

$$\int_\Gamma \nabla b_{\Omega_i} \cdot \nabla b_\Omega \, d\Gamma = \int_{\Gamma \setminus \mathrm{Sk}(\Omega_i)} \nabla b_{\Omega_i} \cdot \nabla b_\Omega \, d\Gamma.$$

This last integral splits into two integrals

$$\int_\Gamma \nabla b_{\Omega_i} \cdot \nabla b_\Omega \, d\Gamma = \int_{\Gamma \cap \Gamma_i} \nabla b_{\Omega_i} \cdot \nabla b_\Omega \, d\Gamma + \int_{\Gamma \setminus (\mathrm{Sk}(\Omega_i) \cup \Gamma_i)} \nabla b_{\Omega_i} \cdot \nabla b_\Omega \, d\Gamma,$$

since $\Gamma \cap \Gamma_i \setminus \mathrm{Sk}(\Omega_i) = \Gamma \cap \Gamma_i$. On $\Gamma \setminus (\mathrm{Sk}(\Omega_i) \cup \Gamma_i)$ ∇b_{Ω_i} exists and, by assumption, on $\Gamma \cap \Gamma_i$ it exists H_{N-1} a.e.. Therefore

$$|\nabla b_\Omega(x) - \nabla b_{\Omega_i}(x)|^2 = 2 - 2 \nabla b_\Omega(x) \cdot \nabla b_{\Omega_i}(x) \quad H_{N-1} \text{ a.e. on } \Gamma. \qquad \blacksquare$$

REMARK 2.1 *Outside of* $\Gamma_i \cap \Gamma$ *the only place where* ∇b_{Ω_i} *is not well-defined is* $\Gamma \cap \mathrm{Sk}(\Omega_i)$. *If it is interpreted as a semi-differentiable term (i.e.* $df(x; v) = \lim_{t \searrow 0} (f(x + tv) - f(x))/t$*)*

$$db_{\Omega_i}(x; \nabla b_\Omega(x)) = \frac{1}{b_{\Omega_i}(x)} \min_{p \in \Pi_{\Gamma_i}(x)} (x - p) \cdot \nabla b_\Omega(x) \tag{14}$$

(see Delfour and Zolésio, 2001, Chapter 5, Thm 2.1 (ii)) and this new term is defined wherever $\nabla b_\Omega(x)$ *is defined. To use this we would also need to make sense of* $|\nabla b_{\Omega_i}|^2$ *on* $\mathrm{Sk}(\Omega_i)$ *and then*

$$\begin{aligned} g_{1,2}([\Omega]) &= \sum_{i=1}^{I} \int_\Gamma |b_{\Omega_i}(x)|^2 + 2 - 2\, db_{\Omega_i}(x; \nabla b_\Omega(x)) \, d\Gamma \\ &= \sum_{i=1}^{I} \int_\Gamma |b_{\Omega_i}(x)|^2 \, d\Gamma + 2\, H_{N-1}(\Gamma) \\ &\quad - 2 \int_{\Gamma \cap \mathrm{Sk}(\Omega_i)} db_{\Omega_i}(x; \nabla b_\Omega(x)) \, d\Gamma - 2 \int_{\Gamma \setminus \mathrm{Sk}(\Omega_i)} \nabla b_{\Omega_i} \cdot \nabla b_\Omega \, d\Gamma. \end{aligned} \tag{15}$$

In practice, the case $H_{N-1}(\Gamma \cap \mathrm{Sk}(\Omega_i)) > 0$ *will seldom occur since it requires that two sets of zero* N*-dimensional Lebesgue measure intersect with a non-zero* $(N-1)$*-dimensional Hausdorff measure.*

2.3. Main issues

The first issue is the choice of the objective function: with or without the gradient, and defined on D or Γ? The ill-definiteness of the integral on Γ could be overcome by *averaging* over the tubular neighborhood $U_h(\Gamma)$

$$g^h_{0,p}([\Omega]) \stackrel{\text{def}}{=} \sum_{i=1}^{I} \frac{1}{2h} \int_{U_h(\Gamma)} |b_\Omega - b_{\Omega_i}|^p \, dx \tag{16}$$

$$g^h_{1,p}([\Omega]) \stackrel{\text{def}}{=} \sum_{i=1}^{I} \frac{1}{2h} \int_{U_h(\Gamma)} |b_\Omega - b_{\Omega_i}|^p + |\nabla b_\Omega - \nabla b_{\Omega_i}|^p \, dx, \tag{17}$$

but some of the advantages of working only on Γ will be lost. Note that the presence of the gradient on Γ in (17) implicitly implies that Γ must be a submanifold of codimension one. When Γ has codimension r strictly greater than one, the objective function (17) would not be suitable and the integral in (16) would have to be divided by h^r. The next issue is the question of the choice of the family of sets in relation to the existence of a minimizing solution. The last issue is to find characterizations of the minimizing solutions and devise schemes to compute them. When the topology of the sets is known (e.g. number of connected components), shape semiderivative can be used to approximate the best shape or at least to decrease the objective function, but other tools could be used. If shape semiderivatives of the objective functions are to be used, one more degree of smoothness will usually be expected from b_Ω and b_{Ω_i} to make sense of the derivatives. Thus the choice of an objective function is critically dependent on the nature of the data available, that is, the properties of the b_{Ω_i}'s, $1 \le i \le I$. If the Ω_i's are polygonal sets in $\mathbf{R}^2$ or Γ_i is made up of triangular facets in $\mathbf{R}^3$, the skeleton $\mathrm{Sk}(\Omega_i)$ will be H_{N-1}-measurable and the set of cracks $\mathrm{C}(\Omega_i)$ will contain all the vertices. So ∇b_{Ω_i} will only exist H_{N-1}-almost everywhere on Γ_i and $D^2 b_{\Omega_i}$ will, at best, be a matrix of bounded measures.

3. Sobolev domains

We recall some recent results from Delfour and Zolésio (2004) on *Sobolev domains* and establish the continuity of the boundary integral with respect to the domain. Given $h > 0$ the *open and closed tubular neighborhoods* of a set A are defined as

$$U_h(A) \stackrel{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}} : d_A(x) < h\right\}, \quad A_h \stackrel{\text{def}}{=} \left\{x \in \mathbf{R}^{\mathrm{N}} : d_A(x) \le h\right\}. \tag{18}$$

Recalling that $d_\Gamma(x) = |b_\Omega(x)|$ we also have $U_h(\Gamma) = \left\{x \in \mathbf{R}^{\mathrm{N}} : |b_\Omega(x)| < h\right\}$.

Definition 3.1 *Given $m > 1$ and $p \ge 1$, a subset Ω of $\mathbf{R}^{\mathrm{N}}$ is said to be an* (m,p)-Sobolev domain *if $\Gamma \neq \varnothing$ and*

$$\exists h > 0 \text{ such that } b_\Omega \in W^{m,p}_{\mathrm{loc}}(U_h(\Gamma)).$$

We use the extension of b_Ω by zero outside of $U_h(\Gamma)$ to $\mathbf{R}^{\rm N}$ introduced in Delfour and Zolésio (2004).

Theorem 3.1 *Given $h > 0$ and a subset Ω of $\mathbf{R}^{\rm N}$ with nonempty boundary Γ, let $\rho_h \in \mathcal{D}(]-h,h[)$ be a non-negative function which is equal to 1 in a neighborhood $V =]-h',h'[$, $0 < h' < h$, of 0. Define the* smooth h-extensions of b_Ω and 1 by zero

$$b_\Omega^h \stackrel{\rm def}{=} \rho_h \circ b_\Omega \, b_\Omega, \quad e_\Omega^h \stackrel{\rm def}{=} \rho_h \circ b_\Omega + b_\Omega \rho_h' \circ b_\Omega. \tag{19}$$

It is readily seen that $b_\Omega^h = b_\Omega$ and $e_\Omega^h = 1$ in the tubular neighborhood $b_\Omega^{-1}(V) = U_{h'}(\Gamma) \subset U_h(\Gamma)$ of Γ. By construction $e_\Omega^h \in C_0^{0,1}(U_h(\Gamma))$. The extension b_Ω^h preserves the smoothness properties of b_Ω in $U_h(\Gamma)$ and e_Ω^h can be viewed as an extension of 1 by zero outside $U_h(\Gamma)$ with the same smoothness as b_Ω in $U_h(\Gamma)$. By construction

$$\nabla b_\Omega^h = [\rho_h \circ b_\Omega + b_\Omega \, \rho_h' \circ b_\Omega] \, \nabla b_\Omega = e_\Omega^h \nabla b_\Omega. \tag{20}$$

If there exist $p \geq 1$ and $h > 0$ such that $\Delta b_\Omega \in L^p_{\rm loc}(U_h(\Gamma))$, then

$$\Delta b_\Omega^h = e_\Omega^h \Delta b_\Omega + \nabla e_\Omega^h \cdot \nabla b_\Omega \in L^p_{\rm loc}(\mathbf{R}^{\rm N}) \quad (L^p(\mathbf{R}^{\rm N}) \text{ if } \Gamma \text{ is bounded}).$$

Theorem 3.2 *Given an integer $N \geq 1$, let Ω be a subset of $\mathbf{R}^{\rm N}$, $\varnothing \neq \Gamma \neq \mathbf{R}^{\rm N}$.*

(i) *If there exist $p \geq 1$ and $h > 0$ such that $\Delta b_\Omega \in L^p_{\rm loc}(U_h(\Gamma))$, then*

$$b_\Omega^h \in W^{2,p}_{\rm loc}(\mathbf{R}^{\rm N}) \text{ and } b_\Omega \in W^{2,p}_{\rm loc}(U_h(\Gamma)) \tag{21}$$

and $\mathrm{m}_N(\Gamma) = 0$. The gradient ∇b_Ω exists in all points of $U_h(\Gamma) \backslash \Gamma$ and $|\nabla b_\Omega| = 1$. If Γ is compact

$$b_\Omega^h \in W_0^{2,p}(\mathbf{R}^{\rm N}) \text{ and } \quad \forall h', \, 0 < h' < h, \quad b_\Omega \in W^{2,p}(U_{h'}(\Gamma)), \tag{22}$$

where $W_0^{2,p}(\mathbf{R}^{\rm N})$ is the closure in the $W^{2,p}$-norm of the space $\mathcal{D}(\mathbf{R}^{\rm N})$ of all infinitely differentiable functions defined on $\mathbf{R}^{\rm N}$ with compact support.

(ii) *If, in addition to the assumptions of part* (i), *$p > N$, then Ω is a Hölderian set of class $C^{1,1-N/p}$ and $b_\Omega \in C^{1,1-N/p}_{\rm loc}(U_h(\Gamma))$.*

(iii) *If, in addition to the assumptions of part* (i), *$p > 1$, Γ is compact, $\{\Omega_n\}$ is a sequence of subsets of $\mathbf{R}^{\rm N}$ such that $b_{\Omega_n} \to b_\Omega$ in $W^{1,p}(U_h(\Gamma))$, and there exists a constant c such that*

$$\forall n, \quad \|\Delta b_{\Omega_n}\|_{L^p(U_h(\Gamma_n))} \leq c,$$

then $\|\Delta b_\Omega\|_{L^p(U_h(\Gamma))} \leq c$ and

$$\Delta b_{\Omega_n} \, \chi_{U_h(\Gamma_n)} \rightharpoonup \Delta b_\Omega \, \chi_{U_h(\Gamma)} \text{ in } L^p(U_h(\Gamma))\text{-weak}.$$

Proof. For (i) and (ii) see Delfour and Zolésio (2004). (iii) For convenience denote b_Ω and b_{Ω_n} by b and b_n. Consider the difference of the Laplacians as distributions

$$\forall \varphi \in \mathcal{D}(U_h(\Gamma)),\ < \Delta b_n - \Delta b, \varphi > = - \int_{U_h(\Gamma)} \nabla b_n \cdot \nabla \varphi \, dx + \int_{U_h(\Gamma)} \nabla b \cdot \nabla \varphi \, dx$$

$$\Rightarrow |< \Delta b_n - \Delta b, \varphi >| \leq \|\nabla(b_n - b)\|_{L^p(U_h(\Gamma))} \, \|\nabla \varphi\|_{L^q(U_h(\Gamma))}$$

which goes to zero as n goes to ∞. For $\varphi \in \mathcal{D}(U_h(\Gamma))$, there exists $\varepsilon > 0$ such that $0 < 3\varepsilon < h$, and $\sup \varphi \subset U_{h-2\varepsilon}(\Gamma)$. Moreover, there exists N such that

$$\forall n \geq N, \quad U_{h-2\varepsilon}(\Gamma_n) \subset U_{h-\varepsilon}(\Gamma) \subset U_h(\Gamma_n).$$

In view of the above identities, for all $n \geq N$,

$$\begin{aligned}
\int_{U_h(\Gamma)} (\nabla b - \nabla b_n) \cdot \nabla \varphi \, dx &= \int_{U_h(\Gamma)} \nabla b \cdot \nabla \varphi \, dx - \int_{U_h(\Gamma_n)} \nabla b_n \cdot \nabla \varphi \, dx \\
&= - \int_{U_h(\Gamma)} \Delta b \, \varphi \, dx + \int_{U_h(\Gamma_n)} \Delta b_n \, \varphi \, dx \\
&= \int_{U_h(\Gamma)} \left[\Delta b_n \, \chi_{U_h(\Gamma_n)} - \Delta b \, \chi_{U_h(\Gamma)} \right] \varphi \, dx.
\end{aligned}$$

This means that for all $\varphi \in \mathcal{D}(U_h(\Gamma))$

$$\left| \int_{U_h(\Gamma)} \left[\Delta b_n \, \chi_{U_h(\Gamma_n)} - \Delta b \, \chi_{U_h(\Gamma)} \right] \varphi \, dx \right| = \left| \int_{U_h(\Gamma)} (\nabla b - \nabla b_n) \cdot \nabla \varphi \, dx \right| \to 0$$

as n goes to 0. But the norms $\|\Delta b_n \chi_{U_h(\Gamma_n)}\|_{L^p(U_h(\Gamma))}$ are uniformly bounded by c and $1 < p < \infty$ implies $1 < q < \infty$. So, by density of $\mathcal{D}(U_h(\Gamma))$ in $L^q(U_h(\Gamma))$,

$$\forall \varphi \in L^q(U_h(\Gamma)), \quad \int_{U_h(\Gamma)} \Delta b_n \, \chi_{U_h(\Gamma_n)} \, \varphi \, dx \to \int_{U_h(\Gamma)} \Delta b \, \chi_{U_h(\Gamma)} \, \varphi \, dx.$$

By reflexivity of $L^p(U_h(\Gamma))$, we get the weak $L^p(U_h(\Gamma))$-convergence. ∎

For $p > N$ the sets Ω such that $\Delta b_\Omega \in L^p(U_h(\Gamma))$ are at least of class C^1. Therefore the boundary integral is well-defined and can be related to the gradient and the Laplacian of b_Ω. Indeed, by Stokes Theorem, for all $\varphi \in \mathcal{D}(\mathbf{R}^{\mathrm{N}})$

$$\begin{aligned}
&\int_\Gamma \varphi \, dH_{N-1} = \int_\Gamma \varphi \nabla b^h_\Omega \cdot n \, dH_{N-1} = \int_\Omega \operatorname{div}(\varphi \nabla b^h_\Omega) \, dx \\
&= \int_\Omega \nabla \varphi \cdot \nabla b^h_\Omega + \varphi \, \Delta b^h_\Omega \, dx = \int_{\mathbf{R}^{\mathrm{N}}} \chi_\Omega \nabla \varphi \cdot \nabla b^h_\Omega + \chi_\Omega \varphi \, \Delta b^h_\Omega \, dx,
\end{aligned} \tag{23}$$

since the exterior unit normal n exists everywhere on Γ and is equal to ∇b_Ω. Identity (23) extends to all $\varphi \in W_0^{1,q}(\mathbf{R}^N)$, $p^{-1}+q^{-1}=1$.

THEOREM 3.3 *Fix an integer $N \geq 1$ and $N < p < \infty$. Let D be a bounded open Lipschitzian hold-all and $\{\Omega_n\}$ be a sequence of subsets of D such that*

$$\exists c > 0, \forall n \geq 1, \quad \|\Delta b_{\Omega_n}\|_{L^p(U_h(\Gamma_n))} \leq c.$$

Assume that $b_{\Omega_n} \to b_\Omega$ in $W^{1,p}(U_h(D))$ and that $\{\varphi_n\}$ is a sequence in $W^{1,q}(D)$, $p^{-1}+q^{-1}=1$, such that $\varphi_n \to \varphi$ in $W^{1,q}(D)$ for some $\varphi \in W^{1,\infty}(D)$. Then

$$\int_{\Gamma_n} \varphi_n \, dH_{N-1} \to \int_\Gamma \varphi \, dH_{N-1}.$$

Proof. In view of the previous discussion for all h', $0 < h' < h$, and n

$$\int_{\Gamma_n} \varphi_n \, dH_{N-1} = \int_D \chi_{\Omega_n} \nabla\varphi_n \cdot \nabla b_{\Omega_n}^{h'} + \chi_{\Omega_n} \varphi_n \, \Delta b_{\Omega_n}^{h'} \, dx.$$

Consider the first term $\chi_{\Omega_n}\nabla\varphi_n \cdot \nabla b_{\Omega_n}^{h'}$ on the right-hand side. By assumption $\nabla b_{\Omega_n} \to \nabla b_\Omega$ in $L^p(U_h(D))^N$ implies that $\nabla b_{\Omega_n}^{h'} \to \nabla b_\Omega^{h'}$ in $L^p(U_h(D))^N$ and $\chi_{\Omega_n} \to \chi_\Omega$ in $L^p(U_h(D))$ since $\mathrm{m}_N(\Gamma_n) = 0$ and $\mathrm{m}_N(\Gamma) = 0$ for $C^{1,1-p/N}$-sets. Therefore $\nabla\varphi_n \cdot \nabla b_{\Omega_n}^{h'} \to \nabla\varphi_n \cdot \nabla b_\Omega^{h'}$ in $L^1(D)$-strong, $\chi_{\Omega_n} \rightharpoonup \chi_\Omega$ in $L^\infty(U_h(D))$-weak*, and the corresponding integrals converge. For the second term we already know from Theorem 3.2 (ii) that

$$\begin{aligned} &\Delta b_{\Omega_n} \chi_{U_h(\Gamma_n)} \rightharpoonup \Delta b_\Omega \chi_{U_h(\Gamma)} \text{ in } L^p(U_h(\Gamma))\text{-weak} \\ &\Rightarrow \Delta b_{\Omega_n}^h \rightharpoonup \Delta b_\Omega^h \text{ in } L^p(D)\text{-weak,} \end{aligned}$$

since $\Delta b_{\Omega_n}^h = e_{\Omega_n}^h \Delta b_{\Omega_n} + (e_{\Omega_n}^h)'$ with $(e_{\Omega_n}^h)' = 2\rho_h' \circ b_{\Omega_n} + b_{\Omega_n} \rho_h'' \circ b_{\Omega_n}$. So it is sufficient to show that $\chi_{\Omega_n}\varphi_n \to \chi_\Omega\varphi$ in $L^q(D)$-strong to get the convergence of the integral of $\chi_{\Omega_n}\varphi_n \, \Delta b_{\Omega_n}^{h'}$ to the integral of $\chi_\Omega\varphi \, \Delta b_\Omega^{h'}$. This follows from the following estimates and the assumptions

$$\begin{aligned} \|\chi_{\Omega_n}\varphi_n - \chi_\Omega\varphi\|_{L^q(D)} &\leq \|(\chi_{\Omega_n} - \chi_\Omega)\varphi\|_{L^q(D)} + \|\chi_{\Omega_n}(\varphi_n - \varphi)\|_{L^q(D)} \\ &\leq \|\chi_{\Omega_n} - \chi_\Omega\|_{L^q(D)} \|\varphi\|_{L^\infty(D)} + \|\varphi_n - \varphi\|_{L^q(D)}, \end{aligned}$$

where the right-hand side goes to zero as n goes to infinity. ■

4. Existence of solution

The objective functions defined on D or $U_h(\Gamma)$ make sense since ∇b_Ω and ∇b_{Ω_i} are well-defined a.e. on D or $U_h(\Gamma)$. Existence results are discussed for D. The functions defined on Γ require that Γ be sufficiently smooth to make sense of the integral with respect to the $(N-1)$-Hausdorff measure. Under this assumption $g_{0,p}$ makes sense, but $g_{1,p}$ further requires that ∇b_Ω and ∇b_{Ω_i} have a trace on Γ. Furthermore we need the continuity of the boundary integral with respect to the set in some appropriate topology. This is a much more demanding problem. Existence results are given for $g_{0,p}$ and $W^{2,p}$-Sobolev domains.

4.1. Objective functions on D

Recall the minimization problems for the functions $f_{0,p}([\Omega])$ and $f_{1,p}([\Omega])$ for a bounded open subset D of $\mathbf{R}^N$

$$f_{0,p}([\Omega]) = \sum_{i=1}^{I} \|b_\Omega - b_{\Omega_i}\|^p_{L^p(D)}, \quad f_{1,p}([\Omega]) = \sum_{i=1}^{I} \|b_\Omega - b_{\Omega_i}\|^p_{W^{1,p}(D)}.$$

The function $f_{0,p}([\Omega])$ is continuous with respect to the $C(\overline{D})$-metric topology associated with b_Ω. For D bounded, $C_b(D)$ is compact in $C(\overline{D})$ and $L^p(D)$ (see Delfour and Zolésio, 2001, Thm 2.2 (ii), Chapter 5) and we have existence of a minimizing $b_\Omega \in C_b(D)$. The function $f_{1,p}([\Omega])$ is continuous with respect to the $W^{1,p}$-metric topology associated with b_Ω in $W^{1,p}(D)$. For D bounded and any compact subfamily of $C_b(D)$ we have existence of a minimizing solution (see Theorem 3.6 in Delfour and Zolésio, 2004, and families of sets of locally bounded curvature with a bound on the norm of the Hessians, the uniform cusp and cone properties in Delfour and Zolésio, 2001).

4.2. Objective functions on Γ

The minimization problem for the functions

$$g_{0,p}([\Omega]) = \int_\Gamma |b_\Omega - b_{\Omega_i}|^p \, dx, \quad g_{1,p}([\Omega]) = \int_\Gamma |b_\Omega - b_{\Omega_i}|^p + |\nabla b_\Omega - \nabla b_{\Omega_i}|^p \, dx \tag{24}$$

is much more delicate since the integrals are over the variable boundary Γ. As in Section 2.2, assumptions on Γ and the Γ_i's are necessary to make sense of the objective functions. For instance assume that the conditions of Theorem 2.1 are satisfied and consider the family of *Sobolev domains* for which $b_\Omega \in W^{2,p}(U_h(\Gamma))$, $p > N$. In view of Theorem 3.2 (ii), they are of class $C^{1,1-N/p}$.

The minimization problems for the function $g_{0,p}([\Omega])$

$$\inf_{\substack{\Omega \subset D,\ b_\Omega \in W^{2,p}(U_h(\Gamma)) \\ \|\Delta b_\Omega\|_{L^p(U_h(\Gamma))} \le c}} \sum_{i=1}^{I} \int_\Gamma |b_{\Omega_i}|^p \, dx. \tag{25}$$

now makes sense and has a solution. Indeed by Theorem 3.3 the objective function $g_{0,p}([\Omega])$ is continuous in the $W^{1,p}$-topology and by Theorem 3.6 in Delfour and Zolésio (2004) we minimize over a compact family. The minimization problems for $g_{1,p}([\Omega])$

$$\inf_{\substack{\Omega\subset D,\ b_\Omega\in W^{2,p}(U_h(\Gamma))\\ \|\Delta b_\Omega\|_{L^p(U_h(\Gamma))}\le c}} \quad \sum_{i=1}^{I}\int_\Gamma |b_{\Omega_i}|^p + |\nabla b_\Omega - \nabla b_{\Omega_i}|^p\,dx. \tag{26}$$

is much more delicate in view of the presence of the gradient terms. The objective function is well-defined since the function b_Ω is at least C^1 on Γ but ∇b_{Ω_i} must be well-defined H_{N-1} a.e. on Γ as was done in the case $p = 2$ in Theorem 2.1 under the assumption $H_{N-1}(\Gamma \cap \mathrm{Sk}(\Omega_i)) = 0$. As for the continuity, from Theorem 3.3, it would require that $b_{\Omega_i} \in W^{2,q}(D)$.

5. Shape semiderivatives and application to b_Ω

In this section the elements of the velocity method and the notion of Eulerian semiderivative are briefly reviewed (see, for instance Delfour and Zolésio, 2001, Chapter 8) and applied to the computation of the semiderivative of $b_\Omega(x)$. From this we show that, under suitable assumptions on the velocity field, the oriented distance function and the projection onto the boundary are solutions of new nonlinear evolution equations for the initial sets with thin boundary.

In shape analysis the derivative of an objective function with respect to a set is obtained by generating perturbations of the set via a non-autonomous velocity field $V : [0,\tau] \times \mathbf{R}^N \to \mathbf{R}^N$, $0 < \tau < \infty$, verifying the conditions

$$\begin{aligned} \forall x \in \mathbf{R}^N, \quad & V(\cdot,x) \in C\big([0,\tau];\mathbf{R}^N\big),\\ \exists c>0,\ \forall x,y \in \mathbf{R}^N, \quad & \|V(\cdot,y)-V(\cdot,x)\|_{C([0,\tau];\mathbf{R}^N)} \le c|y-x|, \end{aligned} \tag{27}$$

where $V(\cdot,x)$ is the function $t \mapsto V(t,x)$. The *parameter* t can be viewed as an artificial time. A point X is moved to the position $x(t) = x(t;X)$ via the equation

$$\frac{dx}{dt}(t) = V(t,x(t)), \quad 0<t<\tau, \quad x(0) = X \in \mathbf{R}^N. \tag{28}$$

It will be convenient to define the velocity fields

$$x \mapsto V(t)(x) \overset{\text{def}}{=} V(t,x) : \mathbf{R}^N \to \mathbf{R}^N, \quad 0 \le t \le \tau. \tag{29}$$

This yields the families of transformations $\{T_t\}$ and perturbations $\{\Omega_t\}$

$$\forall t,\ 0<t<\tau, \quad \left|\begin{array}{l} X \mapsto T_t(X) \overset{\text{def}}{=} x(t) = x(t;X)\\ \forall \Omega \subset \mathbf{R}^N, \quad \Omega_t(V) \overset{\text{def}}{=} T_t(V)(\Omega). \end{array}\right. \tag{30}$$

Definition 5.1 *Given a* shape function f *defined on subsets* Ω *of* $\mathbf{R}^N$ *or* D, f *has a* Eulerian semiderivative *at* Ω *in the direction* V *if the following limit exists*

$$df(\Omega; V) \stackrel{\text{def}}{=} \lim_{t \searrow 0} \frac{f(\Omega_t(V)) - f(\Omega))}{t}.$$

6. Derivatives of the objective functions over D

Introduce the notation

$$b'_\Omega(x) \stackrel{\text{def}}{=} \frac{\partial}{\partial t} b_{\Omega_t}(x)\Big|_{t=0^+} \quad \text{and} \quad (b^2_\Omega)' \stackrel{\text{def}}{=} \frac{\partial}{\partial t} b^2_{\Omega_t}(x)\Big|_{t=0^+}$$

(those derivatives have been computed in Theorem 5.1 of Delfour and Zolésio (2004). From now on assume that the variable set Ω has *thin boundary*. First consider the objective function

$$f_{0,2}([\Omega]) \stackrel{\text{def}}{=} \sum_{i=1}^{I} \int_D |b_\Omega - b_{\Omega_i}|^2 \, dx. \tag{31}$$

By assumptions (27) on V and Theorem 5.1 (ii) from Delfour and Zolésio (2004), its shape semiderivative is

$$df_{0,2}([\Omega]; V) = \sum_{i=1}^{I} \int_D 2\,(b_\Omega - b_{\Omega_i})\, b'_\Omega \, dx = -\sum_{i=1}^{I} \int_D 2\,(b_\Omega - b_{\Omega_i})\, V(0) \cdot \nabla b_\Omega \, dx$$

without assumption on the the sets Ω_i. For the second objective function

$$f_{1,2}([\Omega]) \stackrel{\text{def}}{=} \sum_{i=1}^{I} \int_D |b_\Omega - b_{\Omega_i}|^2 + |\nabla b_\Omega - \nabla b_{\Omega_i}|^2 \, dx \tag{32}$$

we only need to concentrate on the second term

$$k([\Omega]) \stackrel{\text{def}}{=} \sum_{i=1}^{I} \int_D |\nabla b_\Omega - \nabla b_{\Omega_i}|^2 \, dx = \sum_{i=1}^{I} \int_D 2 - 2\, \nabla b_\Omega \cdot \nabla b_{\Omega_i} \, dx.$$

If, for each i, $b_{\Omega_i} \in W^{2,2}(D)$ and $V(0) = 0$ on ∂D, we get

$$df_{1,2}([\Omega]; V) = -2 \sum_{i=1}^{I} \int_D \{b_\Omega - b_{\Omega_i} + \Delta b_{\Omega_i}\}\, \nabla b_\Omega \cdot V(0) \circ p_\Gamma \, dx.$$

Otherwise its Eulerian semiderivative is *formally* given by the expression

$$\begin{aligned} dk([\Omega]; V) &= -2 \sum_{i=1}^{I} \int_D \nabla b_{\Omega_i} \cdot \nabla b'_\Omega \, dx \\ &= 2 \sum_{i=1}^{I} \int_D \nabla b_{\Omega_i} \cdot \nabla \left(\nabla b_\Omega \cdot V(0) \circ p_\Gamma\right) \, dx. \end{aligned} \tag{33}$$

In this computation we have *implicitly* used the property

$$(\nabla b_\Omega)' = \nabla b'_\Omega = -\nabla(\nabla b_\Omega \cdot V(0) \circ p_\Gamma)$$

which means that the terms

$$\nabla(\nabla b_\Omega \cdot V(0) \circ p_\Gamma) = D^2 b_\Omega V(0) \circ p_\Gamma + Dp_\Gamma{}^* DV(0) \circ p_\Gamma \nabla b_\Omega$$
$$Dp_\Gamma = I - \nabla b_\Omega {}^*\nabla b_\Omega - b_\Omega D^2 b_\Omega$$

make sense as vectors and matrices of L^2-functions. If $b_\Omega \in W^{2,2}(D)$:

$$df_{1,2}([\Omega]; V) = 2 \sum_{i=1}^{I} \int_D - \{b_\Omega - b_{\Omega_i}\} \nabla b_\Omega \cdot V(0) \circ p_\Gamma + \nabla b_{\Omega_i} \cdot \left(D^2 b_\Omega V(0) \circ p_\Gamma + Dp_\Gamma{}^* DV(0) \circ p_\Gamma \nabla b_\Omega\right) dx.$$

So, the semiderivative of the objective function $f_{1,2}([\Omega])$ exists if $b_\Omega \in W^{2,2}(D)$ or $b_{\Omega_i} \in W^{2,2}(D)$ for each i. This can be restrictive.

7. Derivative of the objective functions on Γ

Start with the square of the objective function (11) for $p = 2$

$$g_{0,2}([\Omega]) \stackrel{\text{def}}{=} \sum_{i=1}^{I} \int_\Gamma |b_\Omega(x) - b_{\Omega_i}(x)|^2 \, d\Gamma = \sum_{i=1}^{I} \int_\Gamma |b_{\Omega_i}(x)|^2 \, d\Gamma. \tag{34}$$

To compute the semiderivative of this boundary integral, use the following slight generalization from C^2 to $C^{1,1}$-domains of the theorem found in Delfour and Zolésio (2001, Chapter 8, § 4.2, Theorem 4.3, p. 355) and the results of Delfour (2000) for $C^{1,1}$-domains.

Theorem 7.1 *Let Ω be a bounded open subset of $\mathbf{R}^N$ of class $C^{1,1}$ with boundary Γ, that is, there exists $h > 0$ such that $b_\Omega \in C^{1,1}(U_h(\Gamma))$. Assume that, for some $\varepsilon > 0$ and $\tau > 0$, ψ is a function defined on $[0,\tau] \times U_h(\Gamma)$ such that*

$$\psi \in C^1([0,\tau]; H^{1/2+\varepsilon}(U_h(\Gamma))) \cap C^0([0,\tau]; H^{3/2+\varepsilon}(U_h(\Gamma)))$$

and that $V \in C^0([0,\tau]; C^1_{\text{loc}}(\mathbf{R}^N, \mathbf{R}^N))$. The semiderivative of the function

$$J_V(t) \stackrel{\text{def}}{=} \int_{\Gamma_t(V)} \psi(t) \, d\Gamma_t$$

with respect to $t > 0$ in $t = 0$ is given by

$$\begin{aligned} dJ_V(0) &= \int_\Gamma \psi'(0) + \nabla\psi \cdot V(0) + \psi \, (\operatorname{div} V(0) - DV(0)n \cdot n) \, d\Gamma \\ &= \int_\Gamma \psi'(0) + \left(\frac{\partial\psi}{\partial n} + H\psi\right) V(0) \cdot n \, d\Gamma, \end{aligned} \tag{35}$$

where $\psi'(0)(x) \stackrel{\text{def}}{=} \partial\psi/\partial t(0,x)$.

7.1. Objective function $g_{0,2}$

First apply Theorem 7.1 to the objective function $g_{0,2}$ with $\psi(t) = b^2_{\Omega_i}$. Here there is no dependence on t and we get the following theorem with an assumption on $b^2_{\Omega_i}$ in the tubular neighborhood of Γ for all $i \in I$.

THEOREM 7.2 *Let Ω be a bounded open subset of $\mathbf{R}^{\mathrm{N}}$ of class $C^{1,1}$ with boundary Γ, that is, there exists $h > 0$ such that $b_\Omega \in C^{1,1}(U_h(\Gamma))$. Assume that*

$$\forall i \in I, \quad b^2_{\Omega_i} \in H^{3/2+\varepsilon}(U_h(\Gamma))$$

for some $\varepsilon > 0$. Further assume that $V \in C^0([0,\tau]; C^1_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}}))$. Then

$$\begin{aligned} dg_{0,2}([\Omega]; V(0)) &= \sum_{i=1}^{I} \int_\Gamma \nabla b^2_{\Omega_i} \cdot V(0) + b^2_{\Omega_i} \left(\operatorname{div} V(0) - DV(0)\nabla b_\Omega \cdot \nabla b_\Omega\right) d\Gamma \\ &= \sum_{i=1}^{I} \int_\Gamma \left(\frac{\partial}{\partial n} b^2_{\Omega_i} + \Delta b_\Omega \, b^2_{\Omega_i} \right) V(0) \cdot n \, d\Gamma. \end{aligned} \tag{36}$$

The assumption on the $b^2_{\Omega_i}$'s implies that for all $i \in I$ the skeleton $Sk(\Omega_i)$ of Ω_i is outside of the tubular neighborhood $U_h(\Gamma)$ of Γ. It is interesting to observe that the two formulae (36) would still be well-defined or could be relaxed to weaker assumptions. Of course their validity would be contingent to the availability of a new proof of the theorem under such weaker assumptions.

For instance the first formula would still be well-defined for a domain Ω with a thin boundary Γ with normal ($\nabla b_\Omega(x)$ exists H_{N-1} a.e. on Γ) and $V(0)$ of class C^1, provided that $H_{N-1}(\mathrm{Sk}(\Omega_i) \cap \Gamma) = 0$, for all $i \in I$. Furthermore, if $H_{N-1}(\mathrm{Sk}(\Omega_i) \cap \Gamma) > 0$ for some $i \in I$, the term $\nabla b^2_{\Omega_i} \cdot V(0)$ makes sense on $\Gamma \backslash \mathrm{Sk}(\Omega_i)$, but on $\mathrm{Sk}(\Omega_i) \cap \Gamma$ it can be replaced by the semiderivative

$$db^2_{\Omega_i}(x; V(0,x)) = 2 \min_{p_i \in \Pi_{\Gamma_i}(x)} (x - p_i) \cdot V(0, p_i)$$

(see Delfour and Zolésio, 2001, Chapter 8, Definition 2.1 (ii)). Finally, the first formula (36) would become

$$\begin{aligned} & dg_{0,2}([\Omega]; V(0)) \\ &= \sum_{i=1}^{I} \int_\Gamma 2 \min_{p_i \in \Pi_{\Gamma_i}(x)} (x - p_i) \cdot V(0, p_i) + b^2_{\Omega_i} \left(\operatorname{div} V(0) - DV(0)\nabla b_\Omega \cdot \nabla b_\Omega\right) d\Gamma. \end{aligned}$$

The same technique can be applied to the second formula with the assumption that Ω be of class $C^{1,1}$ in which case $D^2 b_\Omega$ and Δb_Ω exist and are bounded H_{N-1} a.e. on Γ (see Delfour, 2000). Indeed on $\Gamma \cap \mathrm{Sk}(\Omega_i)$

$$db^2_{\Omega_i}(x; \nabla b_\Omega(x)) = 2 \min_{p_i \in \Pi_{\Gamma_i}(x)} (x - p_i) \cdot \nabla b_\Omega(x)$$

(see Delfour and Zolésio, 2001, Chapter 8, Definition 2.1 (ii)) and the second formula (36) would yield

$$\begin{aligned} &dg_{0,2}([\Omega]; V(0)) \\ &= \sum_{i=1}^{I} \int_\Gamma \left(2 \min_{p_i \in \Pi_{\Gamma_i}(x)} (x - p_i) \cdot \nabla b_\Omega(x) + \Delta b_\Omega(x)\, b^2_{\Omega_i}(x) \right) V(0)(x) \cdot n(x)\, d\Gamma. \end{aligned}$$

From the computational point of view the projections $p_i = p_{\Gamma_i}$ are as easy to generate as the partial derivative $\partial b^2_{\Omega_i}/\partial n$. Note that the second formula requires more smoothness on Ω than the first one.

7.2. Objective function $g_{1,2}$

Now turn to the objective function (12) with $p = 2$

$$g_{1,2}([\Omega]) \stackrel{\text{def}}{=} \sum_{i=1}^{I} \int_\Gamma b^2_{\Omega_i} + |\nabla b_\Omega - \nabla b_{\Omega_i}|^2\, d\Gamma. \tag{37}$$

In general this function is not well-defined since the gradients may not exist on Γ. It is well-defined under the assumptions of Theorem 2.1 and

$$g_{1,2}([\Omega]) = \sum_{i=1}^{I} \left\{ \int_\Gamma b^2_{\Omega_i}\, d\Gamma + 2\, H_{N-1}(\Gamma) - 2 \int_{\Gamma \setminus \mathrm{Sk}(\Omega_i)} \nabla b_{\Omega_i} \cdot \nabla b_\Omega\, d\Gamma \right\} \tag{38}$$

is well-defined, but for all $i \in I$, $H_{N-1}(\Gamma \cap \mathrm{Sk}(\Omega_i)) = 0$.

There are three terms in the objective function and they are semidifferentiable under different sets of assumptions. The first term has been discussed in Section 7.1. It is differentiable under the assumptions of Theorem 7.2 which means that $\mathrm{Sk}(\Omega_i) \cap \Gamma = \varnothing$ for all $i \in I$. Theorem 7.1 can be applied to the second term $H_{N-1}(\Gamma)$ under the assumption that Ω be of class $C^{1,1}$:

$$\frac{d}{dt} \int_{\Gamma_t} d\Gamma_t \bigg|_{t=0} = \int_\Gamma H\, V(0) \cdot n\, d\Gamma, \quad H \stackrel{\text{def}}{=} \Delta b_\Omega. \tag{39}$$

As for the last term it can be handled without the formulae of Theorem 7.1. We shall consider two cases: $\Gamma \cap \mathrm{Sk}(\Omega_i) = \varnothing$ and, in dimension $N = 2$, the special case where $\Gamma \cap \mathrm{Sk}(\Omega_i)$ is a singleton. We also consider an example of an objective function $g_{1,2}$ which is non-differentiable unless weighted norms are used on V near points a_i of $\mathrm{Sk}(\Omega_i)$ where the Hessian matrix has terms of the form $1/|x - a_i|$. So there are several open issues floating around.

7.2.1. The b_{Ω_i}'s are smooth in $U_h(\Gamma)$

Assume that $b_\Omega \in C^{1,1}(U_h(\Gamma))$ for some $h > 0$. Let e^h_Ω be the extension of the function equal to one by zero outside $U_h(\Gamma)$ as introduced in Definition 3.1. Under this assumption and those of Theorem 2.1

$$\begin{aligned} g_{1,2}([\Omega_t]) &\overset{\text{def}}{=} \sum_{i=1}^{I} \int_{\Gamma_t} b^2_{\Omega_i} + (2 - 2\,\nabla b_{\Omega_t} \cdot \nabla b_{\Omega_i})\, d\Gamma_t \\ &= \sum_{i=1}^{I} \int_{\Gamma_t} b^2_{\Omega_i}\, d\Gamma_t + 2 \int_{\Gamma_t} d\Gamma_t - 2 \int_{\Gamma_t} e^h_\Omega\, \nabla b_{\Omega_i} \cdot n_t\, d\Gamma_t \\ &= \sum_{i=1}^{I} \int_{\Gamma_t} b^2_{\Omega_i}\, d\Gamma_t + 2 \int_{\Gamma_t} d\Gamma_t - 2 \int_{\Omega_t} \operatorname{div}\left(e^h_\Omega\, \nabla b_{\Omega_i}\right) dx. \end{aligned}$$

The semiderivative of the first and second terms have already been computed. Apply the formula for the volume integral (see Delfour and Zolésio, 2001, Chapter 8, Thm 4.2) to the third term

$$\begin{aligned} dg_{1,2}([\Omega]; V) &\overset{\text{def}}{=} \sum_{i=1}^{I} \int_{\Gamma} \left\{ H\, b^2_{\Omega_i} + \frac{\partial}{\partial n} b^2_{\Omega_i} + 2\, H - 2 \operatorname{div}\left(e^h_\Omega\, \nabla b_{\Omega_i}\right) \right\} V(0) \cdot n\, d\Gamma \\ &= \sum_{i=1}^{I} \int_{\Gamma} \left\{ H\, b^2_{\Omega_i} + \frac{\partial}{\partial n} b^2_{\Omega_i} + 2\, H - 2\Delta b_{\Omega_i} \right\} V(0) \cdot n\, d\Gamma, \end{aligned}$$

since $e^h_\Omega = 1$ on Γ. This formula requires that $e^h_\Omega\, \nabla b_{\Omega_i} \in W^{2,1}_{\text{loc}}(\mathbf{R}^{\mathrm{N}})$. For instance this condition will be verified for $b_{\Omega_i} \in W^{3,1}(U_{h'}(\Gamma))$ for some h', $0 < h' \leq h$, and even for $b_{\Omega_i} \in H^{5/2+\varepsilon}(U_{h'}(\Gamma))$ for some $\varepsilon > 0$ by using the derivative of the boundary integral given by Theorem 7.1.

THEOREM 7.3 *Let Ω be a bounded open subset of $\mathbf{R}^{\mathrm{N}}$ with boundary Γ such that $b_\Omega \in C^{1,1}(U_h(\Gamma))$ for some $h > 0$. Assume that*

$$\exists h',\ 0 < h' \leq h, \quad \forall i \in I, \quad b^2_{\Omega_i} \in H^{3/2+\varepsilon}(U_{h'}(\Gamma)) \text{ and } b_{\Omega_i} \in H^{5/2+\varepsilon}(U_{h'}(\Gamma))$$

for some $\varepsilon > 0$. Further assume that $V \in C^0([0,\tau]; C^1_{\text{loc}}(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}}))$. Then

$$\begin{aligned} & dg_{1,2}([\Omega]; V(0)) \\ &= \sum_{i=1}^{I} \int_{\Gamma} \left\{ \frac{\partial}{\partial n} b^2_{\Omega_i} + H\, b^2_{\Omega_i} + 2\,(H - \Delta b_{\Omega_i}) \right\} V(0) \cdot n\, d\Gamma. \end{aligned} \tag{40}$$

7.2.2. The skeleton $\mathrm{Sk}(\Omega_i)$ intersects Γ in dimension $N = 2$

It is instructive to look at what is happening in dimension 2 to the semiderivative of the third term in expression (38) when a component of the skeleton $\mathrm{Sk}(\Omega_i)$

of Ω_i is a segment of smooth curve which crosses Ω in one point $a \in \Gamma \cap \mathrm{Sk}(\Omega_i)$ which is not an end point of $\mathrm{Sk}(\Omega_i)$. Note that this case has been excluded by the assumptions used to get the semiderivative of the first term in Theorem 7.3.

Consider the simple example where Ω_i is a triangle. Its skeleton $\mathrm{Sk}(\Omega_i)$ is made up of the three bisectors as shown in broken lines on Fig. 1. The part of $\mathrm{Sk}(\Omega_i)$ intersecting the smooth domain Ω_t is a segment of line. Put a clockwise orientation on the boundary of Ω_t. By convention the clockwise orientation will correspond to the normal n_t on Γ_t pointing outward of Ω_t.

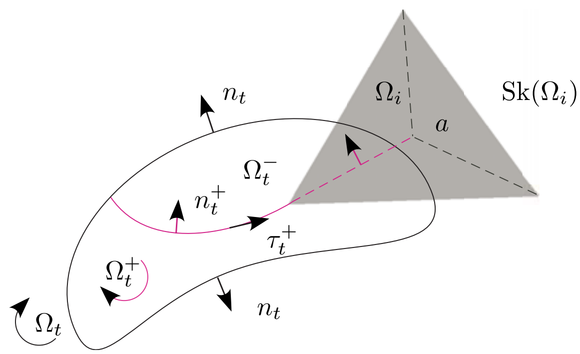

Figure 1. Domains Ω_t, Ω_t^+, Ω_t^-, and Ω_i with clockwise orientation of Ω_t and Ω_t^+.

The piece of skeleton inside Ω_t can be seen as a crack in the domain. What we need is a Stokes formula for a domain with a crack to specify the boundary term on the crack. First extend the piece of $\mathrm{Sk}(\Omega_i)$ in Ω_t to get a smooth interface Σ_t in Ω_t which divides up Ω_t into two domains Ω_t^+ and Ω_t^-. Choose the clockwise orientation on Ω_t^+. This means that on Σ_t we choose the unit exterior normal n_t^+ to Ω_t^+ and the unit tangent vector τ_t^+ in the clockwise direction. Choose as the direction of the normal n_i and the tangent τ_i to the skeleton $\mathrm{Sk}(\Omega_i)$ the ones specified by Ω_t^+ on the piece of boundary $\mathrm{Sk}(\Omega_i) \cap \Sigma_t$ of Γ_t^+. With this construction use Stokes Theorem in Ω_t^+ and Ω_t^-.

$$
\begin{aligned}
\int_{\Omega_t \setminus \mathrm{Sk}(\Omega_i)} \Delta b_{\Omega_i}\, dx &= \int_{\Omega_t^+} \Delta b_{\Omega_i}\, dx + \int_{\Omega_t^-} \Delta b_{\Omega_i}\, dx \\
&= \int_{\Gamma_t^+} \nabla b_{\Omega_i} \cdot n_t^+ \, d\Gamma_t + \int_{\Gamma_t^-} \nabla b_{\Omega_i} \cdot n_t^- \, d\Gamma_t \\
&= \int_{\Gamma_t} \nabla b_{\Omega_i} \cdot n_t \, d\Gamma_t + \int_{\Sigma_t} \nabla b_{\Omega_i}{}^{+} \cdot n_t^+ + \nabla b_{\Omega_i}{}^{-} \cdot n_t^- \, d\Gamma_t \\
&= \int_{\Gamma_t} \nabla b_{\Omega_i} \cdot n_t \, d\Gamma_t + \int_{\Sigma_t} [\nabla b_{\Omega_i}] \cdot n_t^+ \, d\Gamma_t
\end{aligned}
$$

$$= \int_{\Gamma_t} \nabla b_{\Omega_i} \cdot n_t \, d\Gamma_t + \int_{\Omega_t \cap \mathrm{Sk}(\Omega_i)} [\nabla b_{\Omega_i}] \cdot n_i \, d\Gamma_t$$

where ${\nabla b_{\Omega_i}}^+$ and ${\nabla b_{\Omega_i}}^-$ are the vector ∇b_{Ω_i} in the domains Ω_t^+ and Ω_t^-, and

$$[\nabla b_{\Omega_i}] \stackrel{\text{def}}{=} {\nabla b_{\Omega_i}}^+ - {\nabla b_{\Omega_i}}^-$$

is the jump of the ∇b_{Ω_i} across $Sk(\Omega_i)$. Finally we obtain the following identity

$$\begin{aligned} \int_{\Gamma_t} \nabla b_{\Omega_i} \cdot n_t \, d\Gamma_t &= \int_{\Omega_t \setminus \mathrm{Sk}(\Omega_i)} \Delta b_{\Omega_i} \, dx - \int_{\Omega_t \cap \mathrm{Sk}(\Omega_i)} [\nabla b_{\Omega_i}] \cdot n_i \, d\Gamma_t \\ &= \int_{\Omega_t} \Delta b_{\Omega_i} \, dx - \int_{\Omega_t \cap \mathrm{Sk}(\Omega_i)} [\nabla b_{\Omega_i}] \cdot n_i \, d\Gamma_t \end{aligned} \tag{41}$$

since $\mathrm{Sk}(\Omega_i)$ has zero measure. The artificial part of Σ_t has disappeared.

This is the delicate term appearing in the objective function

$$g_{1,2}([\Omega_t]) = \sum_{i=1}^{I} \int_{\Gamma_t} b_{\Omega_i}^2 + 2 \; - 2 \, \nabla b_{\Omega_t} \cdot \nabla b_{\Omega_i} \, d\Gamma_t.$$

We leave aside the first two terms and concentrate on the term involving the inner product of the gradients and use expression (41) to differentiate it

$$k([\Omega_t]) \stackrel{\text{def}}{=} \int_{\Omega_t} \Delta b_{\Omega_i} \, dx - \int_{\Omega_t \cap \mathrm{Sk}(\Omega_i)} [\nabla b_{\Omega_i}] \cdot n_i \, d\Gamma.$$

The first term is a standard integral with an integrand which is independent of t. It is semidifferentiable for $\Delta b_{\Omega_i} \in W^{1,1}(U_h(\Gamma) \setminus \mathrm{Sk}(\Omega_i))$. The second term is an integral over a curve with a free end point $T_t(a)$ on Γ_t since the point $a \in \Gamma \cap \mathrm{Sk}(\Omega_i)$ is not an end point of $\mathrm{Sk}(\Omega_i)$. Using the orientation τ_i along that curve $\mathrm{Sk}(\Omega_i) \cap \Omega_t$

$$dk([\Omega]; V) = \int_{\Gamma} \Delta b_{\Omega_i} \, V(0) \cdot n \, d\Gamma + [\nabla b_{\Omega_i}](a) \cdot n_i(a) \, V(0,a) \cdot \tau_i(a), \tag{42}$$

where τ_i is the unit tangent vector to the skeleton in the direction corresponding to the clockwise orientation of Ω^+ and, a fortiori, of Σ and the piece of skeleton $\mathrm{Sk}(\Omega_i)$ which intersects Ω. This means that τ_i points away from Ω on Γ.

Observe that the expression (42) of the semiderivative is linear in V and that its support is Γ. So k is *differentiable*. For the example of the triangle $\Delta b_{\Omega_i} = 0$ almost everywhere in a neighborhood of the point a on $\mathrm{Sk}(\Omega_i) \cap \Sigma$ since a is not an end point of $\mathrm{Sk}(\Omega_i)$ and the formula reduces to

$$dk([\Omega]; V) = [\nabla b_{\Omega_i}](a) \cdot n_i(a) \, V(a) \cdot \tau_i(a).$$

We have proved the following result:

THEOREM 7.4 *Let Ω be a bounded open subset of $\mathbf{R}^{\mathrm{N}}$ of class $C^{1,1}$ with boundary Γ, that is, there exists $h > 0$ such that $b_\Omega \in C^{1,1}(U_h(\Gamma))$. Assume that $\mathrm{Sk}(\Omega_i) \cap \Omega$ is a smooth curve and that $\mathrm{Sk}(\Omega_i) \cap \Gamma = \{a\}$ is a singleton and that a is not an end point of $\mathrm{Sk}(\Omega_i)$. Further assume that*

$$\forall i \in I, \quad \Delta b_{\Omega_i} \in W^{1,1}(U_h(\Gamma) \setminus \mathrm{Sk}(\Omega_i))$$

and that $V \in C^0([0,\tau]; C^1_{\mathrm{loc}}(\mathbf{R}^{\mathrm{N}}, \mathbf{R}^{\mathrm{N}}))$. Then the derivative is given by

$$dk([\Omega]; V) = \int_\Gamma \Delta b_{\Omega_i}\, V(0) \cdot n\, d\Gamma + [\nabla b_{\Omega_i}](a) \cdot n_i(a)\, V(0,a) \cdot \tau_i(a). \tag{43}$$

REMARK 7.1 *In the theorem the condition $\Delta b_{\Omega_i} \in W^{1,1}(U_h(\Gamma) \setminus \mathrm{Sk}(\Omega_i))$ does not mean that $b_{\Omega_i} \in W^{3,1}(U_h(\Gamma))$ as can be seen from the examples of the triangle or the square. When a is an end point of the smooth segment of $\mathrm{Sk}(\Omega_i)$, the objective function might only be semidifferentiable in some directions V or not be semidifferentiable at all.*

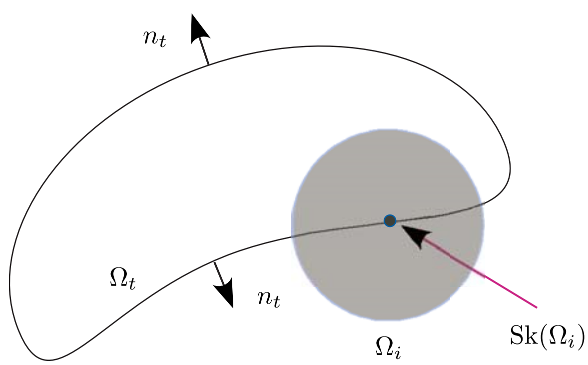

Figure 2. Domains Ω_t and Ω_i and skeleton $\mathrm{Sk}(\Omega_i)$ for the disk

In dimension 2 another special case is when a point a of Γ is also an isolated point of $\mathrm{Sk}(\Omega_i)$ for some $i \in I$. A typical simple example is the disk of center a and radius R whose skeleton is $\{a\}$. As Γ is moved to Γ_t the point $T_t(a)$ is on Γ_t or an isolated point inside or outside of Ω_t. When $a \in \overline{\Omega_t}$, we do not obtain the usual Stokes formula on Γ_t since Δb_i has a singularity in $x = a$. To be more specific, recall from Delfour and Zolésio (2001, Chapter 5, Example 6.2) that for $\Omega_i = B_R(a)$

$$b_{\Omega_i}(x) = |x-a| - R, \quad \nabla b_{\Omega_i}(x) = \frac{x-a}{|x-a|},$$

$$< \Delta b_{\Omega_i}, \varphi >= \int_{\mathbf{R}^2} \frac{1}{|x-a|} \varphi\, dx \quad \Rightarrow \Delta b_{\Omega_i} = \frac{1}{|x-a|}.$$

But, since $a \in \Gamma$, Δb_{Ω_i} is not an L^1-function on the curve Γ and the term

$$\int_\Gamma \Delta b_{\Omega_i} V(0) \cdot n \, d\Gamma = \int_\Gamma \frac{1}{|x-a|} V(0) \cdot n \, d\Gamma$$

which appears in formula (43) can blow up depending on the choice of V! Furthermore it is not clear what will happen to the second term of formula (43). So $g_{1,2}([\Omega])$ is probably not semidifferentiable for all V. Another problematic example is the circle for which Δb_{Ω_i} is a bounded measure

$$b_{\Omega_i}(x) = \big||x-a| - R\big|, \quad \nabla b_{\partial\Omega_i}(x) = \begin{cases} \frac{x-a}{|x-a|}, & |x-a| > R, \\ -\frac{x-a}{|x-a|}, & |x-a| < R, \end{cases}$$

$$< \Delta b_{\Omega_i}, \varphi >= 2 \int_{\Gamma_i} \varphi \, ds - \int_{\Omega_i} \frac{1}{|x-a|} \varphi \, dx + \int_{\complement\Omega_i} \frac{1}{|x-a|} \varphi \, dx$$

$$\Rightarrow \Delta b_{\Omega_i} = 2\delta_{\Gamma_i} - \frac{1}{|x-a|} \chi_{\Omega_i} + \frac{1}{|x-a|} \chi_{\mathbf{R}^N \setminus \Omega_i}.$$

7.2.3. The skeleton $\mathrm{Sk}(\Omega_i)$ intersects Γ in dimension N

In dimension N, one issue is to find Stokes formulas for smooth domains with cracks of codimension 1, 2, 3, etc. When the piece of skeleton which intersects Γ is a piece of smooth submanifold of co-dimension 1 and $\Gamma \cap \mathrm{Sk}(\Omega_i)$ is a piece of smooth submanifold of co-dimension 2, the formula obtained in the special case $N = 2$ will very likely generalize to an expression of the following type

$$\begin{aligned} & dk([\Omega]; V(0)) \\ & = 2 \sum_{i=1}^{I} \int_\Gamma \Delta b_{\Omega_i} V \cdot n \, d\Gamma - \int_{\Gamma \cap \mathrm{Sk}(\Omega_i)} [\nabla b_{\Omega_i}] \cdot n_i \; V(0) \cdot \tau_i \, dH_{N-2}, \end{aligned} \tag{44}$$

where τ_i is the unit vector tangent to $\mathrm{Sk}(\Omega_i)$, normal to $\Omega \cap \mathrm{Sk}(\Omega_i)$ on the $(N-2)$-dimensional submanifold $\Gamma \cap \mathrm{Sk}(\Omega_i)$, and pointing outward of $\Omega \cap \mathrm{Sk}(\Omega_i)$.

8. A simple example for the objective function $g_{0,2}$

In this section we consider the problem of specifying a *nominal object* from I samples of that object using the objective function $g_{0,2}$. Here the object is not assumed to be the boundary of a smooth set. For instance, it can feature cracks in dimension 2. Conversely, the nominal object and the maximum deviation as measured by $g_{0,2}$ can be used as criteria to detect similar objects from several observations.

Since only the boundary Γ of the object Ω appears in $g_{0,2}$, it is sufficient to specify Γ. Further assume that Γ is the union of line segments in $\mathbf{R}^2$. For instance Ω could be a *cracked set* first introduced in Delfour and Zolésio

(2004) in the context of the segmentation problem of Mumford and Shah. Let $x_1, x_2, \dots, x_j, \dots, x_M$ be a sequence of M distinct points in $\mathbf{R}^2$, $A = \{a_{jk}\}$ the *connectivity matrix*, and C_{jk} the line between the points x_j and x_k

$$a_{jk} \overset{\text{def}}{=} \begin{cases} 1, & \text{if } x_j \text{ and } x_k \text{ are connected} \\ 0, & \text{if } x_j \text{ and } x_k \text{ are not connected} \end{cases} \tag{45}$$

$$C_{jk} \overset{\text{def}}{=} \{sx_j + (1-s)x_k : s \in [0,1]\}, \quad \text{if } a_{jk} = 1. \tag{46}$$

From another viewpoint, the above set made up of pieces of lines can also be viewed as an approximation to cracked sets and the material below as a first step towards a *numerical implementation* of a descent method.

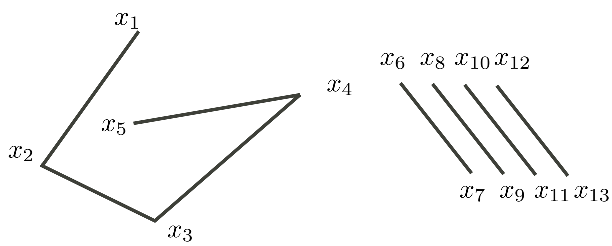

Figure 3. Boundary Γ of a cracked set Ω.

Clearly $a_{jk} = a_{kj}$ and $C_{jk} = C_{kj}$. With the above definitions

$$\Gamma \overset{\text{def}}{=} \bigcup_{\substack{j<k \\ a_{jk}=1}} C_{jk}, \quad g_{0,2} = \sum_{i=1}^{M} \int_\Gamma |b_\Omega - b_{\Omega_i}|^2 \, d\Gamma = \sum_{j=1}^{M} \sum_{\substack{j<k\leq M \\ a_{jk}=1}} \int_{C_{jk}} \sum_{i=1}^{M} b_{\Omega_i}^2 \, d\Gamma. \tag{47}$$

The sample objects can be specified by the sets Ω_i's or their boundaries Γ_i's since $b_{\Omega_i}^2 = d_{\Gamma_i}^2$. For simplicity further introduce the notation $f = \sum_{i=1}^{I} b_{\Omega_i}^2$. Each $b_{\Omega_i}^2$ is semi-differentiable in the sense explained for the formula (14) and $db_{\Omega_i}^2(x;v)$ can be explicitly computed from the set of projections $\Pi_{\Gamma_i}(x)$ of x onto Γ_i

$$db_{\Omega_i}^2(x;v) = \min_{p \in \Pi_{\Gamma_i}(x)} (x-p) \cdot v. \tag{48}$$

Therefore $df(x;v)$ exists and can be explicitly computed.

To complete this section we compute the directional semiderivative of the objective function with respect to node x_j in the direction v. Only the terms connected to x_j will depend on x_j

$$j(x_j) \stackrel{\text{def}}{=} \sum_{\substack{k\neq j\\ a_{jk}=1}} \int_{C_{jk}} f\,d\Gamma = \sum_{\substack{k\neq j\\ a_{jk}=1}} \int_0^1 f(sx_j+(1-s)x_k)\,|x_j-x_k|\,ds. \tag{49}$$

Compute for $t>0$ the k-term of the differential quotient $[j(x_j+tv)-j(x_j)]/t$

$$\begin{aligned}
&\frac{1}{t}\left[\int_0^1 f(s[x_j+tv]+(1-s)x_k)\,|[x_j+tv]-x_k|\right.\\
&\qquad\left. - f(sx_j+(1-s)x_k)\,|x_j-x_k|\,ds\right]\\
&\to \int_0^1 df(sx_j+(1-s)x_k;sv)\,|x_j-x_k|\\
&\qquad + f(sx_j+(1-s)x_k)\,\frac{x_j-x_k}{|x_j-x_k|}\cdot v\,ds
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\; dj(x_j;v) = &\int_0^1 df(sx_j+(1-s)x_k;v)\,|x_j-x_k| s\,ds\\
&+ \frac{x_j-x_k}{|x_j-x_k|}\cdot v\int_0^1 f(sx_j+(1-s)x_k)\,ds
\end{aligned} \tag{50}$$

This expression is made up of a differentiable part and a nondifferentiable part when the skeleton of one of the Ω_i's intersects the curve Γ along one or more segments. It does not require any semidifferentiability assumption on the b_{Ω_i}'s. Since the formula is valid for any semidifferentiable function f, it is interesting to consider the case $f=1$ for which the objective functional is equal to the total length of the boundary Γ. Here the nondifferentiable term is zero and the integral in the second term is one. We are left with the sum of the tangent vectors to all lines connected to x_j.

It is interesting to notice that the above formula can be obtained by the special choice of velocity

$$V_j(x) \stackrel{\text{def}}{=} c_j(x)\,v \tag{51}$$

associated with the point x_j and the direction $v\in\mathbf{R}^2$, where c_j is a continuous piecewise linear function such that at the node x_k

$$c_j(x_k) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } x_k = x_j\\ 0, & \text{if } x_k \neq x_j.\end{cases} \tag{52}$$

This approach was used in Zolésio (1984) to compute the derivative of an objective function that depends on the solution of a finite element problem with respect to the internal nodes of the triangulation of the underlying domain. For t

sufficiently small, the transformation generated by V_j is $T_t(x) = x + t\, c_j(x) v$ that maps lines onto lines. Indeed for the objective function

$$\int_{x_k}^{x_j} f\, d\Gamma,$$

we have

$$\begin{aligned}\delta(t) &\stackrel{\text{def}}{=} \frac{1}{t} \int_{T_t(x_k)}^{T_t(x_j)} f\, d\Gamma - \int_{x_k}^{x_j} f\, d\Gamma \\ &= \frac{1}{t} \int_{x_k}^{x_j} f \circ T_t \frac{|T_t(x_j) - T_t(x_k)|}{|x_j - x_k|} - f\, d\Gamma \\ &= \int_{x_k}^{x_j} \frac{f \circ T_t - f}{t} \frac{|T_t(x_j) - T_t(x_k)|}{|x_j - x_k|} - f \frac{1}{t} \left[\frac{|T_t(x_j) - T_t(x_k)|}{|x_j - x_k|} - 1 \right] d\Gamma.\end{aligned}$$

But

$$\frac{f(T_t(x)) - f(x)}{t} = \frac{f(x + t\, c_j(x) v) - f(x)}{t} \to df(x; c_j(x) v) = c_j(x)\, df(x; v)$$

since $c_j(x)$ is positive. Also for $x_j \neq x_k$

$$\frac{1}{t} \left[\frac{|T_t(x_j) - T_t(x_k)|}{|x_j - x_k|} - 1 \right] \to \frac{x_j - x_k}{|x_j - x_k|} \cdot v \frac{1}{|x_j - x_k|}.$$

Finally we get the intrinsic form of formula (50)

$$\int_{x_k}^{x_j} c_j(x)\, df(x; v) + \frac{x_j - x_k}{|x_j - x_k|} \cdot v\, \frac{1}{|x_j - x_k|} f\, d\Gamma. \tag{53}$$

In the above model the unknowns are the nodes and possibly the connectivity matrix A. In its full generality the numerical minimization will require nondifferentiable optimization techniques and 0-1 combinatorial methods to take care of the matrix A. Implementation of such methods is obviously beyond the scope of this paper.

The above constructions and computations can be extended from piecewise linear curves to piecewise Bézier curves currently used in aeronautics and other areas. For instance, the case of the piecewise second order Bézier curves can readily be obtained by adding to each pair of connected nodes x_j and x_k a *control node* u_{jk} and modifying the definition of the sets C_{jk} from lines to curves as follows

$$C_{jk} \stackrel{\text{def}}{=} \left\{ s^2 x_j + 2s(1-s) u_{jk} + (1-s)^2 x_k : s \in [0,1] \right\}, \quad \text{if } a_{jk} = 1,$$

where $s = 0$ corresponds to the point x_k and $s = 1$ to the point x_j. To complete this section we compute the directional semiderivative of the objective function with respect to node x_j in the direction v. Only the terms that correspond to segments connected to x_j will depend on x_j

$$
\begin{aligned}
j(x_j) &\stackrel{\text{def}}{=} \sum_{\substack{k\neq j\\ a_{jk}=1}} \int_{C_{jk}} f\, d\Gamma \\
&= \sum_{\substack{k\neq j\\ a_{jk}=1}} \int_0^1 f(s^2 x_j + 2s(1-s)u_{jk} \\
&\qquad\qquad + (1-s)^2 x_k)\, 2|s x_j + (1-2s)u_{jk} - (1-s)x_k|\, ds
\end{aligned}
$$

and its semiderivative in the direction v is given by the following expression

$$
\begin{aligned}
dj(x_j;v) = \sum_{\substack{k\neq j\\ a_{jk}=1}} &\int_0^1 df(s^2 x_j + 2s(1-s)u_{jk} + (1-s)^2 x_k; v) \\
&\qquad\qquad |s x_j + (1-2s)u_{jk} - (1-s)x_k|\, 2s^2\, ds \\
&+ \int_0^1 f(s^2 x_j + 2s(1-s)u_{jk} + (1-s)^2 x_k) \\
&\qquad\qquad \frac{s x_j + (1-2s)u_{jk} - (1-s)x_k}{|s x_j + (1-2s)u_{jk} - (1-s)x_k|}\cdot v\, 2s\, ds.
\end{aligned}
$$

Similarly for a pair of indices such that $a_{jk}=1$, the only term depending on the control node u_{jk} is

$$
\begin{aligned}
j(u_{jk}) &\stackrel{\text{def}}{=} \int_{C_{jk}} f\, d\Gamma \\
&= \int_0^1 f(s^2 x_j + 2s(1-s)u_{jk} + (1-s)^2 x_k)\, 2|s x_j + (1-2s)u_{jk} - (1-s)x_k|\, ds
\end{aligned}
$$

and its semiderivative in the direction v is given by the following expression

$$
\begin{aligned}
dj(u_{jk};v) = &\int_0^1 df(s^2 x_j + 2s(1-s)u_{jk} + (1-s)^2 x_k; v) \\
&\qquad\qquad |s x_j + (1-2s)u_{jk} - (1-s)x_k|\, 4s(1-s)\, ds \\
&+ \int_0^1 f(s^2 x_j + 2s(1-s)u_{jk} + (1-s)^2 x_k) \\
&\qquad\qquad \frac{s x_j + (1-2s)u_{jk} - (1-s)x_k}{|s x_j + (1-2s)u_{jk} - (1-s)x_k|}\cdot v\, 2(1-2s) ds.
\end{aligned}
$$

The above formula can also be obtained by introducing the following special velocity field associated with the control node u_{jk} and the direction $v \in \mathbf{R}^2$

$$
V_{jk}(x) \stackrel{\text{def}}{=} c_{jk}(x)\, v, \tag{54}
$$

where c_{jk} is a continuous piecewise linear function such that at each node $x_{k'}$ and control node $u_{j'k'}$

$$
c_{jk}(x_{k'}) \stackrel{\text{def}}{=} 0, \quad c_{jk}(u_{j'k'}) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } u_{j'k'} = u_{jk} \\ 0, & \text{if } u_{j'k'} \neq u_{jk}. \end{cases} \tag{55}
$$

For t sufficiently small, it transforms the triangle $\{x_j, x_k, u_{jk}\}$ onto the triangle $\{T_t(x_j), T_t(x_k), T_t(u_{jk})\}$ and the Bézier curve $s^2 x_j + 2s(1-s)u_{jk} + (1-s)^2 x_k$ that is contained in the triangle $\{x_j, x_k, u_{jk}\}$ onto the Bézier curve $s^2 T_t(x_j) + 2s(1-s)T_t(u_{jk}) + (1-s)^2 T_t(x_k)$ in the triangle $\{T_t(x_j), T_t(x_k), T_t(u_{jk})\}$.

The above constructions and computations extend to three space dimensions by representing Γ by triangular facets or curved triangular surfaces with control nodes.

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