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Inverse shape optimization problems and application to airfoils

by

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Abstract: We consider a set of parameterized planar arcs ($x(t)$, $y(t)$) ($0 \leq t \leq 1$), satisfying certain smoothness, regularity and monotonicity conditions (in particuler $x(t)$ is monotone increasing, and $y(t)$ positive and unimodal), and a functional $\mathcal{J}(y)$ involving an adjustable weighting function $\omega(t)$ and a positive constant $\alpha > 1$. We first prove the strict convexity of the functional for $\alpha \geq 2$. Under the less stringent condition $\alpha > 1$, we derive the stationarity condition and the formal expression for the Hessian, and prove that if a point exists at which the functional is stationary w.r.t. variations in $y = y(t)$, for fixed $x = x(t)$, then it is unique and realizes a global minimum; the functional is then unimodal. We also observe that the stationarity condition (Euler-Lagrange equation) is an integral-differential equation depending only on the arc shape and not on the parameterization *per se*, which gives the variational problem a certain intrinsic character. Then, we solve the inverse problem: given an admissible parameterized arc, we construct a smooth weighting function $\omega(t)$ for which the stationarity condition is satisfied, thus making the functional unimodal, and derive certain asymptotics. A numerical example pertaining to optimum-shape design in aerodynamics is computed for illustration.

Keywords: Partial-Differential Equations (PDEs), computational methods, shape optimization, calculus of variations.

1. Motivation

Simple variational problems are proposed as models, mostly for analysis in shape optimization purposes. In more general and complex settings, the shape would affect the solution of a physically relevant PDE. Utilizing such models is envisaged in future works in particular: *(i)* to test numerical algorithms for shape optimization in a simplified computational context bearing the geometrical characteristics of another, more complex problem, as in Bélahcène and Désidéri

(2003), *(ii)* to adapt a numerical shape-optimization algorithm to a meta model in order to improve its convergence, as in Karakasis and Désidéri (2002), or *(iii)* to construct purely geometrical penalty functions for an inverse-problem formulation.

2. Variational problem and stationarity condition

Consider a planar arc connecting the origin $(0,0)$ to the point $(1,0)$ and admitting the following parameterization:

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \tag{1}$$

in which the regularity of the functions $x(t)$ and $y(t)$ $(0 \leq t \leq 1)$ is momentarily not made precise. Additionally, $y(t) \geq 0$.

The following quantities are introduced:

$$\boxed{\begin{aligned} p &= \int_0^1 \sqrt{x'^2(t) \,+\, y'^2(t)}\,\omega(t)\,dt \\ \mathcal{A} &= \int_0^1 y(t)\,x'(t)\,\omega(t)\,dt \end{aligned}} \tag{2}$$

in which the weighting function $\omega(t)$ is positive and adjustable. In the particular case of $\omega(t) \equiv 1$, the quantities p and $\mathcal{A}$ are respectively the arclength and the area of the region between the arc and the x-axis.

From here on, the function $x(t)$ is fixed, somewhat arbitrarily, say $x \in \mathcal{C}^2([0,1])$, $x(0) = x(1) - 1 = 0$, and the function $y(t)$ is subject to the following convex condition

$$y \in K := \left\{ y \in H^2(0,1)\,,\ y(0) = y(1) = 0\,,\ y(t) \geq 0\ \forall t \right\} \tag{3}$$

and considered as the argument of the following functional to be minimized:

$$\boxed{\mathcal{J}(y) = \frac{p^\alpha}{\mathcal{A}}} \tag{4}$$

in which the exponent α is a fixed real positive number, to be chosen later.

The existence of a minimum in the above setting is questionable. Nevertheless, in our development, the necessary condition for optimality is examined, focusing on the inverse problem for which the regularity of the stationary shape is known.

For the purpose of establishing an existence result, we extend the definition of the functionals p and $\mathcal{A}$ to a larger functional space. We set:

$$M^1_\omega(0,1) = \left\{ (u,v) \in L^1(0,1)^2 \right. \\ \left. \text{s.t.} \ \underset{g\in B_1}{\text{Sup}} \int_0^1 \left[u(t) \Big(\omega(t) g_x(t)\Big)' + v(t) \Big(\omega(t) g_y(t)\Big)' \right] dt < \infty \right\} \quad (5)$$

where $g = (g_x, g_y)$, and

$$B_1 = \left\{ g \in \mathcal{C}^1_{\text{comp}}(]0,1[;\mathbb{R}^2) \, , \ \forall t \, , \ \| \, g(t) \, \|_{\mathbb{R}^2} \leq 1 \right\} \quad (6)$$

where $\mathcal{C}^1_{\text{comp}}(]0,1[;\mathbb{R}^2)$ stands for the subset of functions in $\mathcal{C}^1$ with compact support.

The set $M^1_\omega(0,1)$ generalizes the classical Banach space of bounded vector measures to cases involving a smooth and positive weight ω.

The function $x = x(t)$ being fixed in $\mathcal{C}^2([0,1])$ with $x'(t) > 0$, we have

$$\mathcal{A}(y) = \int_0^1 y(t) \, x'(t) \, \omega(t) \, dt = \| \, y \, x' \, \omega \, \|_{L^1(0,1)} \quad (7)$$

and

$$p(y) = \| \, (x', y') \, \|_{M^1_\omega(0,1)} \quad (8)$$

and

$$\mathcal{J}(y) = \frac{\| \, (x', y') \, \|^\alpha_{M^1_\omega(0,1)}}{\| \, y \, x' \, \omega \, \|_{L^1(0,1)}} \, . \quad (9)$$

Finally, we define:

$$BV_\omega(0,1) = \left\{ \, (x,y) \in L^1(0,1)^2 \ \text{s.t.} \ (x', y') \in M^1_\omega(0,1) \, \right\} . \quad (10)$$

Again, this space generalizes the classical definition of the Banach space of functions with bounded variation to a case involving the weight ω.

As x is fixed, it is clear that the condition $(x,y) \in BV_\omega(0,1)$ is equivalent to $y \in BV_\omega(0,1)$. Thus, the minimization of the function $\mathcal{J}(y)$ w.r.t. $(x,y) \in BV_\omega(0,1)$ is equivalent to the minimization w.r.t. $y \in BV_\omega(0,1)$. Nevertheless, it is necessary to introduce the equivalent norm in $BV_\omega(0,1)$ in order to express the relaxation of the functional $\mathcal{J}(y)$.

A basic question which arises is whether the constraint $y(t) \geq 0$ may be saturated at the minimum. To investigate this question, let us suppose that the function $y_1(t)$ realizes a local minimum of $\mathcal{J}(y)$, and examine the possibility for $y_1(t)$ to be equal to zero at points other than the limits.

First, consider the eventuality of an isolated zero a $(0 < a < 1 \, ; \ y_1(a) = 0)$. Small strictly-positive numbers ε_- and ε_+ can be defined such that $y_1(a - \varepsilon_-) =$

$y_1(a+\varepsilon_+) = \delta > 0$ and $y_1(t) < \delta$ for all t in the open interval $]a-\varepsilon_-, a+\varepsilon_+[$. Then define the new function $y_2(t)$ by

$$y_2(t) = \begin{cases} \delta & \text{if } a-\varepsilon_- \le t \le a+\varepsilon_+ \\ y_1(t) & \text{otherwise}\,. \end{cases} \tag{11}$$

Then, obviously:

$$p(y_2) < p(y_1)\,,\ \mathcal{A}(y_2) > \mathcal{A}(y_1) \tag{12}$$

and consequently:

$$\mathcal{J}(y_2) < \mathcal{J}(y_1) \tag{13}$$

which contradicts the assumption. We conclude that $y_1(t)$ cannot be equal to zero at an isolated point.

Second, consider the eventuality of $y_1(t) \equiv 0$ over a maximal interval $[a,b] \subset [0,1]$. Then neighborhoods of $t = a^-$ and $t = b^+$ exist over which the function $y_1(t)$ is strictly positive because isolated zeros are excluded and $[a,b]$ has the *maximal* extent. Then, let ε_- and ε_+ be strictly-positive numbers chosen such that $a-\varepsilon_-$ and $b+\varepsilon_+$ belong to these neighborhoods, $y_1(a-\varepsilon_-) = y_1(b+\varepsilon_+) = \delta > 0$, and $y_1(t) < \delta$ for all t in the open interval $]a-\varepsilon_-, b+\varepsilon_+[$. Then, let $y_2(t)$ be defined by:

$$y_2(t) = \begin{cases} \delta & \text{if } a-\varepsilon_- \le t \le b+\varepsilon_+ \\ y_1(t) & \text{otherwise}\,. \end{cases} \tag{14}$$

Then again:

$$p(y_2) < p(y_1)\,,\ \mathcal{A}(y_2) > \mathcal{A}(y_1) \tag{15}$$

and consequently:

$$\mathcal{J}(y_2) < \mathcal{J}(y_1) \tag{16}$$

which contradicts the assumption. Therefore, no interval $[a,b]$ exists over which $y_1(t) \equiv 0$.

THEOREM 2.1 *If, for fixed $x = x_1(t)$, the functional $\mathcal{J}(y)$ admits a local minimum for $y = y_1(t)$, then $y_1(t) > 0$ everywhere except at endpoints.*

This result allows us to treat the problem as an *unconstrained* minimization.

LEMMA 2.1 *For fixed $x = x_1(t)$, a necessary condition for the functional $\mathcal{J}(y)$ to admit a global minimum achieved for a finite-valued and smooth function $y = y_1(t)$, is $\alpha > 1$.*

Proof. Assume the existence of such a global minimum and consider the family of admissible parameterizations

$$\begin{cases} x_\lambda(t) = x_1(t) \\ y_\lambda(t) = \lambda y_1(t) \end{cases} \tag{17}$$

where λ is a positive free parameter, and the function

$$j(\lambda) = \mathcal{J}\Big(y_\lambda(t)\Big)\,. \tag{18}$$

Note that

$$p = p\Big(y_\lambda(t)\Big) = \int_0^1 \left(x_1'^2 + \lambda^2 y_1'^2\right)^{\frac{1}{2}} \,\omega\, dt \tag{19}$$

and

$$\mathcal{A} = \mathcal{A}\Big(y_\lambda(t)\Big) = \mathcal{A}_1\,\lambda \tag{20}$$

where $\mathcal{A}_1 = \mathcal{A}\Big(y_1(t)\Big)$. Consequently,

$$j(\lambda) = \frac{\lambda^\alpha}{\mathcal{A}}\,\left(\frac{p}{\lambda}\right)^\alpha = \frac{\lambda^{\alpha-1}}{\mathcal{A}_1}\,\left(\int_0^1 \left(\frac{x_1'^2}{\lambda^2} + y_1'^2\right)^{\frac{1}{2}} \omega\, dt\right)^\alpha\,. \tag{21}$$

From this expression, it appears that the hypothesis $\alpha \le 1$ would imply that $j(\lambda)$ be a monotone-decreasing function of λ achieving its global minimum in the limit $\lambda \longrightarrow \infty$, and not for $\lambda = 1$. The contradiction is removed by rejecting this hypothesis. ■

We now establish the stationarity condition. The functional $\mathcal{J}(y)$ is stationary iff:

$$\alpha\,\frac{\delta p}{p} - \frac{\delta\mathcal{A}}{\mathcal{A}} = 0 \tag{22}$$

or, equivalently:

$$\alpha\,\mathcal{A}\,\delta p = p\,\delta\mathcal{A}\,. \tag{23}$$

There follows:

$$\delta p = \int_0^1 \frac{1}{2}\,({x'}^2 \,+\, {y'}^2)^{-\frac{1}{2}}\,2\,y'\,\delta y'\,\omega(t)\,dt \tag{24}$$

and since $\delta y' = (\delta y)'$, an integration by parts yields:

$$\begin{aligned}\delta p &= \Big[\omega y'\,({x'}^2 \,+\, {y'}^2)^{-\frac{1}{2}}\,\delta y\Big]_0^1 - \int_0^1 \frac{d}{dt}\left(\omega y'\,({x'}^2 \,+\, {y'}^2)^{-\frac{1}{2}}\right)\,\delta y\,dt \\ &= \int_0^1 \Big[-(\omega' y' + \omega y'')\,({x'}^2 \,+\, {y'}^2)^{-\frac{1}{2}} \\ &\qquad + \frac{\omega y'}{2}\,({x'}^2 \,+\, {y'}^2)^{-\frac{3}{2}}(2\,x'\,x'' \,+ 2\,y'\,y'')\Big]\,\delta\,y\,dt\end{aligned} \tag{25}$$

since $\delta y(0) = \delta y(1) = 0$. After simplification, it follows that:

$$\boxed{\begin{aligned} \delta p &= \int_0^1 \phi\, \delta y\, dt \\ \delta \mathcal{A} &= \int_0^1 \psi\, \delta y\, dt \end{aligned}} \tag{26}$$

in which the new symbols are defined as follows:

$$\boxed{\begin{aligned} &\phi(t) = \omega\, \frac{x'(x''y' - x'y'')}{(x'^2 + y'^2)^{\frac{3}{2}}} - \omega'\, \frac{y'}{\sqrt{x'^2 + y'^2}} \\ &\psi(t) = \omega(t)\, x'(t) \end{aligned}} \tag{27}$$

The stationarity condition thus writes:

$$\forall\, \delta y\,, \quad \int_0^1 \left[\alpha \mathcal{A}\, \phi - p \psi\right]\, \delta y\, dt = 0 \tag{28}$$

and this is equivalent to the differential equation [...] $\equiv 0$, of the form (Euler-Lagrange equation):

$$\boxed{\forall t, \quad \frac{x''y' - x'y''}{(x'^2 + y'^2)^{\frac{3}{2}}} - \frac{\omega'}{\omega}\, \frac{y'}{x'\sqrt{x'^2 + y'^2}} = \frac{p}{\alpha \mathcal{A}}\ (= const.)} \tag{29}$$

This integral-differential equation is unsurprinsingly of second order w.r.t. the unknown function $y(t)$ which is subject to the two homogeneous Dirichlet conditions $y(0) = y(1) = 0$. In its expression, one recognizes the local curvature:

$$\frac{1}{r} = \frac{d\varphi}{ds} = -\frac{x''y' - x'y''}{(x'^2 + y'^2)^{\frac{3}{2}}} \tag{30}$$

where the sign convention is made classically: the curvilinear arclength s increases as t increases,

$$\frac{ds}{dt} = +\left(x'^2 + y'^2\right)^{\frac{1}{2}}\ , \tag{31}$$

and φ denotes the angle made locally by the tangent with the x-axis. Hence, for a concave shape, $r < 0$.

3. Gradient, Hessian, convexity and unimodality

3.1. Convexity of pseudo-perimeter p

For fixed $x = x(t)$ and $\omega = \omega(t)$, the expression $f = \sqrt{x'^2 + y'^2}\,\omega(t)$ is convex in terms of y', since

$$\frac{\partial^2 f}{\partial y'^2} = \omega(t)\, x'^2\, \left(x'^2 + y'^2\right)^{-\frac{3}{2}} \geq 0\,. \tag{32}$$

Consequently:

Theorem 3.1 *For fixed $x = x(t)$, the pseudo-perimeter p is a strictly-convex functional of $y = y(t)$.*

3.2. Convexity of the functional $\mathcal{J}(y)$ for $\alpha \geq 2$

The raw expression of the functional $\mathcal{J}(y)$ implies that:

$$\delta\mathcal{J} = \frac{\alpha p^{\alpha-1}}{\mathcal{A}}\,\delta p - \frac{p^\alpha}{\mathcal{A}^2}\,\delta\mathcal{A}\,. \tag{33}$$

Differentiating again, it follows that

$$\delta^2\mathcal{J} = \frac{\alpha p^{\alpha-1}}{\mathcal{A}}\,\delta^2 p + \frac{\alpha(\alpha-1)\,p^{\alpha-2}}{\mathcal{A}}\,(\delta p)^2 - \frac{2\alpha p^{\alpha-1}}{\mathcal{A}^2}\,\delta p\,\delta\mathcal{A} + \frac{2p^\alpha}{\mathcal{A}^3}\,(\delta\mathcal{A})^2 \tag{34}$$

since $\delta^2\mathcal{A} = 0$, because the functional $\mathcal{A}(y)$ is linear.

Since $\delta^2 p > 0$ for all nontrivial δy, because the functional $p(y)$ is strictly convex, a sufficient condition for the functional $\mathcal{J}(y)$ to be strictly convex is that the following quadratic form be positive semi-definite:

$$q(X,Y) = \frac{\alpha(\alpha-1)\,p^{\alpha-2}}{\mathcal{A}}\,X^2 - \frac{2\alpha p^{\alpha-1}}{\mathcal{A}^2}\,XY + \frac{2p^\alpha}{\mathcal{A}^3}\,Y^2 = aX^2 - 2bXY + cY^2 \tag{35}$$

in which the definitions of the constants a, b and c are evident. A sufficient condition for strict convexity is therefore the following:

$$ac \geq b^2 \tag{36}$$

that is:

$$2\alpha(\alpha-1) \geq \alpha^2 \tag{37}$$

or equivalently:

$$\boxed{\alpha \geq 2\,.} \tag{38}$$

THEOREM 3.2 *For $\alpha \geq 2$, the functional $\mathcal{J}(y)$, for fixed $x = x(t)$, is strictly convex.*

Values of α less than 2 remain of interest for inverse problems in which α is not known *a priori*. To cover such cases, the unimodality of the functional $\mathcal{J}(y)$ is examined in general in the next subsection.

3.3. General case, $\alpha > 1$; unimodality

Pursuing the analysis further, we first derive explicit expressions for the gradient and the Hessian of the functional, and establish conditions under which the functional is unimodal.

Considering again the logarithmic first variation,

$$\frac{\delta\mathcal{J}}{\mathcal{J}} = \alpha\frac{\delta p}{p} - \frac{\delta\mathcal{A}}{\mathcal{A}} \tag{39}$$

we get the following expressions for the first variation of the functional $\mathcal{J}(y)$ for fixed $x = x(t)$, and the gradient:

$$\boxed{\delta\mathcal{J} = \delta\mathcal{J}(y,\delta y) = \int_0^1 G\,\delta y\,dt\,; \quad G = G(y) = \mathcal{J}\left(\frac{\alpha}{p}\phi - \frac{1}{\mathcal{A}}\psi\right).} \tag{40}$$

Again holding $x = x(t)$ fixed and varying $y = y(t)$ result in the following expression for the second variation:

$$\boxed{\delta^2\mathcal{J} = \delta^2\mathcal{J}(y,\delta y) = \int_0^1 \delta G\,\delta y\,dt} \tag{41}$$

in which:

$$\delta G = \delta G_1 + \delta G_2 \tag{42}$$

where

$$\begin{cases} \delta G_1 = \delta\mathcal{J}\left(\dfrac{\alpha}{p}\phi - \dfrac{1}{\mathcal{A}}\psi\right) \\ \delta G_2 = \mathcal{J}\left(-\dfrac{\alpha\phi}{p^2}\,\delta p + \dfrac{\psi}{\mathcal{A}^2}\,\delta\mathcal{A} + \dfrac{\alpha}{p}\,\delta\phi\right). \end{cases} \tag{43}$$

It follows that

$$\begin{aligned} \int_0^1 \delta G_1\,\delta y\,dt &= \delta\mathcal{J}\int_0^1 \left(\frac{\alpha}{p}\phi - \frac{1}{\mathcal{A}}\psi\right)\delta y\,dt \\ &= \mathcal{J}\left(\int_0^1 \left(\frac{\alpha}{p}\phi - \frac{1}{\mathcal{A}}\psi\right)\delta y\,dt\right)^2. \end{aligned} \tag{44}$$

The above term vanishes when the functional $\mathcal{J}(y)$ is stationary. For the other term, we have:

$$\delta G_2 = \mathcal{J}\left(A(t)\int_0^1 B(\tau)\,\delta y(\tau)\,d\tau + C(t)\int_0^1 D(\tau)\,\delta y(\tau)\,d\tau + \frac{\alpha}{p}\,\delta\phi\right) \quad (45)$$

in which:

$$A(t) = -\frac{\alpha}{p^2}\,\phi(t)\,, \quad B(\tau) = \phi(\tau)\,, \quad C(t) = \frac{1}{\mathcal{A}^2}\,\psi(t)\,, \quad D(\tau) = \psi(\tau)\,. \quad (46)$$

This yields:

$$\frac{\delta^2\mathcal{J}}{\mathcal{J}} = \left(\int_0^1 \left(\frac{\alpha}{p}\phi - \frac{1}{\mathcal{A}}\psi\right)\delta y\,dt\right)^2 + E + \frac{\alpha}{p}\,X_\phi \quad (47)$$

in which:

$$\begin{aligned} E &= \int_0^1 \left(A(t)\int_0^1 B(\tau)\,\delta y(\tau)\,d\tau + C(t)\int_0^1 D(\tau)\,\delta y(\tau)\,d\tau\right)\delta y(t)\,dt \\ &= \int_0^1 A(t)\,\delta y(t)\,dt \int_0^1 B(t)\,\delta y(t)\,dt + \int_0^1 C(t)\,\delta y(t)\,dt \int_0^1 D(t)\,\delta y(t)\,dt \\ &= -\frac{\alpha}{p^2}\left(\int_0^1 \phi(t)\,\delta y(t)\,dt\right)^2 + \frac{1}{\mathcal{A}^2}\left(\int_0^1 \psi(t)\,\delta y(t)\,dt\right)^2 \quad (48) \end{aligned}$$

and

$$X_\phi = \int_0^1 \delta\phi\,\delta y\,dt\,. \quad (49)$$

Note that if $\mathcal{J}(y)$ is stationary,

$$\phi(t) = \frac{p}{\alpha\mathcal{A}}\,\omega(t)\,x'(t) \quad (50)$$

so that:

$$\begin{aligned} E &= -\frac{\alpha}{p^2}\left(\int_0^1 \frac{p}{\alpha\mathcal{A}}\,\omega x'\,\delta y\,dt\right)^2 + \frac{1}{\mathcal{A}^2}\left(\int_0^1 \omega x'\,\delta y\,dt\right)^2 \\ &= \left(1 - \frac{1}{\alpha}\right)\frac{1}{\mathcal{A}^2}\left(\int_0^1 \omega x'\,\delta y\,dt\right)^2 \quad (51) \end{aligned}$$

which is non-negative under the hypothesis $\alpha \geq 1$. Otherwise, we retain the completely general expression (48).

The term X_ϕ is first split into two:

$$X_\phi = X'_\phi + X_\phi" \quad (52)$$

where:

$$X'_\phi = \int_0^1 \frac{\partial\phi}{\partial y'}\,\delta y'\,\delta y\,dt\,, \quad X''_\phi = \int_0^1 \frac{\partial\phi}{\partial y''}\,\delta y''\,\delta y\,dt\,. \tag{53}$$

The term X'_ϕ is easily integrated by parts:

$$\begin{aligned} X'_\phi &= \int_0^1 \frac{\partial\phi}{\partial y'}\left[\tfrac{1}{2}(\delta y)^2\right]'\,dt = \left[\frac{\partial\phi}{\partial y'}\,\tfrac{1}{2}(\delta y)^2\right]_0^1 - \tfrac{1}{2}\int_0^1 \left(\frac{\partial\phi}{\partial y'}\right)'(\delta y)^2\ dt \\ &= -\tfrac{1}{2}\int_0^1 \left(\frac{\partial\phi}{\partial y'}\right)'(\delta y)^2\ dt \end{aligned} \tag{54}$$

since $y(t)$ is subject to Dirichlet conditions.

The integration by parts of the second term, X''_ϕ yields a contribution of different type:

$$X''_\phi = \int_0^1 \frac{\partial\phi}{\partial y''}\,\delta y\ (\delta y')'\ dt = \left[\frac{\partial\phi}{\partial y''}\,\delta y\,\delta y'\right]_0^1 - \int_0^1 \left(\frac{\partial\phi}{\partial y''}\,\delta y\right)'\delta y'\,dt = -Y_\phi - Z_\phi \tag{55}$$

in which:

$$Y_\phi = \int_0^1 \left(\frac{\partial\phi}{\partial y''}\right)'\delta y\,\delta y'\,dt\,, \quad Z_\phi = \int_0^1 \frac{\partial\phi}{\partial y''}\ (\delta y')^2\ dt\,. \tag{56}$$

Lastly, the term Y_ϕ is integrated by parts in a way similar to the previous integration of the term X'_ϕ, yielding:

$$Y_\phi = -\tfrac{1}{2}\int_0^1 \left(\frac{\partial\phi}{\partial y''}\right)''(\delta y)^2\ dt\,. \tag{57}$$

As a result

$$X'_\phi - Y_\phi = \tfrac{1}{2}\int_0^1 \left[\left(\frac{\partial\phi}{\partial y''}\right)'' - \left(\frac{\partial\phi}{\partial y'}\right)'\right](\delta y)^2\ dt\,. \tag{58}$$

But, the function ϕ has been defined in (27). As a result:

$$\begin{aligned} \frac{\partial\phi}{\partial y'} =& \omega x' x''(x'^2 + y'^2)^{-\frac{3}{2}} + \omega x'(x''y' - y''x')(-\frac{3}{2})(x'^2 + y'^2)^{-\frac{5}{2}}(2y') \\ & - \omega'(x'^2 + y'^2)^{-\frac{1}{2}} - \omega' y'(-\frac{1}{2})(x'^2 + y'^2)^{-\frac{3}{2}}(2y') \end{aligned} \tag{59}$$

and

$$\frac{\partial\phi}{\partial y''} = -\omega x'^2(x'^2 + y'^2)^{-\frac{3}{2}}\,. \tag{60}$$

Hence:

$$\left(\frac{\partial \phi}{\partial y''}\right)' = -\,\omega' x'^2 (x'^2 + y'^2)^{-\frac{3}{2}} - 2\omega x' x'' (x'^2 + y'^2)^{-\frac{3}{2}} \\ -\,\omega x'^2 (-\frac{3}{2})(x'^2 + y'^2)^{-\frac{5}{2}} (2x'x'' + 2y'y'') \quad (61)$$

and finally:

$$\boxed{\left(\frac{\partial \phi}{\partial y''}\right)' - \frac{\partial \phi}{\partial y'} \equiv 0\,.} \quad (62)$$

Hence X_ϕ reduces to the following positive term:

$$X_\phi = -Z_\phi = \int_0^1 \frac{\omega x'^2}{(x'^2 + y'^2)^{\frac{3}{2}}} \, (\delta y')^2 \, dt \quad (63)$$

and the second variation to:

$$\frac{\delta^2 \mathcal{J}}{\mathcal{J}} = \left(\int_0^1 \left(\frac{\alpha}{p}\phi - \frac{1}{\mathcal{A}}\psi\right) \delta y \, dt\right)^2 + E + \frac{\alpha}{p} \int_0^1 \frac{\omega x'^2}{(x'^2 + y'^2)^{\frac{3}{2}}} \, (\delta y')^2 \, dt \quad (64)$$

where E is given by (48) in general, or (51) in case of stationarity.

Now, we distinguish two cases. If the functional is stationary ($\delta\mathcal{J} = 0$), (29), (51) and (64) imply that:

$$\frac{\delta^2 \mathcal{J}}{\mathcal{J}} = \left(1 - \frac{1}{\alpha}\right) \frac{1}{\mathcal{A}^2} \left(\int_0^1 \omega x' \, \delta y \, dt\right)^2 + \frac{\alpha}{p} \int_0^1 \frac{\omega x'^2}{(x'^2 + y'^2)^{\frac{3}{2}}} \, (\delta y')^2 \, dt > 0 \quad (65)$$

that is, strictly-positive provided $\alpha \geq 1$ for all nontrivial $\delta y(t)$ (since these are such that $\delta y'(t) \not\equiv 0$ because of the boundary conditions). This permits us to make the following statement converse to Lemma 2.1:

LEMMA 3.1 *If the functional $\mathcal{J}(y)$ is stationary w.r.t. variations in $y = y(t)$, for fixed $x = x(t)$, and if $\alpha \geq 1$, then it achieves at this point a local minimum.*

If, instead, the functional $\mathcal{J}(y)$ is not locally stationary, the following expression is to be used for the second-variation:

$$\boxed{\frac{\delta^2 \mathcal{J}}{\mathcal{J}} = \left(\int_0^1 \left(\frac{\alpha}{p}\phi - \frac{1}{\mathcal{A}}\psi\right) \delta y \, dt\right)^2 - \frac{\alpha}{p^2} \left(\int_0^1 \phi \, \delta y \, dt\right)^2 + \frac{1}{\mathcal{A}^2} \left(\int_0^1 \psi \, \delta y \, dt\right)^2 + \frac{\alpha}{p} \int_0^1 \frac{\omega x'^2}{(x'^2 + y'^2)^{\frac{3}{2}}} \, (\delta y')^2 \, dt\,.} \quad (66)$$

The above developments indicate that if $\alpha \geq 1$, all the points at which the functional is stationary w.r.t. variations in y, correspond to local minima.

In this framework, let us now make the provisional hypothesis of existence of two distinct stationary points $y_1 = y_1(t)$ and $y_2 = y_2(t)$ for the functional $\mathcal{J}(y)$. Let $\Delta y = y_2 - y_1$, and observe that the following function of the real variable λ,

$$J(\lambda) = \mathcal{J}(y_1 + \lambda \Delta y) \tag{67}$$

admits local minima at $\lambda = 0$ and $\lambda = 1$. Thus the maximum of $J(\lambda)$ $(0 \leq \lambda \leq 1)$ is not achieved at either limit $\lambda = 0$ or $\lambda = 1$, but instead at one intermediate point θ $(0 < \theta < 1)$ at least. This can be seen as a trivial expression of the Mountain-Pass Theorem (Jabri, 2003) for a function of the real variable. But, $J'(\theta)$ is obtained by substituting $y_1 + \lambda \Delta y$ to y, and Δy to δy in the expression of the first variation, (40):

$$J'(\theta) = J(\theta) \int_0^1 \left(\frac{\alpha}{p} \phi - \frac{1}{\mathcal{A}} \psi \right) \Delta y \, dt = 0 \, . \tag{68}$$

Hence, at this particular point y:

$$\frac{\alpha}{p} \int_0^1 \phi \, \Delta y \, dt = \frac{1}{\mathcal{A}} \int_0^1 \psi \, \Delta y \, dt \tag{69}$$

and the same substitution in the second variation, (66), provides the second derivative:

$$\frac{J''(\theta)}{J(\theta)} = \left(1 - \frac{1}{\alpha}\right) \frac{1}{\mathcal{A}^2} \left(\int_0^1 \psi \, \Delta y \, dt \right)^2 + \frac{\alpha}{p} \int_0^1 \frac{\omega x'^2}{(x'^2 + y'^2)^{\frac{3}{2}}} \left(\Delta y'\right)^2 dt > 0 \tag{70}$$

since this expression cannot be equal to zero, because this would require that $\Delta y' \equiv 0$, implying $\Delta y(t) \equiv C$ (a constant), and $C = 0$ to satisfy the homogeneous boundary conditions, thus $y_1 = y_2$. The strict positivity of $J''(\theta)$ contradicts the previous statement according to which $\lambda = \theta$ corresponds to a local maximum of $J(\lambda)$. This contradiction is removed by rejecting the above provisional hypothesis of existence of two distinct stationary points. We thus conclude by the following

Theorem 3.3 (Uniqueness and unimodality) *If $\alpha > 1$ and if the functional $\mathcal{J}(y)$ admits one point of stationarity w.r.t. variations in $y = y(t)$, for fixed $x = x(t)$, then such point is unique and it realizes a global minimum of the functional, which is unimodal.*

In view of Lemma 2.1, this theorem can be formulated alternately as follows:

Corollary 3.1 *Equation (29) admits no solutions for $\alpha \leq 1$; for $\alpha > 1$, if a solution exists, it is unique, and it realizes a global minimum of the functional $\mathcal{J}(y)$, which is unimodal.*

Originally, we attempted to establish the unimodality by a more technical means involving a certain convexity criterion. This route failed to deliver the full proof, but revealed itself to be somewhat instructive; see Appendix A for details.

3.4. Functional behavior as α varies

We now examine the behavior of the functional as the exponent α varies, and for this we momentarily indicate this dependence by a subscript over the functional symbol. At this stage, it is known that the functional is unimodal iff it admits a stationary point, necessarily unique, and a necessary condition is $\alpha > 1$. However this condition is not sufficient in general as the next section will demonstrate, and we rely on the construction of the inverse problem, Section 5, to guarantee the existence.

Hence, the functions $x(t)$ and $\omega(t)$ being given, assume values of the parameter α exist for which the functional $\mathcal{J}_\alpha(y)$ is unimodal, and let this parameter be fixed in what follows, and set

$$\mathcal{J}_\alpha^* = \min_y \mathcal{J}_\alpha(y) \tag{71}$$

so that:

$$\forall y\,,\ \mathcal{J}_\alpha(y) \geq \mathcal{J}_\alpha^*\,. \tag{72}$$

Consider another parameter value $\beta > \alpha$. We have:

$$\mathcal{J}_\beta(y) = \frac{p^\beta}{\mathcal{A}} = p^{\beta-\alpha}\,\mathcal{J}_\alpha(y) \tag{73}$$

so that:

$$\forall y\,,\ \mathcal{J}_\beta(y) \geq p_{\min}^{\beta-\alpha}\,\mathcal{J}_\alpha^* > 0 \tag{74}$$

where $p_{\min} = \int_0^1 x'(t)\,\omega(t)\,dt$; hence the infemum

$$\mathcal{J}_\beta^* = \operatorname*{Inf}_y \mathcal{J}_\beta(y) \tag{75}$$

is strictly positive. Thus, let $\{\, y_n \,\}$ ($n \in \mathbb{N}$) be a minimizing sequence of positive functions:

$$\lim_{n\to\infty} \mathcal{J}_\beta(y_n) = \mathcal{J}_\beta^*\,. \tag{76}$$

We have:

$$\mathcal{A}(y) = \frac{p(y)^\beta}{\mathcal{J}_\beta(y)} = \frac{p(y)^\alpha}{\mathcal{J}_\alpha(y)} \tag{77}$$

so that:

$$p(y) = \left(\frac{\mathcal{J}_\beta(y)}{\mathcal{J}_\alpha(y)}\right)^{\frac{1}{\beta-\alpha}} \tag{78}$$

and:

$$\mathcal{A}(y_n) = \frac{\mathcal{J}_\beta(y_n)^{\frac{\alpha}{\beta-\alpha}}}{\mathcal{J}_\alpha(y_n)^{\frac{\beta}{\beta-\alpha}}}\,. \tag{79}$$

Now, for sufficiently large n:

$$\mathcal{J}_\beta(y_n) \leq 2\,\mathcal{J}_\beta^* \tag{80}$$

so that:

$$\mathcal{A}(y_n) \leq \frac{\left(2\,\mathcal{J}_\beta^*\right)^{\frac{\alpha}{\beta-\alpha}}}{(\,\mathcal{J}_\alpha^*)^{\frac{\beta}{\beta-\alpha}}} \tag{81}$$

which proves that the sequence $\{\, y_n x'\omega \,\}$ is bounded in L^1. Therefore, the norm $\|\,(x', y_n')\,\|_{M^1_\omega(0,1)}$ is bounded, (x, y_n) is bounded in $BV_\omega(0,1)$, and finally y_n is bounded in the usual $BV(0,1)$. By virtue of the classical Helly compactness theorem, we deduce the existence of a subsequence, denoted by the same symbol, (x, y_n) which *strongly* converges in $L^1(0,1)$ to some $(x, y^*) \in BV_\omega(0,1)$, and *weakly* converges (as bounded measures) to (x, y^*). As the $M^1_\omega(0,1)$ norm is lower-semi-continuous w.r.t. the weak convergence, we get:

$$\|\,(x, y^*)\,\|_{M^1_\omega(0,1)} \leq \liminf \|\,(x, y_n)\,\|_{M^1_\omega(0,1)} \tag{82}$$

and

$$\|\,y^* x'\omega\,\|_{L^1(0,1)} = \lim\,\|\,y_n x'\omega\,\|_{L^1(0,1)} \tag{83}$$

and, as $\{\, y_n \,\}$ is a minimizing sequence:

$$\mathcal{J}(y^*) \leq \frac{\liminf \|\,(x, y_n)\,\|_{M^1_\omega(0,1)}}{\lim\,\|\,y_n x'\omega\,\|_{L^1(0,1)}} = \lim\left(\frac{\|\,(x, y_n)\,\|_{M^1_\omega(0,1)}}{\|\,y_n x'\omega\,\|_{L^1(0,1)}}\right) = \operatorname*{Inf}_{y\in BV(0,1)} \mathcal{J}(y)\,. \tag{84}$$

In conclusion, y^* realizes the minimum of the functional $\mathcal{J}(y)$ in $BV(0,1)$.

Therefore, the set of values of the parameter α for which the functional is unimodal is necessarily an interval, and we conclude by the following

THEOREM 3.4 *Let $x(t)$ and $\omega(t)$ be fixed. If a value of the parameter α exists for which the functional is unimodal, there also exists $\alpha_0 \geq 1$ such that:*

1. *for $\alpha < \alpha_0$, there exists no finite minimizing function of the functional $\mathcal{J}(y)$;*
2. *for $\alpha > \alpha_0$, the functional $\mathcal{J}(y)$ is unimodal.*

4. Reference case, $\omega(t) \equiv 1$

It is instructive to examine this fundamental particular case in which the integral-differential equation simplifies to:

$$-\frac{1}{r} = \frac{p}{\alpha \mathcal{A}} \tag{85}$$

and characterizes concave circular arcs of radius $|r| = -r$, whose centers, accounting for the problem symmetry, are located on the line $x = \frac{1}{2}$. New parameters h and θ_0 are introduced (see Fig. 1), and the following geometrical relations hold:

$$\begin{cases} \frac{1}{2} = |r| \cos\theta_0 \\ h = |r| \sin\theta_0 \\ p = (\pi - 2\theta_0)|r| \\ \mathcal{A} = \frac{1}{2}(\pi - 2\theta_0)r^2 - \frac{1}{2}h \times 1\,. \end{cases} \tag{86}$$

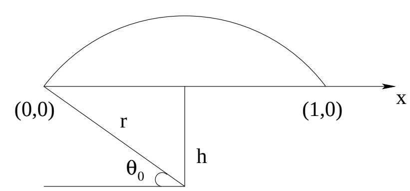

Figure 1. Optimal arc (notations)

Consequently, (29) first becomes:

$$\frac{1}{|r|} = \frac{(\pi - 2\theta_0)|r|}{\alpha\left[\frac{1}{2}(\pi - 2\theta_0)r^2 - r^2 \sin\theta_0 \cos\theta_0\right]} = \frac{\gamma|r|}{\frac{1}{2}\alpha\,(\gamma - \sin\gamma)\,r^2} \tag{87}$$

in which the new definition is made:

$$\gamma = \pi - 2\theta_0 \tag{88}$$

($-\frac{\pi}{2} \le \theta_0 \le \frac{\pi}{2}$ and $0 \le \gamma \le 2\pi$). It follows that

$$\boxed{\rho(\gamma) := \frac{\sin \gamma}{\gamma} = 1 - \frac{2}{\alpha}\,.} \tag{89}$$

The function $\rho(\gamma)$ is represented in Fig. 2. Its range is an interval $[\rho_{\min}, 1]$, where the negative value $\rho_{\min}$ is achieved at a certain abscissa $\gamma_{\min}$ close and inferior to $\frac{3\pi}{2}$, solution of the equation $\rho'(\gamma_{min}) = (\gamma_{\min}\ \cos\gamma_{\min} - \sin\gamma_{\min})/\gamma_{min}^2 = 0$; hence, $\gamma_{\min}$ is the fixed point of the iteration $\gamma_{n+1} = \tan^{-1}\gamma_n + \pi$, which easily yields:

$$\gamma_{\min} \simeq 4.493\,, \qquad \rho_{\min} \simeq -0.217\ . \tag{90}$$

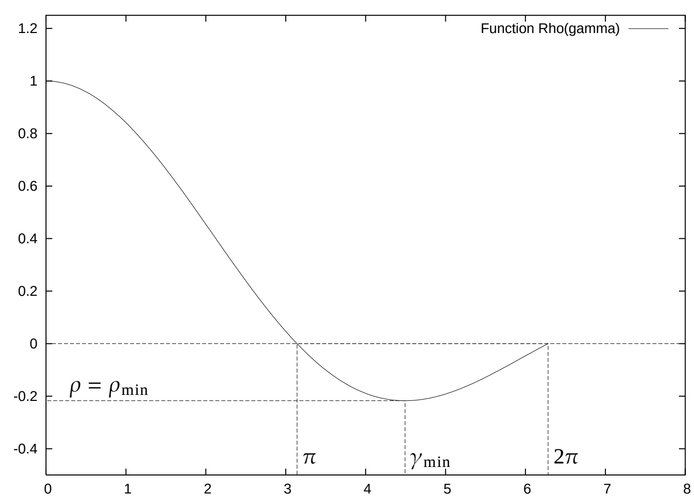

Figure 2. Variations of the function $\rho(\gamma)$.

Consequently, equation (89) admits a solution iff:

$$\boxed{\alpha \ge \alpha_{\min}} \tag{91}$$

in which $1 - 2/\alpha_{\min} = \rho_{\min}$, which gives:

$$\alpha_{\min} \simeq 1.643\ . \tag{92}$$

For $\alpha_{\min} \leq \alpha < 2$, $\pi < \gamma \leq \gamma_{\min}$; hence $\frac{1}{2}(\pi - \gamma_{\min}) \leq \theta_0 < 0$ [1]. In this case, the vector $\overrightarrow{dM}/dt_0$ tangent at the origin to the optimal circular arc points towards $x < 0$, and this corresponds to a circle whose center has a strictly positive ordinate; see Fig. 3. In this configuration, the stationary solution does not correspond to a graph in the (x, y) plane, and the results of Section 3 do not apply straightforwardly.

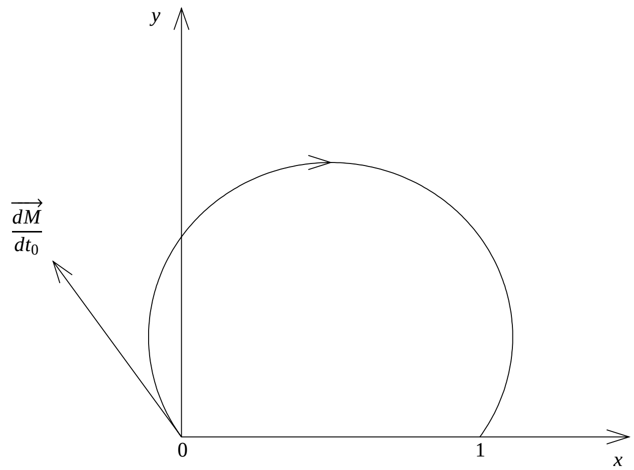

Figure 3. Circular arc satisfying the stationarity condition in the case $\alpha_{\min} \leq \alpha < 2$

But a simple argument permits us to conclude directly: since $\omega(t) \equiv 1$, p and $\mathcal{A}$ are the conventional perimeter and area, and if

$$\begin{cases} x(t) = x_R(t) \\ y(t) = y_R(t) \end{cases}$$

is a parameterization of the circle of radius R attached to the origin and the point $(1, 0)$, then as $R \to \infty$:

$$p\,(y_R) \sim 2\pi\, R\,, \quad \mathcal{A}\,(y_R) \sim \pi\, R^2\,, \quad \mathcal{J}\,(y_R) \sim 2^{\alpha}\pi^{\alpha-1}\, R^{\alpha-2} \longrightarrow 0$$

for all $\alpha < 2$. Therefore, the circular arc of Fig. 3 does not realize a global minimum.

Inversely, for $\alpha \geq 2$ we know the problem to be strictly convex (Theorem 3.2). In particular, for $\alpha = 2$, $\gamma = \pi$; hence $\theta_0 = 0$ and $|r| = \frac{1}{2}$. The optimal circular arc is then the upper half-circle of radius $\frac{1}{2}$, centered at $(\frac{1}{2}, 0)$; see Fig. 4.

[1] $\frac{1}{2}(\pi - \gamma_{\min}) \simeq -39^o$

It follows that

$$\mathcal{J} = \frac{p^2}{\mathcal{A}} = \frac{(\pi/2)^2}{\frac{1}{2}\,\pi\,(\frac{1}{2})^2} = 2\,\pi\,. \tag{93}$$

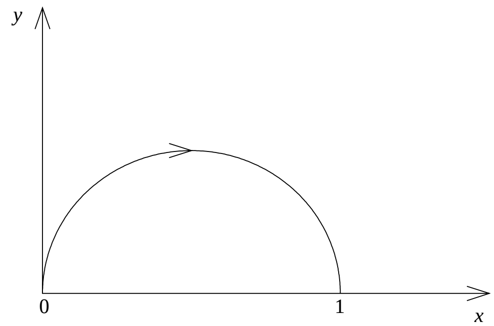

Figure 4. Optimal half-circle corresponding to the case $\alpha = 2$.

Finally, for $\alpha > 2$, $0 < \gamma < \pi$; hence $0 < \theta_0 < \frac{\pi}{2}$. In this case, the vector $\overrightarrow{dM}/dt_0$ tangent at the origin to the optimal circular arc points towards $x > 0$, and this corresponds to a circle whose center has a strictly negative ordinate; see Fig. 5.

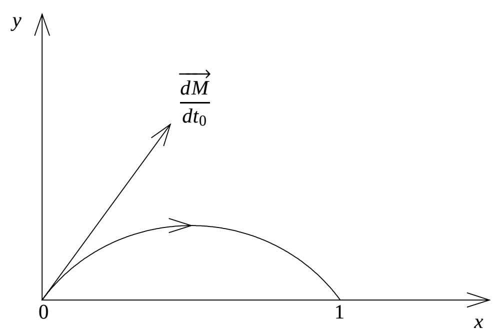

Figure 5. Optimal circular arc corresponding to the case $\alpha > 2$.

5. General case, $\omega(t) \not\equiv 1$; inverse problem

Now we turn to the following inverse problem: is it possible, by a judicious choice of the function $\omega(t)$ and the exponent α to guarantee the existence of a stationary solution, and further to force the solution of the integral-differential equation (29) to present desirable given geometrical characteristics? For example, for application to optimum-shape design in aerodynamics, can it be similar, or better, identical to a specified airfoil shape?

Before answering this question, we introduce the following

DEFINITION 5.1 (*Admissible parameterization) The planar-arc parameterization* $\big(x(t), y(t)\big)$ $(0 \le t \le 1)$ *is said to be admissible iff it satisfies the boundary conditions:*

$$x(0) = y(0) = x(1) - 1 = y(1) = 0 \tag{94}$$

and the following hypotheses:

1. *smoothness and regularity:* $x(t)$ *and* $y(t)$ *are of class* $\mathcal{C}^2$ *(at least) and* $x'(t)^2 + y'(t)^2 > 0$ *uniformly;*
2. *piecewise-monotonic variations:* $x(t)$ *is monotone increasing, and* $y(t)$ *unimodal, and at the unique point* $t = t_0$ $(0 < t_0 < 1)$ *at which* $y(t)$ *is maximum, the curvature is finite and nonzero* ($y'(t_0) = 0$ *and* $y''(t_0) < 0$).

(It is understood that to two different such admissible parameterizations correspond two values of t_0 *that are generally different.)*

Evidently, this definition implies the smoothness of the planar arc and the existence of a (single-valued) function f permitting the shape to be represented by the equation $y = f(x)$ as in the case of Fig. 4 or Fig. 5, but not Fig. 3; however, certain derivatives of the function f may be locally infinite, as in Fig. 4 at the arc endpoints, but also in more general cases.

From here on, the discussion is restricted to *admissible parameterizations*, for which applies the following

THEOREM 5.1 (*Constructive algorithm of the variational problem)*
Let $(x(t), y(t))$ $(0 \le t \le 1)$ *be an admissible parameterization. If the positive constant* $\mathcal{C}$ *is set equal to the opposite of the curvature at* $t = t_0$, t_0 *being the parameter value realizing the maximum ordinate* $y(t)$,

$$\mathcal{C} = \frac{p}{\alpha\mathcal{A}} = \left(\frac{x''y' - x'y''}{(x'^2 + y'^2)^{\frac{3}{2}}}\right)\Bigg|_{t=t_0} = \left(\frac{-y''}{x'^2}\right)\Bigg|_{t=t_0} = \left(-\frac{d^2y}{dx^2}\right)\Bigg|_{t=t_0} > 0\,, \tag{95}$$

the function $\omega(t)$*, deduced by quadrature from (29) with the right-hand side set equal to* $\mathcal{C}$,

$$\ln\omega(t) = \int \left(\frac{x''y' - x'y''}{(x'^2 + y'^2)^{\frac{3}{2}}} - \mathcal{C}\right) . \left(\frac{x'\sqrt{x'^2 + y'^2}}{y'}\right) dt \qquad (\omega(0) = 1) \tag{96}$$

is uniquely defined, of class $\mathcal{C}^1$ *(at least), and uniformly positive. Additionally, if the exponent* α *is calculated from*

$$\alpha = \frac{p}{\mathcal{C}A} \tag{97}$$

the functional $\mathcal{J}(y)$ *is stationary w.r.t. variations in* $y = y(t)$*, for fixed* $x = x(t)$.

Proof. Observe that the specific choice made for the constant $\mathcal{C}$ regularizes the integrand, thus permitting the quadrature to be performed stably. The arbitrary constant of integration appears as a multiplicative constant in the expression of the function $\omega(t)$; it is therefore irrelevant, and without loss of generality, one can set $\omega(0) = 1$ for uniqueness. As a result of this quadrature, the function $\omega(t)$ is uniquely defined, of class $\mathcal{C}^1$ (at least), and uniformly positive. By this construction, the stationarity condition (29) is indeed satisfied, provided the exponent α is calculated according to the above definition. ■

REMARK 5.1 *A unique value for the exponent α results from this algorithm, and at this point, the satisfaction of the condition $\alpha > 1$, necessary to a global minimum of the functional $\mathcal{J}$ (by virtue of Lemma 2.1), is unclear; we shall see that this is indeed the case.*

REMARK 5.2 *Using instead the variables s and φ, the integral-differential equation (29) becomes:*

$$\boxed{-\frac{d\varphi}{ds} - \frac{1}{\omega}\frac{d\omega}{ds}\tan\varphi = \mathcal{C}\,.} \tag{98}$$

In the above construction, the value assigned to the positive constant $\mathcal{C}$ is the opposite of the curvature $1/r = d\varphi/ds < 0$ at the point of maximum ordinate ($\varphi = 0$). Therefore, it does not depend on the arc parameterization per se, but only on the shape itself. Consequently, the equation for stationarity, cast in the form of (98) is intrinsic.

REMARK 5.3 *Letting*

$$\boxed{\omega = \frac{\sigma}{\sin\varphi}} \tag{99}$$

in (98) gives the following more compact expression:

$$\boxed{\frac{d\sigma}{\sigma} = -\frac{\mathcal{C}}{\tan\varphi}\,ds\,.} \tag{100}$$

However, this formula is not necessarily suited for numerical quadrature since as $t \longrightarrow t_0$, $\varphi \longrightarrow 0$, $\sigma \longrightarrow 0$ ($\omega \longrightarrow \omega_0$, a finite limit), and the above expression for $d\sigma/dt$ exhibits a 0/0-type indeterminate form, requiring special treatment.

We now have the elements to prove that $\alpha > 1$ is a consequence of the construction in Theorem 5.1. For this purpose, let $\Big(x_1(t), y_1(t)\Big)$ $(0 \le t \le 1)$ be a given admissible parameterization, and suppose that the function $\omega(t)$ and the exponent α have been calculated according to this construction with $x(t) = x_1(t)$ and $y(t) = y_1(t)$. Hence, this substitution of symbols being made, the Euler-Lagrange stationarity condition, (29), holds; however, it is not presupposed here that the function $y(t) = y_1(t)$ realizes a global, or even local minimum of the functional $\mathcal{J}(y)$.

Nevertheless, with the same formal definition of the function $j(\lambda)$, equation (21) still holds, and implies the following asymptotics:

$$j(\lambda) \sim \frac{c_1}{\lambda} \quad (\text{as } \lambda \longrightarrow 0+)\,, \quad j(\lambda) \sim c_2\, \lambda^{\alpha-1} \quad (\text{as } \lambda \longrightarrow \infty) \tag{101}$$

where $c_1 = \Big(\int_0^1 x_1'\, \omega\, dt\Big)^\alpha / \mathcal{A}_1$ and $c_2 = \Big(\int_0^1 |y_1'|\, \omega\, dt\Big)^\alpha / \mathcal{A}_1$.

Since $\mathcal{J}(y)$ is stationary for $y(t) = y_1(t)$,

$$\delta \mathcal{J}(y_1)(\delta y) = 0\,,\ \forall\, \delta y \tag{102}$$

so that, by the chain rule

$$j'(1) = \delta \mathcal{J}(y_1)(y_1) = 0\,. \tag{103}$$

Now, compute the logarithmic derivative of $j(\lambda)$:

$$\begin{aligned} \frac{j'(\lambda)}{j(\lambda)} &= \frac{\alpha-1}{\lambda} + \alpha \frac{\int_0^1 \frac{1}{2}\left(\frac{x_1'^2}{\lambda^2} + y_1'^2\right)^{-\frac{1}{2}} \left(-\frac{2x_1'^2}{\lambda^3}\right) \omega\, dt}{\int_0^1 \left(\frac{x_1'^2}{\lambda^2} + y_1'^2\right)^{\frac{1}{2}} \omega\, dt} \\ &= \frac{\alpha-1}{\lambda} - \alpha \frac{\int_0^1 \left(x_1'^2 + \lambda^2 y_1'^2\right)^{-\frac{1}{2}} \frac{x_1'^2}{\lambda^2}\, \omega\, dt}{\int_0^1 \left(x_1'^2 + \lambda^2 y_1'^2\right)^{\frac{1}{2}} \frac{1}{\lambda}\, \omega\, dt} \end{aligned} \tag{104}$$

so that:

$$j'(\lambda) = 0 \Longleftrightarrow (\alpha - 1) \int_0^1 \left(x_1'^2 + \lambda^2 y_1'^2\right)^{\frac{1}{2}} \omega\, dt = \alpha \int_0^1 \frac{x_1'^2}{\left(x_1'^2 + \lambda^2 y_1'^2\right)^{\frac{1}{2}}}\, \omega\, dt\,. \tag{105}$$

This expression can now permit us to prove the following

Theorem 5.2 *Let* $\Big(x_1(t), y_1(t)\Big)$ $(0 \le t \le 1)$ *be an admissible parameterization, and let the function* $\omega(t)$ *and the exponent* α *be calculated by the construction in Theorem 5.1 with* $x(t) = x_1(t)$ *and* $y(t) = y_1(t)$*. Then:*

1. $\alpha > 1$;
2. *the function*

$$j(\lambda) = \mathcal{J}\Big(\lambda y_1(t)\Big) \tag{106}$$

for fixed $x(t) = x_1(t)$, *is unimodal; it admits the following asymptotics*

$$j(\lambda) \sim \frac{c_1}{\lambda} \quad (as\ \lambda \longrightarrow 0+)\,, \quad j(\lambda) \sim c_2\,\lambda^{\alpha-1} \quad (as\ \lambda \longrightarrow \infty) \tag{107}$$

where $c_1 = \Big(\int_0^1 x_1'\,\omega\,dt\Big)^\alpha / \mathcal{A}_1$, $c_2 = \Big(\int_0^1 |y_1'|\,\omega\,dt\Big)^\alpha / \mathcal{A}_1$, *and* $\mathcal{A}_1 = \int_0^1 y_1\,x_1'\,\omega\,dt$, *and a unique stationary point,* $\lambda_1 = 1$, *realizing a global minimum.*

Proof. Some of the conclusions of the theorem have already been established, in particular the asymptotics. Additionally, in view of (103), setting $\lambda = 1$ in (105) yields:

$$1 - \frac{1}{\alpha} = \frac{\int_0^1 \frac{x_1'^2}{\left(x_1'^2+y_1'^2\right)^{\frac{1}{2}}}\,\omega\,dt}{\int_0^1 \left(x_1'^2 + y_1'^2\right)^{\frac{1}{2}}\,\omega\,dt} > 0 \Longrightarrow \alpha = \frac{\int_0^1 \left(x_1'^2 + y_1'^2\right)^{\frac{1}{2}}\,\omega\,dt}{\int_0^1 \frac{y_1'^2}{\left(x_1'^2+y_1'^2\right)^{\frac{1}{2}}}\,\omega\,dt} > 1\,. \tag{108}$$

Secondly, in the equation stated in (105), the function of λ appearing on the left-hand side of the equality sign is monotone *increasing*, whereas inversely, the function on the right-hand side is monotone *decreasing*. Therefore, equality holds for at most one λ, hence uniquely for $\lambda = 1$. In summary, the smooth function $j(\lambda)$ admits the cited-above asymptotics with $\alpha > 1$ and a unique stationary point, $\lambda = 1$. Hence, $\lambda = \lambda_1 = 1$ realizes a local minimum, which is also unique and global. The function $j(\lambda)$ is therefore unimodal. ■

Considering now variations in an arbitrary admissible direction, we have the following

THEOREM 5.3 *With the setting of Theorem 5.2, including the calculation of the function* $\omega(t)$ *and the exponent* α *by the construction in Theorem 5.1 with* $x(t) = x_1(t)$ *and* $y(t) = y_1(t)$, *consider another admissible parameterization* $\Big(x_2(t), y_2(t)\Big)$ $(0 \le t \le 1)$. *Then, the function*

$$j_2(\lambda) = \mathcal{J}\Big(\lambda y_2(t)\Big) \tag{109}$$

for fixed $x(t) = x_2(t)$, *is unimodal; it admits the following asymptotics*

$$j_2(\lambda) \sim \frac{c_1'}{\lambda} \quad (as\ \lambda \longrightarrow 0+)\,, \quad j_2(\lambda) \sim c_2'\,\lambda^{\alpha-1} \quad (as\ \lambda \longrightarrow \infty) \tag{110}$$

where $c_1' = \Big(\int_0^1 x_2'\,\omega\,dt\Big)^\alpha / \mathcal{A}_1'$, $c_2' = \Big(\int_0^1 |y_2'|\,\omega\,dt\Big)^\alpha / \mathcal{A}_1'$, *and* $\mathcal{A}_1' = \int_0^1 y_2\,x_2'\,\omega\,dt$, *and a unique stationary point,* $\lambda = \lambda_2$, *realizing a global minimum.*

Proof. The reason for the asymptotics is the same as previously, except that now we know that $\alpha > 1$ from Theorem 5.2. Hence, the smooth function $j_2(\lambda)$ admits a global minimum at which point it is stationary. But an equation similar to (105) indicates that such a stationary point, if it exists, is unique. The remaining follows. ■

6. Illustrative numerical example involving a known airfoil shape

The RAE2822 airfoil is a classical geometry in computational aerodynamics, known for its low-drag performance in the transonic regime. This shape has been tabulated by the European Project ECARP (Périaux et al., 1998) and is also available from the KTH Web site "Pablo":

`http://www.nada.kth.se/~chris/pablo/pablo.html`

The corresponding data are used here to test our construction numerically.

This airfoil is represented on top of Fig. 6 on which it is apparent that if the airfoil's upper surface is concave, the lower surface is not convex. Therefore, our construction is not directly applicable to the lower surface for which $y(x)$ is not unimodal. To overcome this difficulty, it is decided to measure ordinates w.r.t. to the camberline. This is equivalent to treating the case of a symmetrical airfoil of identical distribution of thickness along the chord. This new airfoil is shown on Fig. 6 (bottom).

The symmetrical airfoil upper surface is parameterized first, specifically by a degree-n Bézier parameterization (Farin, 1990)

$$P(t)\ :\quad \begin{cases} x(t) = \displaystyle\sum_{k=0}^{n} \binom{n}{k} t^k\,(1-t)^{n-k}\,x_k \\ y(t) = \displaystyle\sum_{k=0}^{n} \binom{n}{k} t^k\,(1-t)^{n-k}\,y_k \end{cases} \tag{111}$$

where $\binom{n}{k}$ is a binomial coefficient, and

$$P_k = (x_k, y_k) \quad (0 \le k \le n) \tag{112}$$

is an adjustable set of control points.

Applying a technique described in Désidéri (2203) and Bélahcène and Désidéri (2003), we have used specifically the data given in Table 1 defining a suitable control polygon.

This control polygon supports a degree-16 Bézier parameterization and results from the optimization of the two free parameters $x_2^{(4)}$ and $x_3^{(4)}$ associated

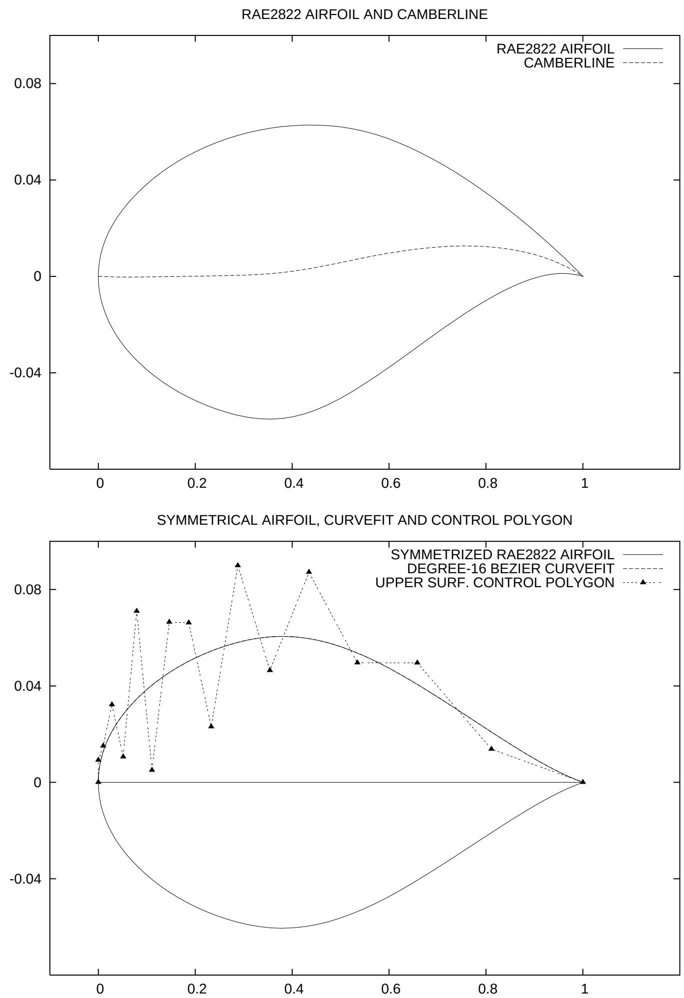

Figure 6. RAE2822 airfoil and camberline (top); symmetrical airfoil of identical distribution of thickness along the chord, superimposed degree-16 Bézier curvefit, and control polygon (bottom); $x : y$ scale 1:5

with a degree-4 Bézier parameterization. The optimization process can be described as follows: given the values for these parameters satisfying the conditions $0 \leq x_2^{(4)} \leq x_3^{(4)} \leq 1$, the support of a degree-4 Bézier parameterization is constructed by completing the sequence at endpoints by 0's and 1:

$$\{ x_k^{(4)} \} = \left\{ 0, 0, x_2^{(4)}, x_3^{(4)}, 1 \right\} . \tag{113}$$

Table 1. Coordinates of control points supporting the degree-16 parameterization of the symmetrized RAE2822 airfoil

k	x_k	y_k
0	0.00000000	0.00000000
1	0.00000000	0.00901285
2	0.01000000	0.01567691
3	0.02735714	0.02838362
4	0.05013187	0.02125740
5	0.07708791	0.04556686
6	0.10769231	0.04674385
7	0.14211538	0.01152551
8	0.18123077	0.11801427
9	0.22661538	-.01510481
10	0.28054945	0.11016884
11	0.34601648	0.03575430
12	0.42670330	0.09234295
13	0.52700000	0.04909957
14	0.65200000	0.05107346
15	0.80750000	0.01403431
16	1.00000000	0.00000000

Then, the classical *degree elevation process* (Farin, 1990) is applied 12 times to obtain a candidate support of a degree-16 Bézier parameterization:

$$\text{For } i = 5, ..., 16 \text{ do :} \quad x_0^{(i)} = 0\,;\; x_k^{(i)} = \frac{k}{i}\, x_{k-1}^{(i-1)} + (1 - \frac{k}{i})\, x_k^{(i-1)} \quad (1 \le k \le i)\,. \tag{114}$$

At level $n = 16$, the superscript $^{(16)}$ is omitted hereafter. By this construction, the sequence $\{x_k\}$ $(0 \le k \le n = 16)$ and by consequence, the function $x(t)$ are monotone increasing; additionally $x'(0) = 0$. For any such sequence, a unique sequence $\{y_k\}$ is then defined such that the corresponding Bézier arc is a least-squares approximation of the airfoil tabulated data. Note that $dy/dt(0) = ny_1$; thus if $y_1 \neq 0$, the shape is tangent to the y-axis at the origin, as airfoils are. Additionally, considering the upper surface, only cases for which $y_1 > 0$ are retained; among all possibilities, the elected parameterization is defined to be the one associated with the least value of the total variation $TV(\{y_k\}) = \sum_{k=1}^{n} |y_k - y_{k-1}|$. By proceeding in this way, excessive variations in the control polygon $P_0\, P_1 \ldots P_n$ have been avoided, while enforcing the geometrical and certain monotonicity constraints.

This cautious construction of the parameterization revealed to be rather effective to improve the iterative performance of the shape optimization algorithm that one possibly applies thereafter (by allowing the y_k's to vary) to minimize a physical criterion such as drag in aerodynamics (Clarich and Désidéri, 2002; Tang and Désidéri, 2002). The corresponding curvefit and control polygon are shown on Fig. 6 (bottom). The differences between the tabulated and parameterized data are of the order of 10^{-6}, thus not visible.

Note that $P(t)$ can be computed by the known de Casteljau algorithm:

$$P_k^j = (1-t)\,P_k^{j-1} + t\,P_{k+1}^{j-1} \quad j = 1, 2, ..., n\,;\; k = 0, 1, ..., n-j \tag{115}$$

where $P_k^0 = P_k$; as a result of this algorithm, $P(t) = P_0^n$. The merit of this algorithm resides in its intrinsic numerical stability (Sederberg, 2004), because it only involves convex combinations of bounded terms. However it is $O(n^2)$ in computing cost, thus not optimal. An alternative is to factor either $(1-t)^n$ if $0 \le t \le \frac{1}{2}$, or t^n if $\frac{1}{2} \le t \le 1$,

$$P(t) = (1-t)^n \sum_{k=0}^{n} \binom{n}{k} x_k\, u^k = t^n \sum_{k=0}^{n} \binom{n}{k} x_k\, v^{n-k} \tag{116}$$

where $u = t/(1-t)$ and $v = (1-t)/t$, and to evaluate the finite sum by the classical Horner factorization algorithm as a polynomial in either u or v. This process is $O(n)$ in cost. Both factorizations were tested throughout the entire interval [0,1] and resulted in very close results without apparent excessive rounding errors. The factorization of t^n, more stable near $t = 1$, was preferred to avoid instabilities in the region of the domain where the cumulative errors associated with the quadratures performed numerically from $t = 0$ could be critical.

A great advantage of the Bézier parameterization is to provide simple formal expressions for the successive derivatives. In particular:

$$\begin{cases} x'(t) = n \displaystyle\sum_{k=0}^{n-1} \binom{n-1}{k} t^k\,(1-t)^{n-1-k}\,(x_{k+1} - x_k) \\ \qquad\;\; = n t^{n-1} \displaystyle\sum_{k=0}^{n-1} \binom{n-1}{k} (x_{k+1} - x_k)\, v^{n-k} \\ x''(t) = n(n-1) \displaystyle\sum_{k=0}^{n-2} \binom{n-2}{k} t^k\,(1-t)^{n-2-k}\,(x_{k+2} - 2x_{k+1} + x_k) \\ \qquad\;\; = n(n-1) t^{n-2} \displaystyle\sum_{k=0}^{n-2} \binom{n-2}{k} (x_{k+2} - 2x_{k+1} + x_k)\, v^{n-k} \end{cases} \tag{117}$$

and similarly for $y'(t)$ and $y''(t)$.

A simple program in the MAPLE language was developed to incorporate the formulas, proceed with both formal and numerical calculations and realize plots. The parameterization of the arc being given, the equation

$$y'(t_0) = 0 \tag{118}$$

is solved first for v_0 giving $t_0 = 1/(v_0+1)$, and this permits to compute the constant $\mathcal{C}$. Then the expression

$$\theta(\tau) = \left(\frac{x''(\tau)y'(\tau) - x'(\tau)y''(\tau)}{x'^2(\tau) + y'^2(\tau)} - \mathcal{C}\sqrt{x'^2(\tau) + y'^2(\tau)} \right) \frac{x'(\tau)}{y'(\tau)} \tag{119}$$

is evaluated and integrated w.r.t. τ from 0 to t to give $\omega(t)$ according to (96); lastly, quadratures are performed to get p and $\mathcal{A}$, and the exponent α, and the functional value is calculated.

Using first the raw data of Table 1, we obtained the results plotted in Fig. 7, and certain surprising computed values indicated in the first row of Table 2.

Table 2. Computational parameters as the height $y_{\max}$ of the half, symmetrized RAE2822 airfoil varies

$y_{\max}$	p	$\mathcal{A}$	α	$\mathcal{J}$	$\omega_{\max}/\omega_{\min}$
.0605[1]	9.869941199	.2300139269	70.89388551	$.1346\ 10^{72}$	67.77046225
0.1	6.076089363	.2285417055	26.59416394	$.3025\ 10^{22}$	41.40837127
0.2	3.311823172	.2244289935	7.380500871	30709.80236	21.85349498*
0.3	2.551429236	.2217957704*	3.835629629	163.8014309	26.55945639
0.4	2.309646110	.2218169974	2.603863915	39.86832902	37.17292189
0.5[2]	2.293345658*	.2249660763	2.039436261	24.15669807	55.28979685
0.6	2.411134451	.2315163309	1.736265759	19.90931635	86.18290342
0.7	2.631214442	.2417460867	1.555343812	18.62635869	139.2398221
0.8	2.945983733	.2560226326	1.438764299	18.48559818*	231.2135672
0.9	3.360657329	.2748430334	1.359016231	18.89445381	392.1145181
1.0	3.889281952	.2988629959	1.301742056	19.60585095	675.9438667
1.1	4.553612828	.3289261951	1.258903982	20.49786634	1180.163909
2.0	25.74741157	1.168458865	1.102091885	30.70043981	236261.7172
10.	$.3815\ 10^{10}$	$.3743\ 10^{8}$	1.019640589	157.2198284	$.6542\ 10^{27}$

[1] raw data, case of Fig. 7
[2] case of Fig. 8
* approximate minimum

Examining more closely these results, several observations can be made. First, with the Bézier parameterization, the functions $x(t)$ and‘ $y(t)$ are smooth, and have smooth and bounded derivatives, whereas the curvefit is such that $y'(x)$ is infinite at the origin, as desirable. However, the results are rather sensitive to the scaling, and in particular with the raw data, the scales are evidently inappropriate. The aspect ratio (chord/height) of the RAE2822 airfoil is close to 20, as typical of standard airfoils. As a result, it appears clearly from Fig. 6,

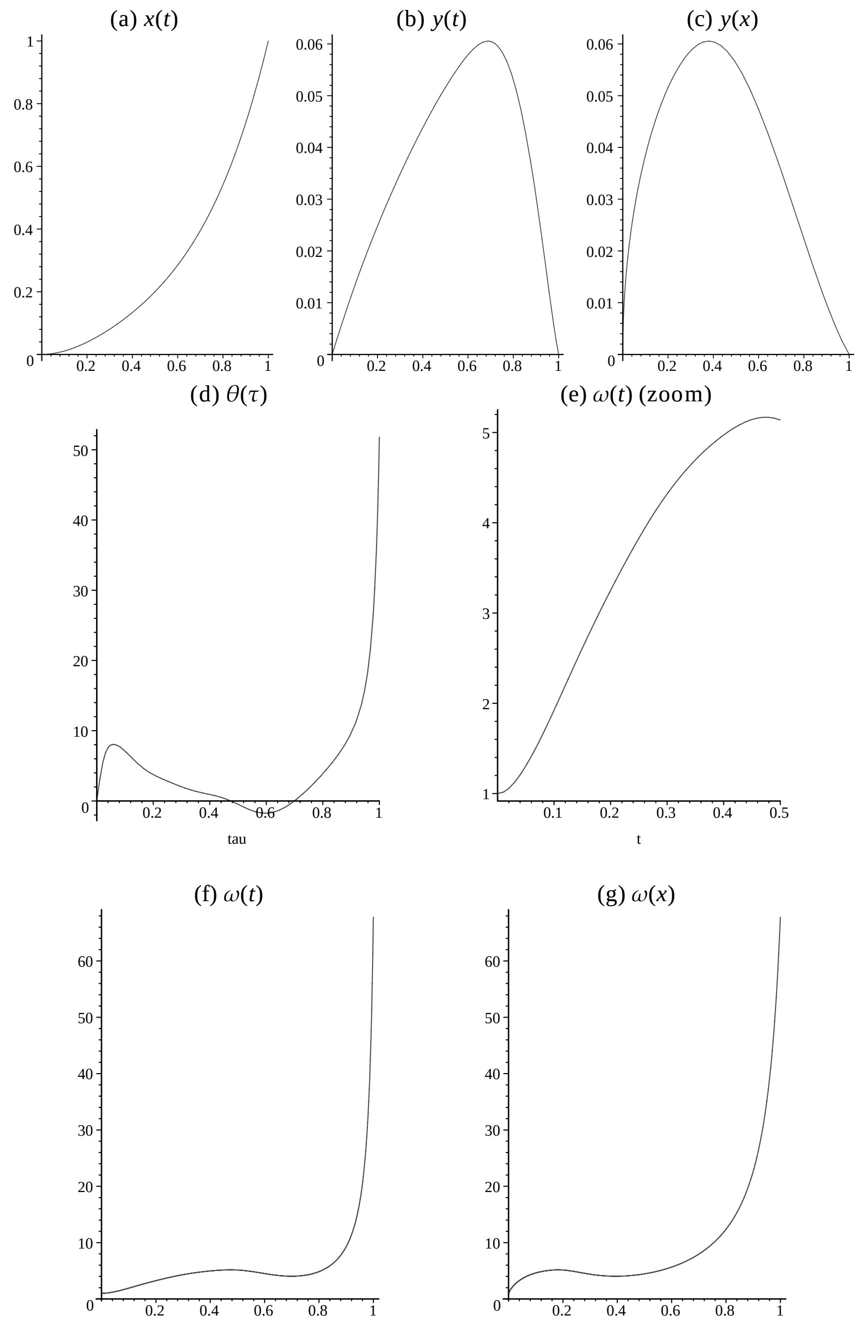

Figure 7. Output of MAPLE program; raw data for symmetrized RAE2822 airfoil

and even though the scales have been chosen in a way that reduces the visibility of this effect, that the curvature at the point of maximum height is very small compared to the curvature at the origin. As a consequence, the constant $\mathcal{C}$ is small ($\mathcal{C} = .6052734906$), and the function $\theta(\tau)$ is found to be positive over a large portion of its domain over which $\omega(t)$ is monotone-increasing; $\omega(x)$ is uniformly greater than 1. Although the integrals p and $\mathcal{A}$ are not excessively different from 1 in magnitude, the resulting exponent α is very large ($\alpha > 70$), and the functional value explodes. This difficulty can easily be attenuated by normalizing the initial data to allow $y(t)$ (or $y(x)$) to vary between 0 and a specified value $y_{\max}$. Repeating the experiment with the airfoil coordinates replaced by normalized data, gives the results indicated in the subsequent rows of Table 2, for various values of the maximum airfoil ordinate $y_{\max}$.

The case $y_{\max} = 0.5$ is in some sense, the closest to the circle; the corresponding data are plotted in Fig. 8. For this case, the constant $\mathcal{C} = 4.998530508$, that is more than 8 times larger than above, and the function $\theta(\tau)$ shows a better balance of positive and negative values; the function $\omega(t)$ decreases before increasing, and the maximum value of $\omega'(x)$ is smaller, although $\omega(x)$ still exhibits a boundary-layer-type shape near the trailing edge. Here, the exponent α resulting from the algorithm is strictly greater than one, which confirms again the theoretical result, but close to 2. The ratio $\omega_{\max}/\omega_{\min}$ is a sort of condition number; it is close to 55 in this case, which allows a tractable numerical treatment.

The remaining data of Table 2 indicate that the quantities p, $\mathcal{A}$, $\mathcal{J}$ and the condition number $\omega_{\max}/\omega_{min}$ are unimodal functions of $y_{\max}$, but each quantity achieves a minimum at a different point. In contrast, the exponent α is monotonic and tends to 1 as $y_{\max} \longrightarrow \infty$. These trends are also shown on Fig. 9. The condition number $\omega_{\max}/\omega_{\min}$, in a logarithmic scale, has an apparent linear portion, indicative of a power law, before achieving its minimum at an angular point; no explanation for this is available.

The last experiment is the most fundamental and is meant to demonstrate that given the function $\omega(t)$ (or $\omega(x)$) and the exponent α computed by our MAPLE program in the case of Fig. 8, the shape can be produced by minimizing the functional $\mathcal{J}$. This experiment was carried out for the symmetrical airfoil upper surface. To simplify the experiment, the sequence of abscissas $\{x_k\}$ ($0 \le k \le n = 16$) was fixed, and only the corresponding ordinates $\{y_k\}$ have been computed by numerical minimization of the functional by the Quasi-Newton Method using the E04JYF procedure of the NAG Library. This computation was carried with equal success for a number of initial conditions, including the rather different following two cases:

$$\begin{aligned} &\text{Test case 1:}\quad \text{initial } y_k = 1 \\ &\text{Test case 2:}\quad \text{initial } y_k = \frac{1}{10} + \left(\frac{k}{15}\right)^4 \end{aligned} \tag{120}$$

$(1 \le k \le n-1 = 15\,;\ y_0 = y_{16} = 0)$.

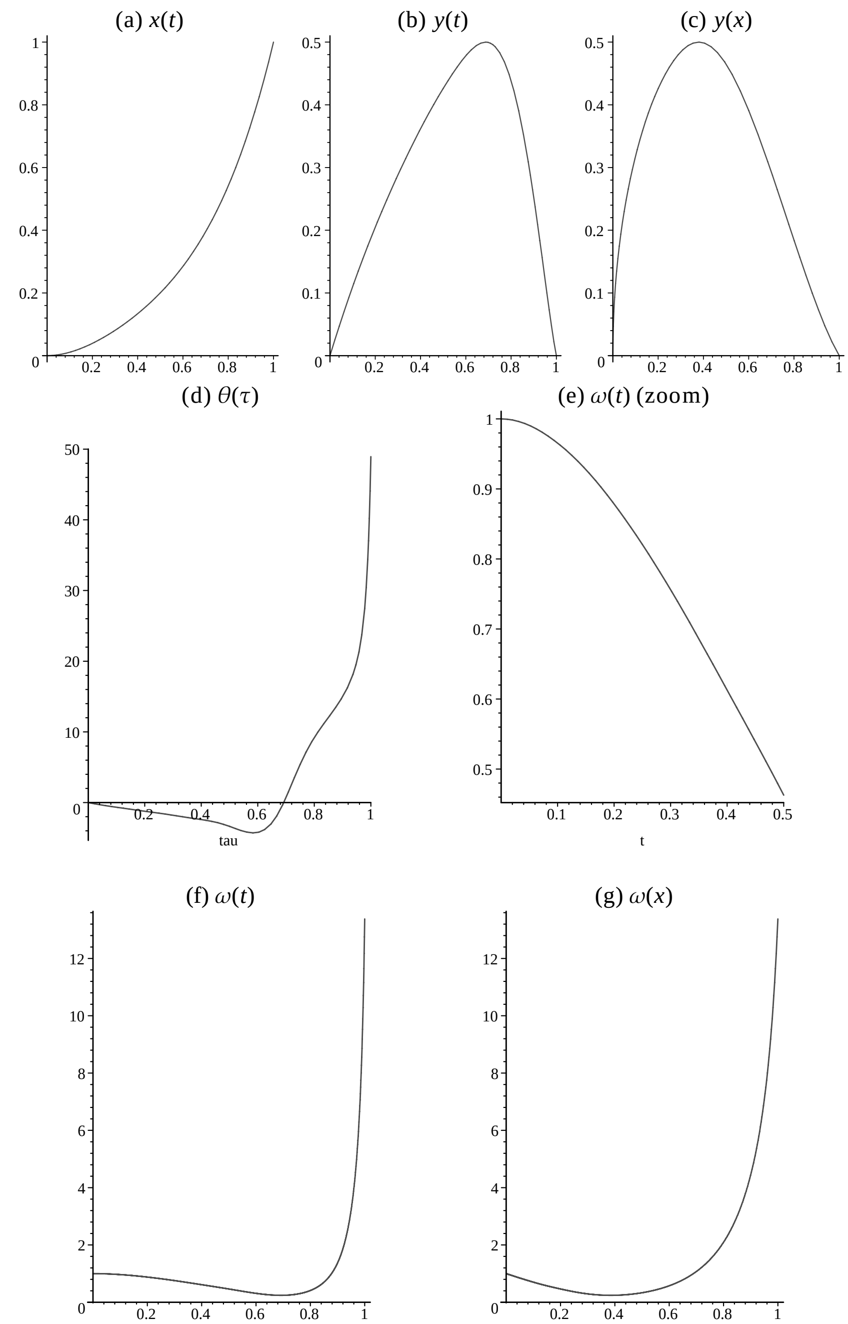

Figure 8. Output of MAPLE program; normalized data for the symmetrized RAE2822 airfoil

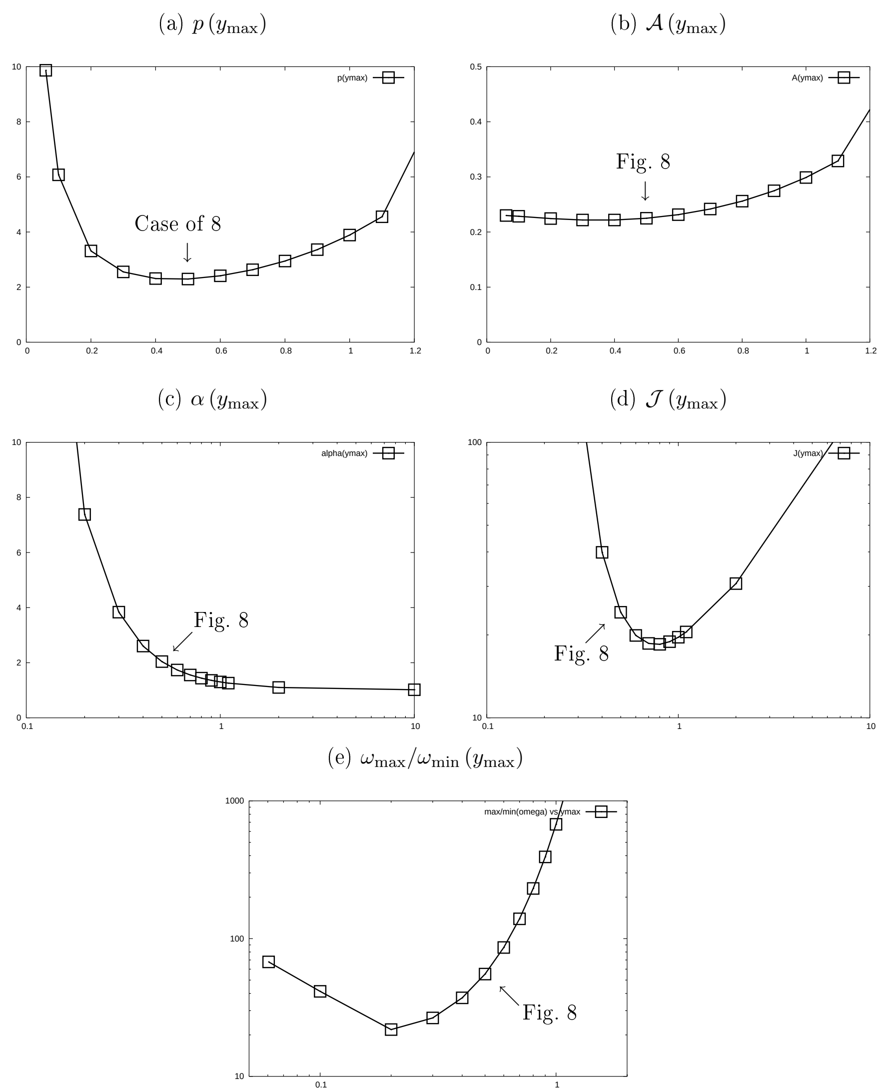

Figure 9. Variations of p, $\mathcal{A}$, α, $\mathcal{J}$ and $\omega_{\max}/\omega_{\min}$ with $y_{\max}$

The evident convergence of the shape with the black-box iterative procedure is shown on Fig. 10, which provides the minimum functional value (top), and the corresponding best-found shape (bottom) in terms of function evaluations. Additionally, this experiment shows that the rise of the function $\omega(x)$ near $x = 1$ with the employed non-optimal normalization of the ordinates, does not cause the minimization procedure serious difficulties to converge.

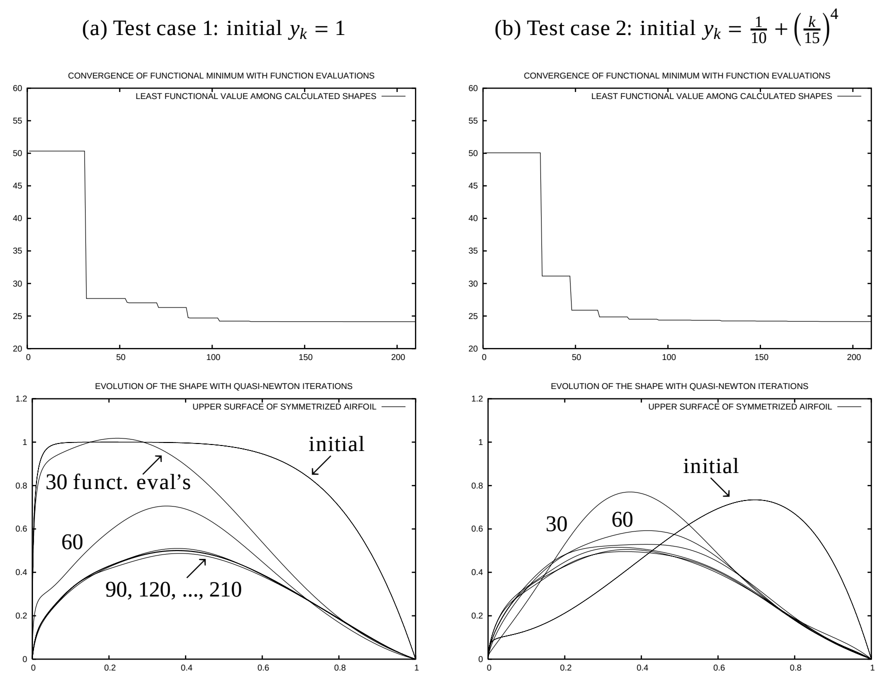

Figure 10. Convergence of the shape with the quasi-Newton iteration (top: minimum functional value; bottom: best-found shape in terms of function evaluations)

7. Conclusions

We have established and illustrated the construction of a unimodal shape optimization model, as a problem of calculus of variations. Besides the theoretical result, we anticipate that the formulation could be more powerful, in inverse problems, than a more conventional approach consisting in minimizing the L^2 (or H^1) norm, $\|y - y_T\|$ (where y_T denotes a target-shape function), by potentially discriminating more the more relevant geometrical elements of the given shape inherited from the optimization of a physical criterion. In this respect, the observed greater numerical sensitivity of the weighting function $\omega(x)$ in the zone just upstream the trailing-edge in the airfoil case, tends to confirm this expectation.

Acknowledgements

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References

Bélahcène, F. and Désidéri, J.-A. (2003) Paramétrisation de Bézier adaptative pour l'optimisation de forme en Aérodynamique. Research Report 4943, INRIA.

Clarich, A. and Désidéri, J.-A. (2002) Self-adaptive parameterisation for aerodynamic optimum-shape design. Research Report 4428, INRIA.

Désidéri, J.-A. (2003) Hierarchical optimum-shape algorithms using embedded Bézier parameterizations. In: Yu. Kuznetsov, P. Neittanmäki and O. Pironneau, eds., *Numerical Methods for Scientific Computing, Variational Problems and Applications*. CIMNE, Barcelona.

Farin, G. (1990) *Curves and Surfaces for Computer Aided Geometric Design – A practical Guide author*. W. Rheinboldt and D. Siewiorek, eds., Computer Science and Scientific Computing, Academic Press, 2nd, Boston.

Jabri, Y. (2003) *The Mountain Pass Theorem, Variants, Generalizations and Some Applications*. Cambridge University Press.

Karakasis, M. and Désidéri, J.-A. (2002) Model Reduction and Adaption of Optimum-Shape Design in Aerodynamics by Neural Networks. Research Report 4503, INRIA.

Périaux, J., Bugeda, G., Chaviaropoulos, P.K., Giannakoglou, K., Lantéri, S. and Mantel, B. (1998) *Optimum Aerodynamic Design & Parallel Navier-Stokes Computations, ECARP European Computational Aerodynamics Research Project*. Notes on Numerical Fluid Mechanics, Vieweg, Braunschweig/Wiesbaden, Germany.

Sederberg, T.W. (2004) *Computer Aided Geometric Design*. CS557 class notes, `http://cagd.cs.byu.edu/~tom/courses.html` Brigham Young University, Utah.

Tang, Z.L. and Désidéri, J.-A. (2002) Towards Self-Adaptive Parameterization of Bézier Curves for Airfoil Aerodynamic Design. Research Report 4572, INRIA.

A practical unimodality criterion

As a step to prove that the local minimum of the functional was indeed global, we attempted first to establish the unimodality by application of the criterion below to a particular case. This route failed to deliver the full proof, but revealed itself to be instructive, and we describe it here for its own sake.

LEMMA 7.1 (*Criterion of unimodality*) *Let f be a strictly-positive, convex and monotone-increasing function of the real variable, and let:*

$$\eta(y) = 1 + \frac{\mathcal{J}(y)\, f''\Big(\mathcal{J}(y)\Big)}{f'\Big(\mathcal{J}(y)\Big)}\,. \tag{121}$$

If the functional $\mathcal{J}(y)$ is stationary w.r.t. variations in $y = y(t)$, for fixed $x = x(t)$, and if the quadratic form

$$\begin{aligned} Q(Y) =& \eta \left(\int_0^1 \left(\frac{\alpha}{p}\phi - \frac{1}{\mathcal{A}}\psi\right) Y\, dt\right)^2 - \frac{\alpha}{p^2}\left(\int_0^1 \phi\, Y\, dt\right)^2 \\ &+ \frac{1}{\mathcal{A}^2}\left(\int_0^1 \psi\, Y\, dt\right)^2 + \frac{\alpha}{p}\int_0^1 \frac{\omega x'^2}{(x'^2+y'^2)^{\frac{3}{2}}}\, Y'^2\, dt \end{aligned} \tag{122}$$

is positive-definite, then it is unimodal.

Proof. The functionals $\mathcal{J}(y)$ and

$$\mathcal{K}(y) = f\Big(\mathcal{J}(y)\Big) \tag{123}$$

have identical directions of variations. Hence a sufficient condition for the functional $\mathcal{J}(y)$ to be unimodal, is that the functional $\mathcal{K}(y)$ be convex. But, injecting the expressions for the first and second variations of the functional $\mathcal{J}(y)$, (40) and (66), in the second variation of the functional $\mathcal{K}(y)$,

$$\delta^2\mathcal{K} = f'(\mathcal{J})\,\delta^2\mathcal{J} + f''(\mathcal{J})\,(\delta\mathcal{J})^2 \tag{124}$$

yields:

$$\delta^2\mathcal{K} = \mathcal{J} f'(\mathcal{J})\left[\frac{\delta^2\mathcal{J}}{\mathcal{J}} + (\eta-1)\left(\frac{\delta\mathcal{J}}{\mathcal{J}}\right)^2\right] = \mathcal{J} f'(\mathcal{J})\, Q(\delta y) \tag{125}$$

which is positive-definite iff the stated criterion is satisfied. ■

Now assume $\alpha > 1$, which is necessary to the existence of a global minimum, and examine whether this criterion is satisfied in some region of the functional space of $y(t)$.

Since Y stands for a free perturbation, we can equivalently replace it by the following expression,

$$Y = \frac{Z}{\psi} \tag{126}$$

and let Z be the new free perturbation. At point y, define the following function of t :

$$F = F(y) = \frac{\alpha}{p}\,\frac{\phi}{\psi} - \frac{1}{\mathcal{A}} \tag{127}$$

so that:

$$\phi = \frac{p\psi}{\alpha}\left(F + \frac{1}{\mathcal{A}}\right) \tag{128}$$

and:

$$\begin{aligned} Q(Y) =&\eta\left(\int_0^1 F\,Z\,dt\right)^2 - \frac{1}{\alpha}\left(\int_0^1 \left(F+\frac{1}{\mathcal{A}}\right) Z\,dt\right)^2 + \frac{1}{\mathcal{A}^2}\left(\int_0^1 Z\,dt\right)^2 \\ &+ \frac{\alpha}{p}\int_0^1 \frac{\omega x'^2}{(x'^2+y'^2)^{\frac{3}{2}}}\,Y'^2\,dt \\ =&R(Z) + S(Z) + \frac{\alpha}{p}\int_0^1 \frac{\omega x'^2}{(x'^2+y'^2)^{\frac{3}{2}}}\,Y'^2\,dt \end{aligned} \tag{129}$$

where:

$$\begin{cases} R(Z) = \left(\eta - \frac{1}{\alpha}\right)\left(\int_0^1 FZdt\right)^2 - \frac{2}{\alpha\mathcal{A}}\left(\int_0^1 FZdt\right) = \left(\eta - \frac{1}{\alpha}\right) I^2 - \frac{2}{\alpha\mathcal{A}} I \\ S(Z) = \left(1 - \frac{1}{\alpha}\right)\frac{1}{\mathcal{A}^2}\left(\int_0^1 Zdt\right)^2 \end{cases} \tag{130}$$

where $I = \int_0^1 F\,Z\,dt$. Evidently, for $\alpha \geq 1$:

$$Q(Y) \geq \min\, R(Z) = -\frac{1}{\alpha^2\mathcal{A}^2\left(\eta - \frac{1}{\alpha}\right)} \geq -\frac{1}{\eta\mathcal{A}^2}\,. \tag{131}$$

Unfortunately, this bound is negative, although arbitrarily small by possible adjustment of η. For this reason and the difficulty to establish bounds on the integral involving Y' in (129), we have not been able to make a definite conclusion on the eventual positivity of $Q(Y)$ itself. Instead, in what follows, we prove that:

$$\min_Z \Big(R(Z) + S(Z)\Big) < 0\,. \tag{132}$$

To achieve this result, we first split the functional space Υ of $y = y(t)$ into two complementary subsets:

$$\Upsilon_1 = \{\, y = y(t) \text{ admissible and such that } F(y) = \text{const. (as a function of t)} \,\} \tag{133}$$

and

$$\Upsilon_2 = \Upsilon - \Upsilon_1 \tag{134}$$

and examine two cases separately.

First, for $y \in \Upsilon_1$,

$$R(Z) + S(Z) = \left(\eta - \frac{1}{\alpha}\right) F^2\zeta^2 - \frac{2}{\alpha\mathcal{A}}\, F\zeta + \left(1 - \frac{1}{\alpha}\right) \frac{1}{\mathcal{A}^2}\, \zeta^2 \tag{135}$$

where

$$\zeta = \int_0^1 Z\, dt \tag{136}$$

can assume any value. Consequently,

$$\begin{aligned} \min_Z \Big(R(Z) + S(Z)\Big) &= \min_\zeta \left[\left(\eta - \frac{1}{\alpha}\right) F^2\zeta^2 - \frac{2}{\alpha\mathcal{A}}\, F\zeta + \left(1 - \frac{1}{\alpha}\right) \frac{1}{\mathcal{A}^2}\, \zeta^2\right] \\ &= -\frac{F^2}{\alpha^2 \left[\left(\eta - \frac{1}{\alpha}\right)\, F^2\mathcal{A}^2 + 1 - \frac{1}{\alpha}\right]}\,. \end{aligned} \tag{137}$$

Regardless of the *a priori* choice of η (positive and large), this minimum is strictly negative, except at points of stationarity ($F \equiv 0$).

Consider now the more general case where $y \in \Upsilon_2$, which implies in particular that $F \not\equiv 0$. Let

$$K = \int_0^1 F\, dt \tag{138}$$

and let us examine whether it is possible that

$$\int_0^1 F^2\, dt = K^2\,. \tag{139}$$

If this equation were satisfied, we could first conclude that $K \neq 0$ (since $F \not\equiv 0$), and:

$$\int_0^1 (F - K)^2\, dt = \int_0^1 F^2 - 2K \int_0^1 F\, dt + K^2 = 0 \tag{140}$$

and this would be in contradiction with the hypothesis of a nonuniform F. Therefore (139) is rejected, and we inversely conclude that:

$$\int_0^1 F^2 \, dt \neq \left(\int_0^1 F \, dt \right)^2 = K^2 \,. \tag{141}$$

Then define the function

$$F^{\perp} = 1 - cF \tag{142}$$

and adjust the constant c to make $F^{\perp}$ orthogonal to F:

$$c = \frac{\int_0^1 F \, dt}{\int_0^1 F^2 \, dt} = \frac{K}{\int_0^1 F^2 \, dt} \,. \tag{143}$$

Additionally,

$$\int_0^1 F^{\perp} \, dt = 1 - c \int_0^1 F = 1 - \frac{K^2}{\int_0^1 F^2 \, dt} \neq 0 \tag{144}$$

by virtue of (141). In summary, the function $F^{\perp}$ which results from this construction satisfies:

$$\int_0^1 F \, F^{\perp} \, dt = 0 \,, \quad \int_0^1 F^{\perp} \, dt \neq 0 \,. \tag{145}$$

Then let

$$Z = \varepsilon F - \varepsilon^{\perp} F^{\perp} \tag{146}$$

in such a way that

$$I = \int_0^1 F \, Z \, dt = \varepsilon \int_0^1 F^2 \, dt = \frac{1}{\alpha \mathcal{A} \left(\eta - \frac{1}{\alpha} \right)} \tag{147}$$

by appropriate choice of the adjustable constant ε. Then $R(Z)$ achieves its negative minimum:

$$R(Z) = - \frac{1}{\alpha^2 \mathcal{A}^2 \left(\eta - \frac{1}{\alpha} \right)} \,. \tag{148}$$

Besides, letting

$$\varepsilon^{\perp} = \varepsilon \frac{\int_0^1 F \, dt}{\int_0^1 F^{\perp} \, dt} \tag{149}$$

gives:

$$\int_0^1 Z\,dt = 0 \tag{150}$$

so that

$$S(Z) = 0\,. \tag{151}$$

Since $S(Z) \geq 0$ in general, this function Z also realizes the minimum of $S(Z)$. Therefore:

$$\min_Z \Big(R(Z) + S(Z)\Big) = -\frac{1}{\alpha^2 \mathcal{A}^2 \left(\eta - \frac{1}{\alpha}\right)} < 0 \tag{152}$$

in this case also.

In conclusion, the test in Lemma 7.1 is not easily applicable to support the proof of unimodality of the functional $\mathcal{J}(y)$, which however, has been established by another argument. The question of convexity for the functional $\mathcal{J}(y)$ as α varies from 1 to 2, remains unclear.