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Shape optimization in problems governed by generalised Navier–Stokes equations: existence analysis

by

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Abstract: We study a shape optimization problem for a paper machine headbox which distributes a mixture of water and wood fibers in the paper manufacturing process. The aim is to find a shape which a priori ensures the given velocity profile on the outlet part. The mathematical formulation leads to an optimal control problem in which the control variable is the shape of the domain representing the header, the state problem is represented by the generalised Navier-Stokes system with nontrivial boundary conditions. The objective of this paper is to prove the existence of an optimal shape.

Keywords: optimal shape design, paper machine headbox, incompressible non-Newtonian fluid, algebraic turbulence model.

1. Introduction

Since very long ago paper has belonged to the most used everyday products. About 19 centuries ago the ancient Chinese developed the paper manufacturing technique using bark and hemp. Since that time many improvements have been made in order to reduce costs and enhance quality, production speed and environmental compatibility. Today paper production has become a complex process.

Recently the paper machine technology has been developing mostly through the experimental work in pilot plants. With increasing speeds and sophisticated machines this approach has become too expensive and time-consuming so that more effective methods must be used to bring further development. One of such methods is mathematical modelling in the framework of continuum mechanics resulting in the numerical simulations for a proposed model. The experimental research is still needed to verify the simulated results.

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The first component in the paper making process is the headbox which is located at the wet end of a paper machine. The headbox shape and the fluid flow phenomena taking place there largely determine the quality of the produced paper. The first flow passage in the headbox is a dividing manifold, called the header. It is designed to distribute the fiber suspension on the wire so that the produced paper has an optimal basis weight and fiber orientation across the whole width of a paper machine. The aim of this work is to find an optimal shape for the back wall of the header so that the outlet flow rate distribution from the headbox results in an optimal paper quality.

The paper making pulp (also called the fibre suspension, furnish or stock) is a mixture of wood fibres, water, filler clays and various chemicals at the concentration of 1% solids to 99 % water by weight. In the large-scale simulation it seems reasonable to model this complex mixture as a single continuum, with the fluid being an incompressible liquid described by the Navier–Stokes equations

$$\rho v_{,t} + \rho \operatorname{div}(v \otimes v) = -\nabla q + \operatorname{div}\left(\mu_0 D(v)\right), \quad \operatorname{div}\, v = 0, \tag{1}$$

where v, q, ρ, μ_0 are, respectively, the velocity, the pressure, the density and the viscosity. The symbol $D(v) = \frac{1}{2}\left(\nabla v + (\nabla v)^T\right)$ means the symmetric part of the gradient of v and $|D(v)| = \left(\frac{1}{2}\sum_{i,j=1}^{2} D_{ij}(v) D_{ij}(v)\right)^{1/2}$ is its norm.

The turbulent character of the flow in the header is a desirable phenomenon in the paper making process. Typically, the input Reynolds number defined as $Re = \frac{\ell V}{\mu_0}$, where V denotes the magnitude of the input velocity and ℓ is the diameter of the input channel, is about 10^6. In the modelling of turbulence, the velocity field v is usually decomposed into the sum of the average velocity u and its fluctuation u'. Averaging of (1) then leads to the system

$$\rho u_{,t} + \rho \operatorname{div}(u \otimes u) = -\nabla p + \operatorname{div}\left(\mu_0 D(u) + R\right), \quad \operatorname{div}\, u = 0, \tag{2}$$

where R denotes the so-called Reynolds tensor given as the average of $-u' \otimes u'$. Since the flow in the header is steady and it is expected that the geometry of the domain changes only in the part of the boundary, we use a classical algebraic model, where

$$R = \rho l_{m,\alpha}^2 |D(u)| D(u) \tag{3}$$

with experimentally determined mixing length $l_{m,\alpha}^2$, specified later. Note that by inserting (3) into (2) we obtain the closed system for unknowns u and p.

Setting $\mu_1 = \rho l_{m,\alpha}^2$ and

$$T\left(p, D(u)\right) = -pI + \left(\mu_0 + \mu_1 |D(u)|\right) D(u) \tag{4}$$

we obtain the model appearing also in non-Newtonian fluid mechanics. The models where the Cauchy stress $T(p, D(u))$ takes the form

$$T(p, D(u)) = -pI + \nu\left(|D(u)|\right) D(u) \tag{5}$$

represent a class of non-Newtonian fluids called the fluids with shear-dependent viscosity. Since in the case (4) the viscosity increases with increasing shear rate (in a simple shear flow), (4) is a model for fluids that have the ability to shear thicken, see Rajagopal (1993), Málek, Rajagopal, Růžička (1995) and Málek, Nečas, Rokyta, Růžička (1996) for more details on non-Newtonian fluids and their mathematical analysis.

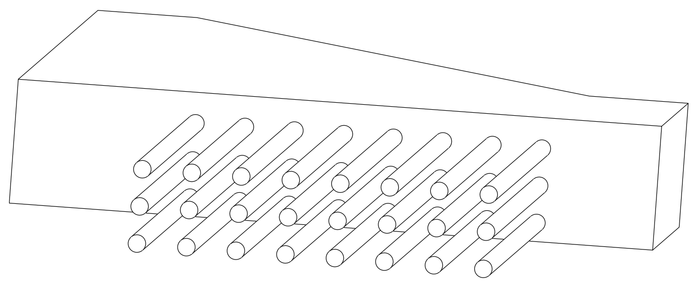

Figure 1. The header

Fig. 1 shows the geometry of the header. The inlet is on the left and the so-called recirculation on the right hand side. Typically about 10 % of the fluid flows out through the recirculation. The main outlet is formed by a number (usually several hundreds or thousands) of small tubes. This fact presents a difficulty in the numerical simulation and thus the complicated geometry of the tube bank is replaced by an effective medium using the homogenization technique. It introduces a nonstandard boundary condition of the form

$$T_{22} = -\sigma |u_\nu| u_\nu, \tag{6}$$

where T, u_ν, σ are the stress tensor, normal component of the velocity and the coefficient of suction, respectively.

This work was motivated by some previous papers: The fluid flow model which is used here has been derived and studied numerically in Hämäläinen (1993). The shape optimization problem has also been solved numerically and the results are presented in Hämäläinen, Mäkinen and Tarvainen (2000), see also Haslinger and Mäkinen (2003). Both the fluid flow model and the shape optimization problem have been studied there formally without establishing existence results. Therefore our goal is to give the theoretical analysis of the flow equations and of the whole optimization problem.

The text is organized as follows. In Section 2 we present the fluid flow model and analyze the existence of a solution. The existence proof is based on the appropriate energy estimates and the Galerkin method. A shape optimization problem is then formulated in Section 3 and the existence of an optimal shape is established. The continuous dependence of solutions to state problems with respect to shape variations is the most important result of this part.

Shape sensitivity analysis for this problem can be done using differential calculus developed in Sokolowski and Zolésio (1992). This, together with an approximation and convergence analysis will be published later on.

2. Steady flow of a non-Newtonian fluid

For describing the fluid flow in the header we shall use a two–dimensional stationary model. First we define the geometry of the problem.

2.1. Description of admissible domains

Let $L_1, L_2, L_3 > 0$, $H_1 \geq H_2 > 0$, $\alpha_{max} \geq \alpha_{min} > 0$, $\gamma > 0$ be given and suppose that $\alpha \in \mathcal{U}_{ad}$, where

$$\mathcal{U}_{ad} = \Big\{\alpha \in C^{0,1}([0,L]);\ \alpha_{min} \leq \alpha \leq \alpha_{max}, \\ \alpha|_{[0,L_1]} = H_1,\ \alpha|_{[L_1+L_2,L]} = H_2, |\alpha'| \leq \gamma \text{ a.e. in } [0,L]\Big\}. \quad (7)$$

Here $C^{0,1}([0,L])$ denotes the set of Lipschitz continuous functions on $[0,L]$ and $L = L_1 + L_2 + L_3$. With any $\alpha \in \mathcal{U}_{ad}$ we associate the domain $\Omega(\alpha)$, see Fig. 2:

$$\Omega(\alpha) = \Big\{(x_1,x_2) \in \mathbb{R}^2; 0 < x_1 < L, 0 < x_2 < \alpha(x_1)\Big\} \quad (8)$$

and introduce the system of admissible domains

$$\mathcal{O} = \big\{\Omega;\ \exists\ \alpha \in \mathcal{U}_{ad}:\ \Omega = \Omega(\alpha)\big\}.$$

Further we shall need the domains $\widehat{\Omega} = (0,L) \times (0,\alpha_{max})$ and $\Omega_0 = \big((0,L_1) \times (0,H_1)\big) \cup \big((0,L) \times (0,\alpha_{min})\big) \cup \big((L_1+L_2,L) \times (0,H_2)\big)$. Notice that $\Omega_0 \subset \Omega \subset \widehat{\Omega}$ for all $\Omega \in \mathcal{O}$.

Clearly $\Omega(\alpha) \in C^{0,1}$ for all $\alpha \in \mathcal{U}_{ad}$, where $C^{0,1}$ is the system of bounded domains with Lipschitz continuous boundaries. We shall denote the parts of the boundary $\partial\Omega(\alpha)$ as follows (see Fig. 2):

$$\begin{aligned}
\Gamma_D &= \Big\{(x_1,x_2) \in \partial\Omega(\alpha); x_1 = 0 \text{ or } x_1 = L\Big\} \\
\Gamma_{out} &= \Big\{(x_1,x_2) \in \partial\Omega(\alpha); L_1 \leq x_1 \leq L_1 + L_2, x_2 = 0\Big\} \\
\Gamma_\alpha &= \Big\{(x_1,x_2) \in \partial\Omega(\alpha); L_1 \leq x_1 \leq L_1 + L_2, x_2 = \alpha(x_1)\Big\} \\
\Gamma_f &= \partial\Omega(\alpha) \setminus \big(\Gamma_D \cup \Gamma_{out} \cup \Gamma_\alpha\big).
\end{aligned}$$

The components Γ_D, Γ_{out} and Γ_f are fixed for every $\alpha \in \mathcal{U}_{ad}$.

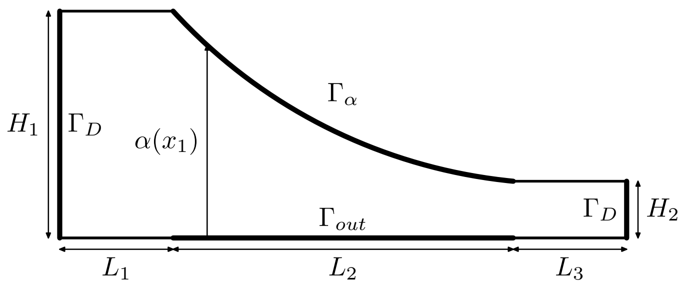

Figure 2. Geometry of $\Omega(\alpha)$ and parts of the boundary $\partial\Omega(\alpha)$

2.2. Classical formulation of the state problem

The fluid motion in $\Omega(\alpha)$ is described by the generalised Navier–Stokes system

$$\left.\begin{array}{rcl} -\mathrm{div}\ T(p, D(u)) + \rho \mathrm{div}(u \otimes u) & = & 0 \\ \mathrm{div}\ u & = & 0 \end{array}\right\} \text{in } \Omega(\alpha). \tag{9}$$

Here u means the velocity, p the pressure, ρ is the density of the fluid and the stress tensor T is defined by the following formulae:

$$T_{ij}(p, D(u)) = -p\delta_{ij} + 2\mu(|D(u)|)D_{ij}(u),\ i, j = 1, 2,$$

$$\mu(|D(u)|) := \mu_0 + \mu_t(|D(u)|) = \mu_0 + \rho l^2_{m,\alpha}|D(u)|,\ \mu_0 > 0,$$

where μ_0 is a constant laminar viscosity and $\mu_t(|D(u)|)$ stands for a turbulent viscosity. The function $l_{m,\alpha}$ represents a mixing length in the algebraic model of turbulence and it has the following form (see Hämäläinen, Mäkinen and Tarvainen, 2000, for more details):

$$l_{m,\alpha}(x) = \frac{1}{2}\alpha(x_1)\left[0.14 - 0.08\left(1 - \frac{2d_\alpha(x)}{\alpha(x_1)}\right)^2 - 0.06\left(1 - \frac{2d_\alpha(x)}{\alpha(x_1)}\right)^4\right],$$

where $d_\alpha(x) = \min\{x_2, \alpha(x_1) - x_2\}, x \in \Omega(\alpha)$.

The equations are completed by the following boundary conditions:

$$\begin{array}{rcll} u & = & 0 & \text{on } \Gamma_f \cup \Gamma_\alpha, \\ u & = & u_D & \text{on } \Gamma_D, \\ u \cdot \tau = u_1 & = & 0 & \text{on } \Gamma_{out}, \\ \sum_{i,j=1}^2 T_{ij}\nu_i\nu_j = T_{22} & = & -\sigma|u_2|u_2 & \text{on } \Gamma_{out}, \end{array} \tag{10}$$

where ν, τ stands for the unit normal, tangential vector, respectively and $\sigma > 0$ is a given suction coefficient. The condition $(10)_4$ originates from the homogenization of a complex geometry (for more details we refer to Hämäläinen, 1993).

By a classical solution we mean any velocity field $u \in (\mathcal{C}^2(\Omega(\alpha)))^2 \cap (\mathcal{C}^1(\overline{\Omega(\alpha)}))^2$ and a pressure $p \in \mathcal{C}^1(\Omega(\alpha)) \cap \mathcal{C}(\overline{\Omega(\alpha)})$ satisfying (9) and (10).

2.3. Weak formulation of the state problem

Throughout the paper we assume that there exists a function $u_0 \in \left(W^{1,3}(\Omega_0)\right)^2$, which satisfies the Dirichlet boundary conditions in the sense of traces, i.e.

$$u_0|_{\Gamma_D} = u_D, \quad u_0|_{\partial\Omega_0 \setminus (\Gamma_D \cup \Gamma_{out})} = 0, \quad u_0 \cdot \tau|_{\Gamma_{out}} = 0$$

and, in addition, div $u_0 = 0$ in Ω_0. We extend u_0 by zero on $\widehat{\Omega} \setminus \Omega_0$. Then, due to the boundary conditions, $u_0 \in \left(W^{1,3}(\widehat{\Omega})\right)^2$ and div $u_0 = 0$ in $\widehat{\Omega}$.

2.3.1. Function spaces

For any $\alpha \in \mathcal{U}_{ad}$ we introduce the following function spaces:

$$\begin{aligned}
\mathcal{V}(\alpha) &:= \Big\{\varphi \in (\mathcal{C}^\infty(\overline{\Omega(\alpha)}))^2;\ \text{div}\ \varphi = 0 \text{ in } \Omega(\alpha)\Big\},\\
\mathcal{V}_0(\alpha) &:= \Big\{\varphi = (\varphi_1, \varphi_2) \in \mathcal{V}(\alpha);\ \varphi_1 \in \mathcal{C}_0^\infty(\Omega(\alpha)),\\
&\qquad \text{dist}(\text{supp}(\varphi_2), \partial\Omega(\alpha) \setminus \Gamma_{out}) > 0\Big\},\\
W(\alpha) &:= \overline{\mathcal{V}(\alpha)}^{\|\cdot\|_\alpha},\\
W_0(\alpha) &:= \overline{\mathcal{V}_0(\alpha)}^{\|\cdot\|_\alpha},\\
W_{u_0}(\alpha) &:= \big\{v \in W(\alpha);\ v - u_0 \in W_0(\alpha)\big\},
\end{aligned}$$

where the norm $\|\cdot\|_\alpha$ is defined by

$$\|v\|_\alpha := \|v\|_{1,2,\Omega(\alpha)} + \|M_\alpha |D(v)|\|_{3,\Omega(\alpha)},$$

with

$$M_\alpha(x) := \Big(l_{m,\alpha}(x)\Big)^{2/3}, x \in \Omega(\alpha). \tag{11}$$

Here we use standard notations: the norm in $L^s(\Omega(\alpha))$, $W^{k,s}(\Omega(\alpha))$ will be denoted by $\|\ \|_{s,\Omega(\alpha)}$, $\|\ \|_{k,s,\Omega(\alpha)}$, respectively, in what follows. We shall also use the Einstein summation convention, i.e. $a_i b_i := \sum_{i=1}^n a_i b_i$.

LEMMA 2.1 *$W(\alpha)$ and $W_0(\alpha)$ are separable reflexive Banach spaces.*

DEFINITION 2.1 *Define the operator $A_\alpha : W(\alpha) \to \left(W(\alpha)\right)^*$ by the formula*

$$\big\langle A_\alpha(v), w\big\rangle_\alpha := \int_{\Omega(\alpha)} M_\alpha^3 |D(v)| D_{ij}(v) D_{ij}(w) dx;\ v, w \in W(\alpha).$$

Here $\langle\cdot,\cdot\rangle_\alpha$ denotes the duality pairing between $\left(W(\alpha)\right)^*$ and $W(\alpha)$.

REMARK 2.1 *Since $M_\alpha = 0$ on $\partial\Omega(\alpha) \setminus \Gamma_D$, it can be extended by zero on $\widehat{\Omega} \setminus \Omega(\alpha)$. The resulting function, which is continuous in $\overline{\widehat{\Omega}}$ and which will be used in the subsequent analysis, will be denoted by $\tilde{M}_\alpha$.*

LEMMA 2.2 *(Some properties of M_α and A_α, $\alpha \in \mathcal{U}_{ad}$)*

(i) If $\alpha_n \rightrightarrows \alpha$ in $[0, L]$ then $\tilde{M}_{\alpha_n} \rightrightarrows \tilde{M}_\alpha$ in $\overline{\widehat{\Omega}}$.

(ii) A_α is monotone in $W(\alpha)$:

$$\big\langle A_\alpha(v) - A_\alpha(w), v - w \big\rangle_\alpha \geq 0 \ \forall v, w \in W(\alpha),$$

and strictly monotone in $W_0(\alpha)$, i.e. the previous inequality is sharp for $v \neq w$, where $v, w \in W_0(\alpha)$.

(iii) A_α is continuous in $W(\alpha)$.

DEFINITION 2.2 *For every $u, v, w \in \big(W^{1,2}(\Omega(\alpha))\big)^2$ we define the trilinear form b_α:*

$$b_\alpha(u, v, w) := \int_{\Omega(\alpha)} u_j \frac{\partial v_i}{\partial x_j} w_i \ dx.$$

REMARK 2.2 *The same analysis can be done for any weight function $M_\alpha : \overline{\widehat{\Omega}} \mapsto \mathbb{R}$ satisfying the following conditions:*

(i) $\forall \alpha \in \mathcal{U}_{ad} \quad M_\alpha \in \mathcal{C}(\overline{\widehat{\Omega}})$;

(ii) $\forall \alpha \in \mathcal{U}_{ad}$ it holds that $M_\alpha|_{\Omega(\alpha)} > 0$;

(iii) $\forall \alpha_n, \alpha \in \mathcal{U}_{ad} \quad \alpha_n \rightrightarrows \alpha$ in $[0, L] \Rightarrow M_{\alpha_n} \rightrightarrows M_\alpha$ in $\overline{\widehat{\Omega}}$.

2.3.2. Definition of a weak solution

We are now ready to give the weak formulation of the state problem. It can be formally derived by multiplying the equations (9) by a smooth solenoidal test function φ and integrating over $\Omega(\alpha)$ with the use of the Green theorem.

DEFINITION 2.3 *A function $u := u(\alpha)$ is said to be a weak solution of the state problem $(\mathcal{P}(\alpha))$ iff*

- *$u \in W_{u_0}(\alpha)$,*
- *for every $\varphi \in W_0(\alpha)$ there holds:*

$$2\mu_0 \int_{\Omega(\alpha)} D_{ij}(u) D_{ij}(\varphi) \ dx + 2\rho \big\langle A_\alpha(u), \varphi \big\rangle_\alpha + \rho b_\alpha(u, u, \varphi) + \sigma \int_{\Gamma_{out}} |u_2| u_2 \varphi_2 \ dS = 0. \quad (12)$$

REMARK 2.3 *Since $\varphi = 0$ on $\partial\Omega(\alpha) \setminus \Gamma_{out}$ and div $\varphi = 0$ in $\Omega(\alpha)$, the pressure disappeared from the weak formulation.*

In the following subsections the existence of a weak solution to $(\mathcal{P}(\alpha))$ on a fixed domain $\Omega(\alpha)$, $\alpha \in \mathcal{U}_{ad}$ will be examined. Thus for simplicity of notations the letter α in the argument will be omitted (we shall write $\Omega := \Omega(\alpha)$, $W := W(\alpha)$, $b := b_\alpha$ etc. in what follows).

2.4. Energy estimates

Recall that the function u_0 is now defined in the whole $\widehat{\Omega}$ and it does not depend on $\alpha \in \mathcal{U}_{ad}$.

THEOREM 2.1 *Let*

$$\|\nabla u_0\|_{3,\widehat{\Omega}} < C \qquad \text{and} \qquad \sigma > \frac{\rho}{2}, \tag{13}$$

where $C > 0$ is specified in (18) *below. Then there exists a constant $C_E := C_E(\|\nabla u_0\|_{3,\widehat{\Omega}})$ such that for any weak solution u of $(\mathcal{P}(\alpha))$ the following estimate holds:*

$$\|\nabla u\|_{2,\Omega}^2 + \|M|D(u)|\|_{3,\Omega}^3 + \int_{\Gamma_{out}} |u_2|^3 \; dS \le C_E. \tag{14}$$

REMARK 2.4 *From the proof it will be seen that the estimate* (14) *holds with a constant C_E independent of $\alpha \in \mathcal{U}_{ad}$.*

Proof of Theorem 2.1. We use $\varphi := u - u_0$ as a test function in $(\mathcal{P}(\alpha))$ and estimate each term on the left of $(\mathcal{P}(\alpha))$ from below.

(i) The first term can be estimated by means of Hölder's, Young's and Korn's inequalities:

$$\int_\Omega D_{ij}(u) D_{ij}(u - u_0) dx \ge \frac{C_{Korn}^2}{2} \|\nabla u\|_{2,\Omega}^2 - \frac{1}{2}\|\nabla u_0\|_{2,\widehat{\Omega}}^2. \tag{15}$$

The Korn inequality is applied to the zero extension of u from Ω to $\widehat{\Omega}$ with a constant $C_{Korn} := C_{Korn}(\widehat{\Omega})$ which is independent of $\alpha \in \mathcal{U}_{ad}$.

(ii) The second term can be estimated by using Hölder's and Young's inequality:

$$\langle A(u), u - u_0 \rangle \ge \frac{1}{3}\|M|D(u)|\|_{3,\Omega}^3 - \frac{1}{3}\|M|D(u_0)|\|_{3,\widehat{\Omega}}^3. \tag{16}$$

(iii) The convective term can be rearranged as follows:

$$b(u, u, u - u_0) = \underbrace{b(u, u - u_0, u - u_0)}_{J_1} + \underbrace{b(u, u_0, u)}_{J_2} - \underbrace{b(u, u_0, u_0)}_{J_3}.$$

Since $u = u_0$ on $\partial\Omega \setminus \Gamma_{out}$ and div $u = 0$ in Ω, we have:

$$J_1 = \int_\Omega u_j \frac{\partial}{\partial x_j}\left(\frac{|u-u_0|^2}{2}\right)dx = \int_{\partial\Omega}(u\cdot\nu)\frac{|u-u_0|^2}{2}dS$$
$$-\int_\Omega \text{div } u \ \frac{|u-u_0|^2}{2}dx \geq -\frac{1+\eta_1}{2}\int_{\Gamma_{out}}|u_2|^3 \ dS - C_{\eta_1}\int_{\Gamma_{out}}|u_{02}|^3 \ dS$$

for any $\eta_1 > 0$ with $C_{\eta_1} > 0$ depending only on η_1.
The term J_2 can be estimated using the embedding $\tilde{W}^{1,2}(\widehat{\Omega}) \hookrightarrow L^3(\widehat{\Omega})$:

$$J_2 \geq -\|\nabla u_0\|_{3,\widehat{\Omega}}\|u\|^2_{3,\Omega} \geq -C^2_{Imb}\|\nabla u_0\|_{3,\widehat{\Omega}}\|\nabla u\|^2_{2,\Omega}, \tag{17}$$

where $\tilde{W}^{1,2}(\widehat{\Omega})$ is the subspace of functions from $W^{1,2}(\widehat{\Omega})$, which are equal to zero on the top of $\widehat{\Omega}$, i.e. on $\widehat{\Gamma} = (0, L)\times\{\alpha_{max}\}$ and C_{Imb} is the norm of this embedding.
Further

$$J_3 \geq -\|u\|_{3,\Omega}\|\nabla u_0\|_{3,\widehat{\Omega}}\|u_0\|_{3,\widehat{\Omega}} \geq -\eta_2\|\nabla u\|^2_{2,\Omega} - C_{\eta_2}\|\nabla u_0\|^4_{3,\widehat{\Omega}}$$

holds for any $\eta_2 > 0$ with $C_{\eta_2} > 0$ depending only on η_2, by making use of the Friedrichs inequality on $\tilde{W}^{1,2}(\widehat{\Omega})$ and the embedding of $\tilde{W}^{1,2}(\widehat{\Omega})$ into $L^3(\widehat{\Omega})$.

(iv) Finally the boundary term can be estimated as follows:

$$\int_{\Gamma_{out}}|u_2|u_2(u_2-u_{02}) \ dS \geq (1-\eta_3)\int_{\Gamma_{out}}|u_2|^3 \ dS - C_{\eta_3}\int_{\Gamma_{out}}|u_{02}|^3 \ dS$$

holds for any $\eta_3 > 0$ with $C_{\eta_3} > 0$ depending only on η_3.

Multiplying each term by the respective physical constant and summing them up we obtain that

$$\left(\mu_0 C^2_{Korn} - \rho C^2_{Imb}\|\nabla u_0\|_{3,\widehat{\Omega}} - \rho\eta_2\right)\|\nabla u\|^2_{2,\Omega} + \frac{2}{3}\rho\|M|D(u)|\|^3_{3,\Omega}$$
$$+\left((1-\eta_3)\sigma - \rho\frac{1+\eta_1}{2}\right)\int_{\Gamma_{out}}|u_2|^3 \ dS$$
$$\leq C_E\left(\|\nabla u_0\|_{2,\widehat{\Omega}}, \|M|D(u_0)|\|_{3,\widehat{\Omega}}, \int_{\Gamma_{out}}|u_{02}|^3 \ dS, \|\nabla u_0\|_{3,\widehat{\Omega}}\right)$$

holds for any $\eta_1, \eta_2, \eta_3 > 0$ with a constant C_E, which depends on the indicated arguments. Choosing

$$\|\nabla u_0\|_{3,\widehat{\Omega}} < \frac{\mu_0}{\rho}\left(\frac{C_{Korn}}{C_{Imb}}\right)^2, \ \frac{\rho}{2} < \sigma \tag{18}$$

we finally arrive at (14). Here we also used the fact that all arguments appearing in C_E can be estimated by $\|\nabla u_0\|_{3,\widehat{\Omega}}$. ■

Remark 2.5 *Let us comment on the assumptions of Theorem 2.1.*

(i) The condition $\sigma > \frac{\rho}{2}$ can be possibly satisfied by adjusting the outflow properties of the headbox.

(ii) Assume that there exists a constant $\overline{C} > 0$ such that

$$\forall \alpha \in \mathcal{U}_{ad} \quad \|M_\alpha^{-1}\|_{2,\Omega(\alpha)} \leq \overline{C}. \tag{19}$$

Then Theorem 2.1 holds for any $\|\nabla u_0\|_{3,\widehat{\Omega}}$ with a constant $C'_E > 0$ independent of α, provided that $\sigma > \frac{\rho}{2}$.

Proof of (ii). We proceed in the same way as in the proof of Theorem 2.1 with the following minor change concerning the term J_2. Using the embedding $W^{1,\frac{6}{5}}(\widehat{\Omega}) \hookrightarrow L^3(\widehat{\Omega})$ with the norm $C'_{Imb} > 0$ we have the following lower estimate:

$$J_2 \geq -C'^2_{Imb} \|\nabla u\|^2_{\frac{6}{5},\Omega} \|\nabla u_0\|_{3,\widehat{\Omega}}. \tag{20}$$

Korn's inequality on $\tilde{W}^{1,\frac{6}{5}}(\widehat{\Omega})$ with the constant $C'_{Korn} > 0$ and Hölder's inequality yield

$$\|\nabla u\|_{\frac{6}{5},\Omega} \leq C'^{-1}_{Korn} \|M^{-1} M D(u)\|_{\frac{6}{5},\Omega} \leq \frac{\overline{C}}{C'_{Korn}} \|M|D(u)|\|_{3,\Omega}. \tag{21}$$

Inserting (21) into (20) and using Young's inequality we then obtain for any $\eta > 0$:

$$J_2 \geq -\eta \|M|D(u)|\|^3_{3,\Omega} - C_\eta \|\nabla u_0\|^3_{3,\widehat{\Omega}}$$

with a constant $C_\eta > 0$ depending only on η. Finally, summing up all the terms multiplied by the respective constants, the expression

$$\left(\mu_0 C^2_{Korn} - \rho\eta_2\right) \|\nabla u\|^2_{2,\Omega} + 2\rho\left(\frac{1}{3} - \eta\right) \|M|D(u)|\|^3_{3,\Omega} + \left((1-\eta_3)\sigma - \rho\frac{1+\eta_1}{2}\right) \int_{\Gamma_{out}} |u_2|^3 \, dS$$

appears on the left. Choosing $\eta_1, \eta_2, \eta_3, \eta > 0$ small enough and $\sigma > \frac{\rho}{2}$ we obtain the result. ■

Remark 2.6 *A direct calculation shows that the function M_α defined in* (11) *does not satisfy condition* (19) *since $M_\alpha \approx x_2^{2/3}$ in vicinity of $\partial\Omega(\alpha) \setminus \overline{\Gamma_D}$. This condition will be satisfied if M_α decays as $x_2^{1/2-\epsilon}$ with $\epsilon > 0$ arbitrarily small.*

2.5. Existence and uniqueness

THEOREM 2.2 (Existence of a weak solution) *Let the assumptions of Theorem 2.1 be satisfied. Then there exists a weak solution of* $(\mathcal{P}(\alpha))$.

Proof. Will be done in two steps (for the sake of simplicity of notations we set $2\mu_0 = 2\rho = \sigma = 1$):

(i) Galerkin approximations

Let $\{\omega^s\}_{s=1}^{\infty}$ be a dense set of linearly independent functions in W_0 and denote by $K_N := \text{span}\{\omega^1, ..., \omega^N\}$ the finite-dimensional subspace of W_0. For every $N = 1, 2, ...$ we solve the Galerkin problem:
Find $u^N \in W$ such that $u^N - u_0 \in K_N$ and u^N satisfies equation (12) for all $\varphi \in K_N$.
Define a mapping $P_N : \mathbb{R}^N \to \mathbb{R}^N$ by

$$P_N(d^N)_s := \int_\Omega D_{ij}(u^N) D_{ij}(\omega^s) dx + \langle A(u^N), \omega^s \rangle + \\ + \frac{1}{2} b(u^N, u^N, \omega^s) + \int_{\Gamma_{out}} |u_2^N| u_2^N \omega_2^s \, dS; \; s = 1, ...N,$$

where $u^N(x) := u_0(x) + \sum_{r=1}^N d_r^N \omega^r(x)$. Then the Galerkin problem is equivalent to finding $\overline{d}^N \in \mathbb{R}^N$ such that

$$P_N(\overline{d}^N) = 0. \tag{22}$$

Next we show that this nonlinear algebraic system has a solution: Clearly the mapping P_N is continuous. Moreover, there exists $R > 0$ such that

$$\forall d^N \in \mathbb{R}^N, |d^N| = R : \; P_N(d^N) \cdot d^N > 0$$

as follows from the energy estimates and positive definiteness of the Gramm matrix. From Brouwer's theorem the existence of $\overline{d}^N \in \mathbb{R}^N$ solving (22) follows.

(ii) Limit passages

Energy estimate (14) holds for every u^N with the same constant:

$$C_E(\|\nabla u_0\|_{3,\widehat{\Omega}}).$$

From this it follows that there exists $u \in W$ such that (a chosen subsequence is denoted again by the same index N)

$$u^N \rightharpoonup u \text{ weakly in } W \text{ as } N \to \infty. \tag{23}$$

Clearly $u \in W_{u_0}$. Further

$$\begin{aligned} &u^N \to u \text{ strongly in } L^q(\Omega), \\ &u^N \to u \text{ strongly in } L^q(\partial\Omega) \text{ as } N \to \infty, \end{aligned} \tag{24}$$

as follows from the compact embedding of $W^{1,2}(\Omega)$ into $L^q(\Omega)$, and $L^q(\partial\Omega)$, $q \in [1, +\infty)$. Let $\varphi \in W_0$ be given. Then

$$\int_\Omega D_{ij}(u^N) D_{ij}(\varphi) dx \to \int_\Omega D_{ij}(u) D_{ij}(\varphi) dx,$$

$$\int_{\Gamma_{out}} |u_2^N| u_2^N \varphi_2 \, dS \to \int_{\Gamma_{out}} |u_2| u_2 \varphi_2 \, dS,$$

$$b(u^N, u^N, \varphi) = b(u^N - u, u^N, \varphi) + b(u, u^N, \varphi) \to b(u, u, \varphi), \ N \to \infty,$$

making use of (23) and (24).
It remains to analyze the second term in (12). We show that

$$A(u^N) \rightharpoonup A(u) \text{ in } W^*, \ N \to \infty.$$

From the energy estimates the boundedness of $\{A(u^N)\}$ in W^* follows:

$$\left| \left\langle A(u^N), \varphi \right\rangle \right| \leq C \|\varphi\|_\alpha,$$

where C does not depend on N and therefore $A(u^N) \rightharpoonup \chi$ in W^*. To prove that $\chi = A(u)$ we use the monotonicity of A:

$$\begin{aligned} \forall \psi \in W: \ 0 &\leq \left\langle A(u^N) - A(\psi), u^N - \psi \right\rangle \\ &= \left\langle A(u^N), u^N - u_0 \right\rangle - \left\langle A(\psi), u^N - \psi \right\rangle - \left\langle A(u^N), \psi - u_0 \right\rangle. \end{aligned} \tag{25}$$

Since $u^N - u_0 \in K_N$, the term $\left\langle A(u^N), u^N - u_0 \right\rangle$ can be expressed by the remaining terms of the Galerkin identity. Therefore (25) reads as follows:

$$\begin{aligned} &\int_\Omega D_{ij}(u^N) D_{ij}(u^N - u_0) dx \leq -\frac{1}{2} b(u^N, u^N, u^N - u_0) \\ &- \int_{\Gamma_{out}} |u_2^N| u_2^N (u_2^N - u_{02}) \, dS - \left\langle A(\psi), u^N - \psi \right\rangle - \left\langle A(u^N), \psi - u_0 \right\rangle. \end{aligned}$$

Letting $N \to \infty$ and using weak lower semi-continuity of the left hand side of the previous inequality and continuity of the remaining terms we obtain:

$$\begin{aligned} \int_\Omega D_{ij}(u) D_{ij}(u - u_0) dx &\leq -\frac{1}{2} b(u, u, u - u_0) \\ &- \int_{\Gamma_{out}} |u_2| u_2 (u_2 - u_{02}) \, dS - \left\langle A(\psi), u - \psi \right\rangle - \left\langle \chi, \psi - u_0 \right\rangle. \end{aligned} \tag{26}$$

Further, we use $u^L - u_0, \ L \leq N$ as a test function in the Galerkin identity for u^N. Passing then to the limit with $N \to \infty$ and then with $L \to \infty$ we have:

$$\int_\Omega D_{ij}(u)D_{ij}(u-u_0)dx + \big\langle \chi, u-u_0 \big\rangle \\ + \frac{1}{2}b(u,u,u-u_0) + \int_{\Gamma_{out}} |u_2|u_2(u_2-u_{02})\; dS = 0. \quad (27)$$

From (26) and (27) we arrive at the inequality

$$0 \le \big\langle \chi - A(\psi), u-\psi \big\rangle \quad \forall \psi \in W. \quad (28)$$

We now use the so-called Minty trick. Instead of ψ we insert a function $u \pm \lambda\xi$ into (28), where $\lambda > 0,\ \xi \in W$:

$$0 \le \big\langle \chi - A(u \pm \lambda\xi), \mp\lambda\xi \big\rangle.$$

Dividing this inequality by λ we obtain for $\lambda \to 0+$:

$$0 \le \pm\big\langle \chi - A(u), \xi \big\rangle \ \forall \xi \in W,$$

making use of radial continuity of A. Thus $\chi = A(u)$.
It remains to verify that u is a weak solution: Choose $\varphi \in W_0$. This function can be approximated by a sequence $\{\varphi^L\}$, $\varphi^L \in K_L$: $\varphi^L \to \varphi$ in W_0, $L \to \infty$. Inserting φ^L into the Galerkin identity for u^N, $N \ge L$ and letting $N \to \infty$ and next $L \to \infty$, we see that (12) is satisfied for every $\varphi \in W_0$. Therefore u is a weak solution. ∎

Theorem 2.3 (Uniqueness) *Let all the assumptions of Theorem 2.1 be satisfied and $\|\nabla u_0\|_{3,\widehat{\Omega}}$ be small enough. Then there exists a unique solution to $(\mathcal{P}(\alpha))$.*

Proof. Let u and v be two solutions of $(\mathcal{P}(\alpha))$. We subtract the weak formulations for u and v with $\varphi = u - v \in W_0$ as a test function. We obtain:

$$2\mu_0\|D(u-v)\|^2_{2,\Omega} + \underbrace{2\rho\big\langle A(u) - A(v), u-v \big\rangle}_{\ge\, 0} \\ +\sigma \int_{\Gamma_{out}} \underbrace{(|u_2|u_2 - |v_2|v_2)(u_2-v_2)}_{\ge\, 0}\; dS = \rho b(v-u, v, v-u) + \rho b(u, v-u, v-u).$$

The terms on the right hand side can be estimated making use of the Hölder inequality, the imbedding of $\tilde{W}^{1,2}(\widehat{\Omega})$ into $L^4(\widehat{\Omega})$ and the energy estimates:

$$b(v-u,v,v-u) \le \|\nabla v\|_{2,\Omega}\|u-v\|^2_{4,\Omega} \le C_E C^2_{Imb}\|\nabla(u-v)\|^2_{2,\Omega},$$

$$b(u,v-u,v-u) \le \|u\|_{4,\Omega}\|\nabla(u-v)\|_{2,\Omega}\|u-v\|_{4,\Omega} \le C_E C^2_{Imb}\|\nabla(u-v)\|^2_{2,\Omega},$$

where C_{Imb} is the norm of the respective embedding and $C_E := C_E\left(\|\nabla u_0\|_{3,\widehat{\Omega}}\right)$ is the constant from the energy estimates. Applying the Korn inequality on the left hand side, we finally obtain:

$$\mu_0 C_{Korn}^2 \|\nabla(u-v)\|_{2,\Omega}^2 \leq 2\rho C_E C_{Imb}^2 \|\nabla(u-v)\|_{2,\Omega}^2,$$

from which it follows that $u = v$ a.e. in Ω if $2C_E < \frac{\mu_0}{\rho}\left(\frac{C_{Korn}}{C_{Imb}}\right)^2$. ■

Remark 2.7 *Let us observe that the bound guaranteeing uniqueness of the solution to* $(\mathcal{P}(\alpha))$ *is also independent of* $\alpha \in \mathcal{U}_{ad}$.

3. Shape optimization problem

The aim of this part is to formulate a shape optimization problem and to prove the existence of its solution.

3.1. Formulation of the problem

We proved that, under certain assumptions, which do not depend on a particular choice of $\Omega(\alpha) \in \mathcal{O}$, there exists at least one weak solution of the state problem $(\mathcal{P}(\alpha))$. Let $\mathcal{G}$ be the graph of the control–to–state (generally multi-valued) mapping:

$$\mathcal{G} := \{(\alpha, u);\ \alpha \in \mathcal{U}_{ad},\ u \text{ is a weak solution of } (\mathcal{P}(\alpha))\}.$$

Further, let us define the cost functional $J : \mathcal{G} \to \mathbb{R}$ by

$$J : (\alpha, u) \mapsto \int_{\tilde{\Gamma}} |u_2 - z_D|^2 \, dS,\ u = (u_1, u_2), \tag{29}$$

where $z_D \in L^2(\tilde{\Gamma})$ is a given function representing the desired outlet velocity profile and $\tilde{\Gamma} \subset \Gamma_{out}$. This choice of J reflects the optimization goal formulated in Section 1

We now formulate the following problem:

$$\textit{Find } (\alpha^*, u^*) \in \mathcal{G} \textit{ such that } J(\alpha^*, u^*) \leq J(\alpha, u)\ \forall (\alpha, u) \in \mathcal{G}. \tag{$\mathbb{P}$}$$

Next we introduce convergence of a sequence of domains.

Definition 3.1 *Let* $\{\Omega(\alpha_n)\},\ \alpha_n \in \mathcal{U}_{ad}$ *be a sequence of domains. We say that* $\{\Omega(\alpha_n)\}$ *converges to* $\Omega(\alpha)$*, shortly* $\Omega(\alpha_n) \rightsquigarrow \Omega(\alpha)$*, iff* $\alpha_n \rightrightarrows \alpha$ *in* $[0, L]$.

As a direct consequence of the Arzelà–Ascoli theorem we have the following compactness result.

Lemma 3.1 *System* $\mathcal{O}$ *is compact with respect to convergence introduced in Definition* 3.1.

3.2. Existence of an optimal shape

First let us recall that the function u_0 which realizes the boundary conditions is the same for all domains $\Omega \in \mathcal{O}$. We now rewrite $(\mathcal{P}(\alpha))$, $\alpha \in \mathcal{U}_{ad}$ using the formulation on the fixed domain $\widehat{\Omega}$:

$$2\mu_0 \int_{\widehat{\Omega}} D_{ij}(\tilde{u}(\alpha)) D_{ij}(\tilde{\varphi})\, dx + 2\rho \big\langle \tilde{A}_\alpha(\tilde{u}(\alpha)), \tilde{\varphi} \big\rangle_{\widehat{\Omega}} + \rho b_{\widehat{\Omega}}(\tilde{u}(\alpha), \tilde{u}(\alpha), \tilde{\varphi}) + \sigma \int_{\Gamma_{out}} |\tilde{u}_2(\alpha)| \tilde{u}_2(\alpha) \tilde{\varphi}_2 \, dS = 0 \quad \forall \varphi \in W_0(\alpha), \qquad (\widehat{\mathcal{P}}(\alpha))$$

where the symbol $\tilde{}$ stands for the zero extension of functions from $\Omega(\alpha)$ on $\widehat{\Omega}$,

$$\big\langle \tilde{A}_\alpha(\tilde{u}(\alpha)), \tilde{\varphi} \big\rangle_{\widehat{\Omega}} := \int_{\widehat{\Omega}} \tilde{M}_\alpha^3 |D(\tilde{u}(\alpha))| D_{ij}(\tilde{u}(\alpha)) D_{ij}(\tilde{\varphi})\, dx,$$

$$b_{\widehat{\Omega}}(\tilde{u}(\alpha), \tilde{u}(\alpha), \tilde{\varphi}) := \int_{\widehat{\Omega}} \tilde{u}_j(\alpha) \frac{\partial \tilde{u}_i(\alpha)}{\partial x_j} \tilde{\varphi}_i \, dx.$$

Further let

$$\widehat{W}(\alpha) := \Big\{ v \in \big(W^{1,2}(\Omega(\alpha))\big)^2;\ \text{div } v = 0 \text{ in } \Omega(\alpha),\ M_\alpha |D(v)| \in L^3(\Omega(\alpha)) \Big\}$$

and define

$$\widehat{W}_{u_0}(\alpha) := \Big\{ v \in \widehat{W}(\alpha);\ v \text{ satisfies the Dirichlet conditions } (10)_1 - (10)_3 \text{ on } \partial\Omega(\alpha) \Big\}.$$

Remark 3.1 *It holds that $W_{u_0}(\alpha) \subseteq \widehat{W}_{u_0}(\alpha)$. The question arises whether these spaces are identical. This is in fact the density problem. For the moment we do not know the answer.*

Theorem 2.1 gives the following energy estimate:

$$\|\nabla \tilde{u}(\alpha)\|^2_{2,\widehat{\Omega}} + \|\tilde{M}_\alpha |D(\tilde{u}(\alpha))|\|^3_{3,\widehat{\Omega}} + \int_{\Gamma_{out}} |u_2(\alpha)|^3 \, dS \le C_E(\|\nabla u_0\|_{3,\widehat{\Omega}}) \qquad (30)$$

for every $(\alpha, u(\alpha)) \in \mathcal{G}$ with the constant $C_E(\|\nabla u_0\|_{3,\widehat{\Omega}})$ independent of α provided that (13) is satisfied.

Theorem 3.1 *Let $\alpha_n \rightrightarrows \alpha$ in $[0, L]$, $\alpha_n, \alpha \in \mathcal{U}_{ad}$ and $u_n := u(\alpha_n)$ be a solution of $(\mathcal{P}(\alpha_n))$. Then there exists $\widehat{u} \in \big(W^{1,2}(\widehat{\Omega})\big)^2$ and a subsequence of $\{\tilde{u}_n\}$ (denoted by the same symbol) such that*

$$\begin{aligned} &\tilde{u}_n \rightharpoonup \widehat{u} \text{ in } \big(W^{1,2}(\widehat{\Omega})\big)^2 \\ &\tilde{M}_{\alpha_n} D(\tilde{u}_n) \rightharpoonup \tilde{M}_\alpha D(\widehat{u}) \text{ in } \big(L^3(\widehat{\Omega})\big)^{2\times 2},\ n \to \infty. \end{aligned} \qquad (31)$$

In addition, the function $u(\alpha) := \widehat{u}|_{\Omega(\alpha)}$ solves $(\mathcal{P}(\alpha))$ provided that $u(\alpha) \in W_{u_0}(\alpha)$.

Proof. Let us denote $\tilde{M}_n := \tilde{M}_{\alpha_n}$, $\Omega_n := \Omega(\alpha_n)$, $\langle\cdot,\cdot\rangle_n := \langle\cdot,\cdot\rangle_{\alpha_n}$ etc.

From energy estimate (30) it follows that

$$\|\tilde{u}_n\|_{1,2,\widehat{\Omega}} \le C,\ \|\tilde{M}_n D(\tilde{u}_n)\|_{3,\widehat{\Omega}} \le C, \tag{32}$$

where $C > 0$ does not depend on n. Therefore we can pass to a subsequence of $\{\tilde{u}_n\}$ (denoted again by the same symbol) so that

$$\begin{aligned} &\tilde{u}_n \rightharpoonup \widehat{u} && \text{in } \big(W^{1,2}(\widehat{\Omega})\big)^2,\\ &\tilde{M}_n D(\tilde{u}_n) \rightharpoonup \widehat{z} && \text{in } \big(L^3(\widehat{\Omega})\big)^{2\times 2},\ n\to\infty. \end{aligned} \tag{33}$$

The following properties of $\widehat{u}$ and $\widehat{z}$ are easily verified:
(i) $\widehat{u} = 0$ in $\widehat{\Omega}\setminus\overline{\Omega}(\alpha)$; $\widehat{z} = 0$ in $\widehat{\Omega}\setminus\overline{\Omega}(\alpha)$;
(ii) $\widehat{z} = \tilde{M}_\alpha D(\widehat{u})$ in $\widehat{\Omega}$;
(iii) div $\widehat{u} = 0$ in $\widehat{\Omega}$;
(iv) $\widehat{u}$ satisfies the required Dirichlet boundary conditions on $\partial\Omega(\alpha)$.

We prove (ii). Since $\mathcal{C}^\infty(\overline{\widehat{\Omega}})$ is dense in $L^{\frac{3}{2}}(\widehat{\Omega})$, it is sufficient to show that

$$\int_{\widehat{\Omega}} \tilde{M}_n D_{ij}(\tilde{u}_n)\psi_{ij}\ dx \to \int_{\widehat{\Omega}} \tilde{M}_\alpha D_{ij}(\widehat{u})\psi_{ij}\ dx,\ n\to\infty$$

holds for every $\psi \in \big(\mathcal{C}^\infty(\overline{\widehat{\Omega}})\big)^{2\times 2}$. Indeed:

$$\begin{aligned} &\left|\int_{\widehat{\Omega}} \Big(\tilde{M}_n D_{ij}(\tilde{u}_n)\psi_{ij} - \tilde{M}_\alpha D_{ij}(\widehat{u})\psi_{ij}\Big)\ dx\right| \le \\ &\le \int_{\widehat{\Omega}} |\tilde{M}_n - \tilde{M}_\alpha||D_{ij}(\tilde{u}_n)\psi_{ij}|\ dx + \left|\int_{\widehat{\Omega}} \tilde{M}_\alpha\left(D_{ij}(\tilde{u}_n) - D_{ij}(\widehat{u})\right)\psi_{ij}\ dx\right| \to 0, \end{aligned}$$

making use of the fact that $\tilde{M}_n \rightrightarrows \tilde{M}_\alpha$ in $\widehat{\Omega}$, $(33)_1$ and that $\tilde{M}_\alpha\psi_{ij} \in L^2(\widehat{\Omega})$.

Let $u(\alpha) := \widehat{u}|_{\Omega(\alpha)}$. Then (i)-(iv) imply that $u(\alpha) \in \widehat{W}_{u_0}(\alpha)$. Next we prove that $u(\alpha)$ solves $(\mathcal{P}(\alpha))$ if $u(\alpha) \in W_{u_0}(\alpha)$. We start from the definition of $(\mathcal{P}(\alpha_n))$:

$$\begin{aligned} 2\mu_0 \int_{\widehat{\Omega}} D_{ij}(\tilde{u}_n)D_{ij}(\tilde{\varphi})\ dx + 2\rho\langle \tilde{A}_n(\tilde{u}_n), \tilde{\varphi}\rangle_{\widehat{\Omega}} + \rho b_{\widehat{\Omega}}(\tilde{u}_n, \tilde{u}_n, \tilde{\varphi})&\\ + \sigma\int_{\Gamma_{out}} |\tilde{u}_{n2}|\tilde{u}_{n2}\tilde{\varphi}\ dS = 0 \quad \forall\varphi \in W_0(\alpha_n).& \end{aligned} \tag{34}$$

Let $\varphi \in \mathcal{V}_0(\alpha)$ be an arbitrary function. Then $\tilde{\varphi}|_{\Omega_n} \in \mathcal{V}_0(\alpha_n)$ for n sufficiently large so that it can be used as a test function in (34). The limit passage in the first, third and fourth term in (34) is a classical one:

$$\begin{aligned}&\int_{\widehat{\Omega}} D_{ij}(\tilde{u}_n)D_{ij}(\tilde{\varphi})\, dx \to \int_{\widehat{\Omega}} D_{ij}(\tilde{u}(\alpha))D_{ij}(\tilde{\varphi})\, dx,\\ &\int_{\Gamma_{out}} |\tilde{u}_{n2}|\tilde{u}_{n2}\tilde{\varphi}_2\, dS \to \int_{\Gamma_{out}} |\tilde{u}_2(\alpha)|\tilde{u}_2(\alpha)\tilde{\varphi}_2\, dS,\\ &b_{\widehat{\Omega}}(\tilde{u}_n,\tilde{u}_n,\tilde{\varphi}) \to b_{\widehat{\Omega}}(\tilde{u}(\alpha),\tilde{u}(\alpha),\tilde{\varphi}),\ n\to\infty.\end{aligned} \tag{35}$$

The most difficult is to handle the second term. Let $B_n \in \big(W_0(\alpha)\big)^*$ be the functional defined by

$$\begin{aligned}2\rho\big\langle B_n,\psi\big\rangle_\alpha := -2\mu_0\int_{\widehat{\Omega}} D_{ij}(\tilde{u}_n)D_{ij}(\tilde{\psi})\, dx - \rho b_{\widehat{\Omega}}(\tilde{u}_n,\tilde{u}_n,\tilde{\psi})&\\ - \sigma\int_{\Gamma_{out}} |\tilde{u}_{n2}|\tilde{u}_{n2}\tilde{\psi}\, dS \quad \forall\psi\in W_0(\alpha).&\end{aligned}$$

From the energy estimate (30) it follows that $\|B_n\|_{\left(W_0(\alpha)\right)^*} \le C$ for all $n\in\mathbb{N}$. Thus, there exists $B\in\big(W_0(\alpha)\big)^*$ such that

$$B_n \rightharpoonup B, n\to\infty. \tag{36}$$

In addition, if $\psi\in\mathcal{V}_0(\alpha)$ then $\tilde{\psi}|_{\Omega_n}\in\mathcal{V}_0(\alpha_n)$ for n large enough and

$$\big\langle B_n,\psi\big\rangle_\alpha = \big\langle \tilde{A}_n(\tilde{u}_n),\tilde{\psi}\big\rangle_{\widehat{\Omega}}. \tag{37}$$

Due to monotonicity of A_n on $W(\alpha_n)$ we have for any $\psi\in W(\alpha_n)$:

$$\begin{aligned}0\le \big\langle A_n(u_n)-A_n(\psi),u_n-\psi\big\rangle_n = \big\langle A_n(u_n),u_n-u_0\big\rangle_n -&\\ -\big\langle A_n(\psi),u_n-\psi\big\rangle_n - \big\langle A_n(u_n),\psi-u_0\big\rangle_n.&\end{aligned} \tag{38}$$

In what follows we use ψ of the form $\tilde{\psi} = u_0+\tilde{\varphi}$, where $\varphi\in\mathcal{V}_0(\alpha)$ is fixed. Then

$$\big\langle A_n(u_n),\psi-u_0\big\rangle_n = \big\langle \tilde{A}_n(\tilde{u}_n),\tilde{\varphi}\big\rangle_{\widehat{\Omega}} = \big\langle B_n,\varphi\big\rangle_\alpha \tag{39}$$

for n large enough making use of (37). Since $u_n\in W_{u_0}(\alpha_n)$, the definition of $(\mathcal{P}(\alpha_n))$, (38) and (39) yield:

$$\begin{aligned}2\mu_0\int_{\widehat{\Omega}} D_{ij}(\tilde{u}_n)D_{ij}(\tilde{u}_n-u_0)\, dx \le -\rho b_{\widehat{\Omega}}(\tilde{u}_n,\tilde{u}_n,\tilde{u}_n-u_0)&\\ -\sigma\int_{\Gamma_{out}} |\tilde{u}_{n2}|\tilde{u}_{n2}(\tilde{u}_{n2}-u_{02})\, dS - \big\langle A_n(\psi),u_n-\psi\big\rangle_n - \big\langle B_n,\varphi\big\rangle_\alpha.&\end{aligned} \tag{40}$$

Letting $n \to \infty$ in (40) we obtain:

$$2\mu_0 \int_{\widehat{\Omega}} D_{ij}(\tilde{u}(\alpha)) D_{ij}(\tilde{u}(\alpha) - u_0)\ dx \leq -\rho b_{\widehat{\Omega}}(\tilde{u}(\alpha), \tilde{u}(\alpha), \tilde{u}(\alpha) - u_0)$$
$$- \sigma \int_{\Gamma_{out}} |\tilde{u}_2(\alpha)| \tilde{u}_2(\alpha)(\tilde{u}_2(\alpha) - u_{02})\ dS - \big\langle A_\alpha(\psi), u(\alpha) - \psi \big\rangle_\alpha - \big\langle B, \varphi \big\rangle_\alpha \tag{41}$$

using that

$$\big\langle A_n(\psi), u_n \big\rangle_n \to \big\langle A_\alpha(\psi), u(\alpha) \big\rangle_\alpha = \big\langle \tilde{A}_\alpha(\tilde{\psi}), \tilde{u}(\alpha) \big\rangle_{\widehat{\Omega}}, n \to \infty. \tag{42}$$

Indeed: from $(33)_2$ and (ii) we know that

$$\tilde{M}_n D(\tilde{u}_n) \rightharpoonup \tilde{M}_\alpha D(\tilde{u}(\alpha)) \text{ in } \big(L^3(\widehat{\Omega})\big)^{2\times 2} \tag{43}$$

as $\widehat{u} = \tilde{u}(\alpha)$. Further

$$\tilde{M}_n^2 |D(\tilde{\psi})| D(\tilde{\psi}) \to \tilde{M}_\alpha^2 |D(\tilde{\psi})| D(\tilde{\psi}) \text{ in } \big(L^{\frac{3}{2}}(\widehat{\Omega})\big)^{2\times 2}, n \to \infty$$

since $M_n \rightrightarrows M_\alpha$ in $\widehat{\Omega}$ and $\tilde{\psi} \in \Big(W^{1,3}(\widehat{\Omega})\Big)^2$. From this and (43) we obtain (42). By assumption there exists $w(\alpha) \in W_0(\alpha)$ such that $u(\alpha) = u_0 + w(\alpha)$. From the definition of $W_0(\alpha)$ it follows that one can find a sequence $\{w_k\}$, $w_k \in \mathcal{V}_0(\alpha)$ such that

$$w_k \to w(\alpha) \text{ in } W(\alpha),\ k \to \infty. \tag{44}$$

Let $k \in \mathbb{N}$ be fixed. Then $\tilde{w}_k|_{\Omega_n} \in \mathcal{V}_0(\alpha_n)$ for n large enough. Therefore $\tilde{w}_k|_{\Omega_n}$ can be used as a test function in $(\mathcal{P}(\alpha_n))$. Inserting $\tilde{w}_k$ into $(\mathcal{P}(\alpha_n))$ and passing to the limit with $n \to \infty$ and then $k \to \infty$ we obtain:

$$2\mu_0 \int_{\widehat{\Omega}} D_{ij}(\tilde{u}(\alpha)) D_{ij}(\tilde{w}(\alpha))\ dx + \big\langle B, w(\alpha) \big\rangle_\alpha + \rho b_{\widehat{\Omega}}(\tilde{u}(\alpha), \tilde{u}(\alpha), \tilde{w}(\alpha))$$
$$+ \sigma \int_{\Gamma_{out}} |\tilde{u}_2(\alpha)| \tilde{u}_2(\alpha) \tilde{w}_2(\alpha)\ dS = 0 \tag{45}$$

making use of (35), (36) and (44). From (41) and (45) we have:

$$-\big\langle \tilde{A}_\alpha(\tilde{\psi}), \tilde{u}(\alpha) - \tilde{\psi} \big\rangle_{\widehat{\Omega}} - \big\langle B, \varphi \big\rangle_\alpha + \big\langle B, w(\alpha) \big\rangle_\alpha \geq 0 \tag{46}$$

using that $\tilde{w}(\alpha) = \tilde{u}(\alpha) - u_0$. Since $\tilde{u}(\alpha) - \tilde{\psi} = \tilde{w}(\alpha) - \tilde{\varphi}$ we see that (46) can be written as follows:

$$\big\langle B - A_\alpha(\psi), w(\alpha) - \varphi \big\rangle_\alpha \geq 0 \quad \forall \varphi \in \mathcal{V}_0(\alpha). \tag{47}$$

From (47), density of $\mathcal{V}_0(\alpha)$ in $W_0(\alpha)$, continuity of A_α and the fact that $\psi = u_0|_{\Omega(\alpha)} + \varphi$, $\varphi \in \mathcal{V}_0(\alpha)$, we obtain:

$$\left\langle B - A_\alpha(u_0 + z), w(\alpha) - z \right\rangle_\alpha \geq 0 \quad \forall z \in W_0(\alpha). \tag{48}$$

Let $z \in W_0(\alpha)$ be of the form $z = w(\alpha) \pm \lambda\theta$, $\lambda > 0$, where $\theta \in W_0(\alpha)$ is arbitrary. Then

$$\left\langle B - A_\alpha(u_0 + w(\alpha) + \lambda\theta), \pm\lambda\theta \right\rangle_\alpha \geq 0.$$

Dividing this inequality by λ and passing to the limit $\lambda \to 0+$ we finally obtain

$$B = A_\alpha(u_0 + w(\alpha)) = A_\alpha(u(\alpha)). \tag{49}$$

This, together with $(35)_1$-$(35)_3$ gives

$$2\mu_0 \int_{\widehat{\Omega}} D_{ij}(\tilde{u}(\alpha)) D_{ij}(\tilde{\varphi}) \, dx + 2\rho \langle \tilde{A}_\alpha(\tilde{u}(\alpha)), \tilde{\varphi} \rangle_{\widehat{\Omega}} \\ + \rho b_{\widehat{\Omega}}(\tilde{u}(\alpha), \tilde{u}(\alpha), \tilde{\varphi}) + \sigma \int_{\Gamma_{out}} |\tilde{u}_2(\alpha)| \tilde{u}_2(\alpha) \tilde{\varphi}_2 \, dS = 0, \tag{50}$$

for every $\varphi \in \mathcal{V}_0(\alpha)$ and consequently also for $\varphi \in W_0(\alpha)$. ■

Remark 3.2 *Under the assumptions which guarantee uniqueness of the solution to $(\mathcal{P}(\alpha))$ the whole sequence $\{\tilde{u}_n\}$ tends to $\tilde{u}(\alpha)$ in the sense of Theorem 3.1.*

Remark 3.3 *If $W_{u_0}(\alpha) = \widehat{W}_{u_0}(\alpha)$, the assumption $u(\alpha) \in W_{u_0}(\alpha)$ is automatically satisfied.*

Theorem 3.2 (Existence of an optimal shape) *Let there exist a minimizing sequence $\{(\alpha_n, u_n)\}$, $(\alpha_n, u_n) \in \mathcal{G}$, of $(\mathbb{P})$ with an accumulation point (α^*, u^*) such that $u^*|_{\Omega(\alpha^*)} \in W_{u_0}(\alpha^*)$. Then $(\alpha^*, u^*|_{\Omega(\alpha^*)})$ is an optimal pair for $(\mathbb{P})$.*

Proof. Without loss of generality we may assume that $\alpha_n \rightrightarrows \alpha^* \in \mathcal{U}_{ad}$ in $[0, L]$. From the assumptions on the sequence $\{(\alpha_n, u_n)\}$ it follows that there exists its accumulation point (α^*, u^*) such that $(\alpha^*, u^*|_{\Omega(\alpha^*)}) \in \mathcal{G}$. Further

$$q = \inf_{(\alpha, u(\alpha)) \in \mathcal{G}} J(\alpha, u(\alpha)) = \lim_{n \to \infty} J(\alpha_n, u_n) = J(\alpha^*, u^*|_{\Omega(\alpha^*)}) \geq q$$

making use of continuity of J. ■

4. Conclusion

The paper consists of two parts. The first one deals with the existence proof for the generalised steady-state Navier–Stokes system. In the second part the shape optimization problem with the Navier–Stokes system as a state constraint is studied.

Due to an algebraic turbulence model the weak formulation of the state problem involves the weighted Sobolev spaces. The existence and uniqueness of a solution is proved for small data and with a constraint imposed on the model parameters by using energy estimates, the monotone operator theory and the Galerkin method. The analysis of the state problem shares many similarities with the techniques presented in Ladyzhenskaya (1968, 1969), Lions (1969) and Parés (1992).

The proof of the continuous dependence of solutions on boundary variations is the key result in the shape optimization part. This property is proved under an additional assumption, namely that a limit function of a minimizing sequence belongs to an appropriate space meaning that the existence of an optimal shape is conditional. The paper suggests, however, a way of getting an unconditional type of result.

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