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Reachability and minimum energy control of positive 2D systems with delays

by

Tadeusz Kaczorek

Warsaw University of Technology
Institute of Control and Industrial Electronics,
02-662 Warsaw, Koszykowa 75, Poland
e-mail: kaczorek@isep.pw.edu.pl

Abstract: The notion of internally positive 2D model (system) with delays is introduced. Solution of the 2D linear models with delays is derived. Necessary and sufficient conditions for the internal positivity and reachability of the models are established. The Cayley-Hamilton theorem for the 2D linear models with delays is extended. The minimum energy control for the internally positive 2D models with delays is formulated and solved.

Keywords: 2D positive system, delay, reachability, solution, minimum energy control.

1. Introduction

The most popular models of two-dimensional (2D) linear systems are the models introduced by Roesser (1975), Fornasini-Marchesini (1976, 1978) and Kurek (1985). The models have been extended for positive systems in Kaczorek (1996, 2002), Valcher (1997), Xie and Wang (2003). The overviews of some recent results in positive systems have been given in the monographs of Farina and Rinaldi (2000), Kaczorek (2002), and in papers by Kaczorek (2003), Xie and Wang (2003). The upper bound for for the reachability index of the positive 2D general model has been considered in Kaczorek (2004). The reachability and minimum energy control of positive discrete-time systems with one delay have been analyzed in Kaczorek and Busłowicz (2004).

In this paper the notion of internally positive 2D model with delays will be introduced and necessary and sufficient conditions for the interval positivity and reachability will be established. The minimum energy control for this class of 2D systems will be formulated and solved. To the best knowledge of the author these problems for positive 2D systems with delays have not been considered yet.

2. Preliminaries

Consider the 2D general model (without delays), Kurek (1985), Kaczorek (2002),

$$
\begin{aligned}
&x_{i+1,j+1} = A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1} \\
&y_{i,j} = C x_{ij} + D u_{ij} \quad i,j \in \mathbb{Z}_+ \quad \text{(the set of nonnegative integers)}
\end{aligned} \tag{1}
$$

where $x_{ij} \in \mathbb{R}^n$, $u_{ij} \in \mathbb{R}^m$, $yij \in \mathbb{R}^p$ are the state, input and output vectors and $A_k \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times m}$, $k = 0,1,2$, $C_k \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$.
The boundary conditions for (1) are given by

$$
x_{i0},\ i \in \mathbb{Z}_+ \ \text{ and } \ x_{0j},\ j \in \mathbb{Z}_+ \tag{2}
$$

where x_{i0} and x_{0j} are known.
Let $\mathbb{R}_+^{m\times n}$ be a set of $m \times n$ real matrices with nonnegative entries and $\mathbb{R}_+^m = \mathbb{R}_+^{m\times 1}$.

Definition 2.1 *(Kaczorek, 2002) The model (system) (1) is called internally positive (shortly positive) if for any boundary conditions $x_{i0} \in \mathbb{R}_+^n$, $x_{0j} \in \mathbb{R}_+^n$, $i,j \in \mathbb{Z}_+$ and every input sequence $u_{ij} \in \mathbb{R}_+^m$, $i,j \in \mathbb{Z}_+$ we have $x_{ij} \in \mathbb{R}_+^n$ and $y_{ij} \in \mathbb{R}_+^p$ for all $i,j \in \mathbb{Z}_+$*

Theorem 2.1 *(Kaczorek, 2002) The model (1) is positive if and only if*

$$
A_k \in \mathbb{R}^{n\times n}, \quad B_k \in \mathbb{R}^{n\times m}, \quad \textit{for } k = 0,1,2 \quad \textit{and } C \in \mathbb{R}_+^{p\times n}, \quad D \in \mathbb{R}_+^{p\times m}. \tag{3}
$$

Consider the 2D general model with delays

$$
\begin{aligned}
\bar{x}_{i+1,j+1} &= A_{00}\bar{x}_{ij} + A_{10}\bar{x}_{i+1,j} + A_{20}\bar{x}_{i,j+1} + \\
&+ \sum_{k=1}^{h} (A_{0k}\bar{x}_{i-d_{ik},j-d_{jk}} + A_{1k}\bar{x}_{i-d_{ik}+1,j-d_{jk}} + A_{2k}\bar{x}_{i-d_{ik},j-d_{jk}+1}) + \\
&+ B_{00}u_{ij} + B_{10}u_{i+1,j} + B_{20}u_{i,j+1} \\
y_{ij} &= C_0\bar{x}_{ij} + D_0 u_{ij}, \quad i,j \in \mathbb{Z}_+
\end{aligned} \tag{4}
$$

where $\bar{x}_{ij} \in \mathbb{R}^{\bar{n}}$, $\bar{u}_{ij} \in \mathbb{R}^{\bar{m}}$, $\bar{y}_{ij} \in \mathbb{R}^{\bar{p}}$ are the state, input and output vectors, $A_{0k}, A_{1k}, A_{2k} \in \mathbb{R}^{\bar{n}\times\bar{n}}$, $k = 0,1,\dots,h$, $B_{00}, B_{10}, B_{20} \in \mathbb{R}^{\bar{n}\times\bar{m}}$, $C_0 \in \mathbb{R}^{\bar{p}\times\bar{n}}$, $D_0 \in \mathbb{R}^{\bar{p}\times\bar{m}}$ and d_{ik}, d_{jk}, $k = 1,\dots,h$ are delays (nonnegative integer).

To simplify the notation we shall consider the model with only one delay ($d_{i1} = d_1 \geq 1$, $d_{j1} = d_2 \geq 1$) and $B_0 = B_{00}$, $B_{10} = B_{20} = 0$ of the form

$$
\begin{aligned}
\bar{x}_{i+1,j+1} &= A_{00}\bar{x}_{ij} + A_{10}\bar{x}_{i+1,j} + A_{20}\bar{x}_{i,j+1} + A_{01}\bar{x}_{i-d_1+1,j-d_2} + \\
&+ A_{11}\bar{x}_{i-d_1+1,j-d_2} + A_{21}\bar{x}_{i-d_1,j-d_2+1} + B_0 u_{ij} \\
y_{ij} &= C_0\bar{x}_{ij} + D_0 u_{ij}, \quad i,j \in \mathbb{Z}_+ .
\end{aligned} \tag{5}
$$

We shall establish the necessary and sufficient condtions under which the model (5) is positive and reachable. The Cayley-Hamilton theorem will be extended for the models with delays. The minimum energy control problem will be formulated and solved.

3. Positivity of the model with delays

By defining

$$x_{ij} = \begin{bmatrix} \bar{x}_{ij} \\ \bar{x}_{i-1,j} \\ \bar{x}_{i,j-1} \\ \bar{x}_{i-1,j-1} \\ \cdots\cdots\cdots \\ \bar{x}_{i-d_1+1,j-d_2} \\ \bar{x}_{i-d_1,j-d_2} \end{bmatrix} \in \mathbb{R}^n,\ A_0 = \begin{bmatrix} A_{00} & 0 & \dots & 0 & A_{01} \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ I & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times n},\ B = \begin{bmatrix} B_0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} A_{10} & 0 & \dots & 0 & A_{11} \\ 0 & 0 & \dots & 0 & 0 \\ I & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad A_2 = \begin{bmatrix} A_{20} & 0 & \dots & 0 & A_{21} \\ I & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 \end{bmatrix} \in \mathbb{R}^{n\times n},$$

$$C = \begin{bmatrix} C_0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{p\times n}, \quad D = D_0 \in \mathbb{R}^{p\times m} \tag{6}$$

we can write equation (5) in the form (1).

Applying Theorem 2.1 to the obtained equivalent model yields the following theorem.

Theorem 3.1 *The model (5) is positive if and only if*

$$A_{ki} \in \mathbb{R}_+^{n\times n},\ k = 0,1,2,\ i = 0,1 \quad B_0 \in \mathbb{R}_+^{n\times m},\ C_0 \in \mathbb{R}_+^{p\times n},\ \text{and}\ \ D_0 \in \mathbb{R}_+^{p\times m}. \tag{7}$$

In a similar way we may establish that the general model with delays (1) is positive if and only if all its matrices have nonnegative entries.

4. Solution of the model with delay

The solution to the equation

$$\begin{aligned} x_{i+1,j+1} = A_{00}x_{ij} + A_{10}x_{i+1,j} + A_{20}x_{i,j+1} + A_{01}x_{i-d_1,j-d_2} + \\ + A_{11}x_{i-d_1+1,j-d_2} + A_{21}x_{i-d_1,j-d_2+1} + B_0u_{ij} \end{aligned} \tag{8}$$

with boundary conditions

$$x_{i0},\ x_{0j},\ i,j \in \mathbb{Z}_+ \ \text{ and } \ k = -1,\dots,-d_1;\ l = -1,\dots,-d_2 \tag{9}$$

and given input u_{ij}, $i,j \in \mathbb{Z}_+$ will be derived by the use of the 2D $\mathcal{Z}$ transform.

Taking into account that

$$\begin{aligned}
\mathcal{Z}[x_{i+1,j+1}] &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{i+1,j+1} z_1^{-i} z_2^{-j} = z_1 z_2 \left[X(z_1,z_2) - X(z_1) - X(z_2) + x_{00}\right] \\
\mathcal{Z}[x_{i-d_1,j-d_2}] &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{i-d_1,j-d_2} z_1^{-i} z_2^{-j} = \\
&= z_1^{-d_1} z_2^{-d_2} \left[X(z_1,z_2) + \sum_{k=-1}^{-d_1}\sum_{l=-1}^{-d_2} z_1^{-k} z_2^{-l}\right] \\
\mathcal{Z}[x_{i+1,j}] &= z_1\left[X(z_1,z_2) - X(z_2)\right], \qquad \mathcal{Z}[x_{i,j+1}] = z_2\left[X(z_1,z_2) - X(z_1)\right]
\end{aligned}$$

where

$$X(z_1,z_2) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{ij} z_1^{-i} z_2^{-j}, \quad X(z_1) = \sum_{i=0}^{\infty} x_{i0} z_1^{-i}, \quad X(z_2) = \sum_{j=0}^{\infty} x_{0j} z_2^{-j}$$

we may write the equation (8) as follows

$$M(z_1,z_2)X(z_1,z_2) = N(z_1,z_2) + B_0 U(z_1,z_2) \tag{10}$$

where

$$\begin{aligned}
M(z_1,z_2) &= [I z_1 z_2 - A_{00} - A_{10} z_1 - A_{20} z_2 - A_{01} z_1^{-d_1} z_2^{-d_2} + \\
&\qquad - A_{11} z_1^{1-d_1} z_2^{-d_2} - A_{21} z_1^{-d_1} z_2^{1-d_2}] \\
N(z_1,z_2) &= z_1 z_2 \left(X(z_1) + X(z_2) - x_{00}\right) - A_{10} z_1 X(z_2) - A_{20} z_2 X(z_1) + \\
&\quad + z_1^{-d_1} z_2^{-d_2} \Bigg[\sum_{k=-d_1}^{-1}\sum_{l=-d_2}^{\infty} A_{01} x_{kl} z_1^{-k} z_2^{-l} + \sum_{k=0}^{\infty}\sum_{l=-d_2}^{-1} A_{01} x_{kl} z_1^{-k} z_2^{-l} + \\
&\quad + \sum_{k=1-d_1}^{-1}\sum_{l=-d_2}^{\infty} A_{11} x_{kl} z_1^{1-k} z_2^{-l} + \sum_{k=0}^{\infty}\sum_{l=-d_2}^{-1} A_{11} x_{kl} z_1^{1-k} z_2^{-l} + \\
&\quad + \sum_{k=-d_1}^{-1}\sum_{l=1-d_2}^{\infty} A_{21} x_{kl} z_1^{-k} z_2^{1-l} + \sum_{k=0}^{\infty}\sum_{l=1-d_2}^{-1} A_{21} x_{kl} z_1^{-k} z_2^{1-l}\Bigg] \\
U(z_1,z_2) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} u_{ij} z_1^{-i} z_2^{-j}.
\end{aligned}$$

Let

$$M^{-1}(z_1,z_2) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Phi_{ij} z_1^{-(i+1)} z_2^{-(j+1)}. \tag{11}$$

Comparison of the matrix coefficients at the same powers of z_1 and z_2 of the equality

$$M(z_1,z_2)\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Phi_{ij}z_1^{-(i+1)}z_2^{-(j+1)}\right]=\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Phi_{ij}z_1^{-(i+1)}z_2^{-(j+1)}\right]!M(z_1,z_2)=I \tag{12}$$

yields

$$\Phi_{ij}=\begin{cases} I \quad \text{(the identity matrix)} & \text{for } i=j=0\\ A_{00}\Phi_{i-1,j-1}+A_{10}\Phi_{i,j-1}+ & \\ +A_{20}\Phi_{i-1,j}+A_{01}\Phi_{i-d_1-1,j-d_2-1}+ & \\ +A_{11}\Phi_{i-d_1,j-d_2-1}+A_{21}\Phi_{i-d_1-1,j-d_2} & \text{for } i\ge 0\ j\ge 0\ i+j>0\\ 0 \quad \text{(the zero matrix)} & \text{for } i<0 \text{ or/and } j<0\,. \end{cases} \tag{13}$$

From (12) it follows that

$$\begin{aligned} &A_{00}\Phi_{i-1,j-1}+A_{10}\Phi_{i,j-1}+A_{20}\Phi_{i-1,j}+A_{01}\Phi_{i-d_1-1,j-d_2-1}+\\ &\quad +A_{11}\Phi_{i-d_1,j-d_2-1}+A_{21}\Phi_{i-d_1-1,j-d_2}=\\ &=\Phi_{i-1,j-1}A_{00}+\Phi_{i,j-1}A_{10}+\Phi_{i-1,j}A_{20}+\\ &\quad +\Phi_{i-d_1-1,j-d_2-1}A_{01}+\Phi_{i-d_1,j-d_2-1}A_{11}+\Phi_{i-d_1-1,j-d_2}A_{21}. \end{aligned} \tag{14}$$

From (13) for (8) satisfying (7) we have $\Phi_{ij}\in\mathbb{R}_+^{n\times n}$ for $i,j\in\mathbb{Z}_+$.

Knowing the matrices A_{00} A_{10} A_{20} A_{01} A_{11} A_{21} of the equation (8) and using (13) we may find the transition matrices Φ_{ij} for $i,j\in\mathbb{Z}_+$.

From (10) and (11) we have

$$\begin{aligned} X(z_1,z_2)&=M^{-1}(z_1,z_2)\,[N(z_1,z_2)+B_0U(z_1,z_2)]= \\ &=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Phi_{ij}z_1^{-(i+1)}z_2^{-(j+1)}\,[N(z_1,z_2)+B_0U(z_1,z_2)] \end{aligned} \tag{15}$$

and using the inverse 2D $\mathcal{Z}$ transform to (15) we obtain

$$x_{ij}=x_{ij}^{bc}+\sum_{k=0}^{i-1}\sum_{l=0}^{j-1}\Phi_{i-k-1,j-l-1}B_0u_{kl} \qquad \text{for } i,j\in\mathbb{Z}_+ \tag{16}$$

where

$$
\begin{aligned}
x_{ij}^{bc} &= \sum_{k=0}^{i} \left(\Phi_{i-k,j} - \Phi_{i-k-1,j} A_{20}\right) x_{k0} + \sum_{l=0}^{j} \left(\Phi_{i,j-l} - \Phi_{i,j-l-1} A_{10}\right) x_{0l} + \\
&- \Phi_{i,j} x_{00} + \sum_{k=-d_1}^{-1} \sum_{l=-d_2}^{\infty} \Phi_{i-d_1-k-1,j-d_2-l-1} A_{01} x_{kl} + \\
&+ \sum_{k=0}^{\infty} \sum_{l=-d_2}^{-1} \Phi_{i-d_1-k-1,j-d_2-l-1} A_{01} x_{kl} + \sum_{k=1-d_1}^{-1} \sum_{l=-d_2}^{\infty} \Phi_{i-d_1-k,j-d_2-l-1} A_{11} x_{kl} + \\
&+ \sum_{k=0}^{\infty} \sum_{l=-d_2}^{-1} \Phi_{i-d_1-k,j-d_2-l-1} A_{11} x_{kl} + \sum_{k=-d_1}^{-1} \sum_{l=1-d_2}^{\infty} \Phi_{i-d_1-k-1,j-d_2-l} A_{21} x_{kl} + \\
&+ \sum_{k=0}^{\infty} \sum_{l=1-d_2}^{-1} \Phi_{i-d_1-k-1,j-d_2-l} A_{21} x_{kl}
\end{aligned}
\tag{17}
$$

is the transient component depending on the boundary conditions (9). Therefore the following theorem has been proved.

THEOREM 4.1 *The solution satisfying the boundary condition (9) of the equation (8) has the form (16), (4).*

In a similar way the solution to the model (1) can be derived.

5. Generalization of the Cayley-Hamilton theorem

Note that the inverse matrix $M^{-1}(z_1, z_2)$ can be always written in the form

$$
M^{-1}(z_1, z_2) = \frac{H(z_1, z_2)}{d(z_1, z_2)} \tag{18}
$$

where

$$
H(z_1, z_2) = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} H_{ij} z_1^i z_2^j, \qquad H_{ij} \in \mathbb{R}^{n \times n}
$$

is a 2D polynomial matrix and

$$
d(z_1, z_2) = \sum_{k=0}^{N_1} \sum_{l=0}^{N_2} d_{ij} z_1^k z_2^l, \qquad d_{ij} \in \mathbb{R}
$$

is a 2D polynomial.

THEOREM 5.1 *Let $M^{-1}(z_1, z_2)$ have the form (18) and Φ_{ij} be the transition matrices defined by (13). Then*

$$\sum_{i=0}^{N_1}\sum_{j=0}^{N_2} d_{ij}\Phi_{i+k_1,j+k_2} = 0, \qquad \textit{for } k_1, k_2 \in \mathbb{Z}_+. \tag{19}$$

Proof. Using (18) and (11) we may write

$$H(z_1, z_2) = \left(\sum_{k=0}^{N_1}\sum_{l=0}^{N_2} d_{ij} z_1^k z_2^l\right)\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Phi_{ij} z_1^{-(i+1)} z_2^{-(j+1)}\right). \tag{20}$$

Comparison of the matrix coefficients at the same powers of $z_1^{-v} z_2^{-w}$ for $v, w \in \mathbb{Z}_+$, $(v + w > 0)$ of the equality (20) yields (19). ■

Theorem 5.1 is an extension of the well-known Cayley-Hamilton theorem for the 2D systems with delays.

EXAMPLE 5.1 *Consider equation (8) with*

$$\begin{aligned} &A_{10} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_{20} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \quad A_{01} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ &A_{00} = A_{11} = A_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \textit{and } d_1 = d_2 = 1. \end{aligned} \tag{21}$$

Using (13) we obtain

$$\begin{aligned} &\Phi_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \Phi_{22} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}, \ \Phi_{24} = \Phi_{42} = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}, \ \Phi_{33} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, \\ &\Phi_{35} = \Phi_{53} = \begin{bmatrix} 26 & 0 \\ 0 & 26 \end{bmatrix}, \ \Phi_{44} = \begin{bmatrix} 35 & 0 \\ 0 & 35 \end{bmatrix}, \ \Phi_{55} = \begin{bmatrix} 106 & 0 \\ 0 & 106 \end{bmatrix}. \end{aligned} \tag{22}$$

In this case the inverse matrix (18) has the form

$$M^{-1}(z_1, z_2) = \begin{bmatrix} z_1 z_2 + z_2 - z_1^{-1} z_2^{-1} & -z_1 \\ -z_1 - z_2 & z_1 z_2 - z_2 - z_1^{-1} z_2^{-1} \end{bmatrix}^{-1} = \frac{H(z_1, z_2)}{d(z_1, z_2)}$$

where

$$\begin{aligned} &H(z_1, z_2) = \begin{bmatrix} (z_1 z_2)^3 - z_1^2 z_2^3 - z_1 z_2 & z_1^3 z_2^2 \\ z_1^3 z_2^2 + z_1^2 z_2^3 & (z_1 z_2)^3 + z_1^2 z_2^3 - z_1 z_2 \end{bmatrix} \\ &d(z_1, z_2) = (z_1 z_2)^4 - 2(z_1 z_2)^2 - z_1^2 z_2^4 - z_1^4 z_2^2 - (z_1 z_2)^3 + 1. \end{aligned} \tag{23}$$

Using (19), (23) and (5) we obtain for $k_1 = k_2 = 0$

$$\Phi_{44} - 2\Phi_{22} - \Phi_{24} - \Phi_{42} - \Phi_{33} + I =$$
$$= \begin{bmatrix} 35 & 0 \\ 0 & 35 \end{bmatrix} - 2\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and for $k_1 = k_2 = 1$

$$\Phi_{55} - 2\Phi_{33} - \Phi_{35} - \Phi_{53} - \Phi_{44} + \Phi_{11} =$$
$$= \begin{bmatrix} 106 & 0 \\ 0 & 106 \end{bmatrix} - 2\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} - \begin{bmatrix} 26 & 0 \\ 0 & 26 \end{bmatrix} - \begin{bmatrix} 26 & 0 \\ 0 & 26 \end{bmatrix} - \begin{bmatrix} 35 & 0 \\ 0 & 35 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

6. Reachability of the model with delay

Consider the positive 2D model with delays described by the equation (8).

Definition 6.1 *The positive 2D model (8) is called reachable in the rectangle*

$$D_{qt} = \{(i, j) : i, j \in \mathbb{Z}_+,\ 0 \le i < q,\ 0 \le j < t\}$$

if for every state $x_f \in \mathbb{R}_+^n$ there exists an input sequence $u_{ij} \in \mathbb{R}_+^m$ for $(i, j) \in D_{qt}$ which transfers the model from zero boundary condition to the desired state x_f i.e. $x_{qt} = x_f$.

Definition 6.2 *The positive 2D model (8) is called reachable if for every $x_f \in \mathbb{R}_+^n$ there exist a rectangle D_{qt} and an input sequence $u_{ij} \in \mathbb{R}_+^m$ for $(i, j) \in D_{qt}$ which transfers the model from zero boundary conditions to the desired state x_f, i.e. $x_{qt} = x_f$.*

A column is called monomial if and only if it contains only one positive entry and the remaining entries are zero.

Theorem 6.1 *The positive 2D model (8) is reachable in the rectangle D_{qt} if and only if the reachability matrix*

$$R_{qt} = [\Phi_{q-1,t-1}B_0,\ \ \Phi_{q-2,t-1}B_0,\ \ \Phi_{q-1,t-2}B_0, \ldots, \Phi_{01}B_0,\ \ \Phi_{00}B_0] \tag{24}$$

contains n linearly independent monomial columns.

Proof. From (16) for $i = q,\ j = t$ and zero boundary conditions we have

$$x_f = x_{qt} = R_{qt} u_{qt}^{00} \tag{25}$$

where $u_{qt}^{00} = \left[u_{00}^T,\ u_{10}^T,\ u_{01}^T, \ldots, u_{q-2,t-1}^T,\ u_{q-1,t-1}^T\right]^T$ and T denotes transpose.

From (25) it follows that there exists for every $x_f \in \mathbb{R}_+^n$ an input sequence $u_{ij} \in \mathbb{R}_+^m,\ (i, j) \in D_{qt}$ if and only if the matrix (24) contains n linearly independent monomial columns. ■

Theorem 6.2 *The positive 2D model (8) is reachable if*

$$rank\, R_{qt} = n \tag{26}$$

and

$$R_{qt}^T \left[R_{qt}\, R_{qt}^T\right]^{-1} \in \mathbb{R}_+^{mqt\times n}. \tag{27}$$

Moreover, the input sequence that steers the model from zero boundary condition to $x_f \in \mathbb{R}_+^n$ is given by

$$u_{qt}^{00} = R_{qt}^T \left[R_{qt}\, R_{qt}^T\right]^{-1} x_f \tag{28}$$

where R_{qt} is defined by (24).

Proof. If the condition (26) is satisfied then the matrix $R_{qt}R_{qt}^T$ is invertible and from (25) we have

$$u_{qt}^{00} = R_{qt}^T \left[R_{qt}R_{qt}^T\right]^{-1} x_f \in \mathbb{R}_+^{mqt}$$

since (27) holds. ■

7. Minimum energy control

Consider the positive 2D model (8) and the performance index

$$I(u) = \sum_{i=0}^{q-1}\sum_{j=0}^{t-1} u_{ij}^T Q u_{ij} \tag{29}$$

where $Q \in \mathbb{R}^{m\times m}$ is a symmetric positive definite weighting matrix. The minimum energy control problem for the positive 2D model (8) can be stated as follows: Given the matrices $A_{00},\ A_{10},\ A_{20},\ A_{01},\ A_{11},\ A_{21} \in \mathbb{R}_+^{n\times n},\ B_0 \in \mathbb{R}_+^{n\times m}$, the positive integers $q,\ t$ (defining the rectangle D_{qt}), the final state $x_f \in \mathbb{R}_+^n$ and the weighting matrix Q, find an input sequence $u_{ij} \in \mathbb{R}_+^m$ for $(i,j) \in D_{qt}$ which transfers the model (8) from zero boundary condition to the desired final state x_f and minimizes the performance index (29). The problem of minimal energy control for standard (nonpositive) system was formulated and solved by J. Klamka (1991) and for positive 1D system with delays in Kaczorek and Busłowicz (2004).

To solve the problem we define the matrix

$$W = R_{qt}\bar{Q}_{qt}R_{qt}^T \in \mathbb{R}^{n\times n} \tag{30}$$

where R_{qt} is the reachability matrix of the form (24) and

$$\bar{Q}_{qt} = diag\ \left[Q^{-1}, \ldots, Q^{-1}\right] \in \mathbb{R}^{mqt\times mqt}.$$

From (30) it follows that the matrix W is nonsingular if and only if (26) holds.

Define the input sequence $\hat{u}_{ij}$ for $(i,j) \in D_{qt}$ by

$$\hat{u}_{qt}^{00} = \left[\hat{u}_{00}^T,\ \hat{u}_{10}^T,\ \hat{u}_{01}^T, \ldots, \hat{u}_{q-2,t-1}^T,\ \hat{u}_{q-1,t-1}^T\right]^T = \bar{Q}_{qt} R_{qt}^T W^{-1} x_f. \tag{31}$$

From (31) it follows that $\hat{u}_{qt}^{00} \in \mathbb{R}_+^{mqt}$ for any $x_f \in \mathbb{R}_+^n$ if and only if

$$\bar{Q}_{qt} R_{qt}^T W^{-1} \in \mathbb{R}_+^{mqt \times n}. \tag{32}$$

Note that if $W^{-1} \in \mathbb{R}_+^{n \times n}$ and $Q^{-1} \in \mathbb{R}_+^{m \times m}$ then the condition (32) is satisfied since $R_{qt} \in \mathbb{R}_+^{n \times mqt}$.

THEOREM 7.1 *Let us assume that*
i) the model (8) is reachable in the rectangle D_{qt}
ii) the condition (32) is satisfied
and $\bar{u}_{ij} \in \mathbb{R}_+^m$ for $(i,j) \in D_{qt}$ is any input sequence that transfers the model from zero boundary condition to the desired final state $x_f \in \mathbb{R}_+^n$. Then the input sequence $\hat{u}_{ij} \in D_{qt}$ defined by (31) transfers also the model from zero boundary conditions to the state x_f and minimizes the performance index (29), i.e.

$$I(\hat{u}) \leq I(\bar{u}). \tag{33}$$

Moreover, the minimal value of (29) is given by

$$I(\hat{u}) = x_f^T W^{-1} x_f. \tag{34}$$

Proof. If the model (8) is reachable in the rectangle D_{qt} and (32) holds, then for any $x_f \in \mathbb{R}_+^n$ we have $\hat{u}_{ij} \in \mathbb{R}_+^m$, $(i,j) \in D_{qt}$.

We shall show that the input sequence (31) steers the model from zero boundary conditions to the state x_f, i.e. $x_{qt} = x_f$. From (16) for $i = q,\ j = t,\ x_{ij}^{bc} = 0$ and (24), (30), (31) we have

$$x_{qt} = R_{qt} \hat{u}_{qt}^{00} = R_{qt} \bar{Q}_{qt} R_{qt}^T W^{-1} x_f = x_f$$

since $R_{qt} \bar{Q}_{qt} R_{qt}^T W^{-1} = I$.

Both input sequences $\bar{u}_{ij}$ and $\hat{u}_{ij}$, $(i,j) \in D_{qt}$ transfer the model (8) from zero boundary conditions to x_f. Hence $R_{qt} \bar{u}_{qt}^{00} = R_{qt} \hat{u}_{qt}^{00}$ and

$$R_{qt}\left(\hat{u}_{qt}^{00} - \bar{u}_{qt}^{00}\right) = 0. \tag{35}$$

From (31) we have $R_{qt}^T W^{-1} x_f = \bar{Q}_{qt}^{-1} \hat{u}_{qt}^{00}$ and from (35) we obtain

$$\left(\hat{u}_{qt}^{00} - \bar{u}_{qt}^{00}\right)^T R_{qt}^T W^{-1} x_f = \left(\hat{u}_{qt}^{00} - \bar{u}_{qt}^{00}\right)^T \hat{Q}_{qt} \hat{u}_{qt}^{00} = 0 \tag{36}$$

where

$$\hat{Q}_{qt} = \bar{Q}_{qt}^{-1} = diag\ [Q, \ldots, Q] \in \mathbb{R}^{mqt \times mqt}.$$

Using (36) it is easy to show that

$$(\bar{u}_{qt}^{00})^T \hat{Q}_{qt} \bar{u}_{qt}^{00} = (\hat{u}_{qt}^{00})^T \hat{Q}_{qt} \hat{u}_{qt}^{00} + \left(\bar{u}_{qt}^{00} - \hat{u}_{qt}^{00}\right)^T \hat{Q}_{qt} \left(\bar{u}_{qt}^{00} - \hat{u}_{qt}^{00}\right). \tag{37}$$

The inequality (33) holds since the last term in (37) is always nonnegative.

To obtain the minimum value of (29) we substitute (31) into (29)

$$I(\hat{u}) = I(\hat{u}_{qt}^{00}) = \sum_{i=0}^{q-1} \sum_{j=0}^{t-1} \hat{u}_{ij}^T Q \hat{u}_{ij} = \left(\bar{Q}_{qt} R_{qt}^T W^{-1} x_f\right)^T \hat{Q}_{qt} \left(\bar{Q}_{qt} R_{qt}^T W^{-1} x_f\right) =$$
$$= x_f^T W^{-1} R_{qt} \bar{Q}_{qt} R_{qt}^T W^{-1} x_f = x_f^T W^{-1} x_f$$

since $\hat{Q}_{qt} \bar{Q}_{qt} = I$ and $W^{-1} R_{qt} \bar{Q}_{qt} R_{qt}^T = I$. ■

Theorem 7.2 *Let the weighting matrix Q have the form $Q = Ia,\ a > 0$. Then*

$$\hat{u}_{qt}^{00} = R_{qt}^T \left[R_{qt},\ R_{qt}^T\right]^{-1} x_f \tag{38}$$

minimizes the performance index (29) for $Q = Ia,\ a > 0$ and its minimal value is given by

$$I(\hat{u}) = a x_f^T \left[R_{qt},\ R_{qt}^T\right]^{-1} x_f \tag{39}$$

Proof. If $Q = Ia$, then $\bar{Q}_{qt} = Ia^{-1} \in \mathbb{R}_+^{mqt \times mqt}$ and $W = a^{-1}\left[R_{qt},\ R_{qt}^T\right] \in \mathbb{R}^{n \times n}$. From (31) we have

$$\hat{u}_{qt}^{00} = \bar{Q}_{qt} R_{qt}^T W^{-1} x_f = I a^{-1} R_{qt}^T a \left[R_{qt},\ R_{qt}^T\right]^{-1} x_f = R_{qt}^T \left[R_{qt},\ R_{qt}^T\right]^{-1} x_f \tag{40}$$

Substitution of (38) into (29) for $Q = Ia$ yields (39). ■

Example 7.1 *Consider the positive 2D model (8) with the matrices*

$$A_{10} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A_{20} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ A_{01} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{41}$$
$$A_{00} = A_{11} = A_{21} = 0,\ B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},\ d_1 = d_2 = 1.$$

Find the optimal input sequence that transfers the model from zero boundary conditions to the final state $x_f = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and minimizes the performance index (29) with $q = t = 2,\ Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

Using (13) and (24) we obtain the reachability matrix of the form

$$
\begin{aligned}
R_{22} &= [\Phi_{11}B_0,\ \Phi_{01}B_0,\ \Phi_{10}B_0,\ \Phi_{00}B_0] = \\
&= [(A_{00} + A_{10}A_{20} + A_{20}A_{10})B_0,\ \ A_{10}B_0,\ \ A_{20}B_0,\ \ B_0] = \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

By Theorem 5 the model is reachable in the rectangle

$$D_{22} = \{(i,j)\ :\ i,j \in \mathbb{Z}_+,\ 0 \le i < 2,\ 0 \le j < 2\}.$$

Taking into account that $Q = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and using (38) we obtain the desired optimal input sequence of the form

$$\hat{u}_{22}^{00} = [\hat{u}_{00},\ \hat{u}_{10},\ \hat{u}_{01}]^T = R_{22}^T\left[R_{22}\ R_{22}^T\right]^{-1} x_f = [1,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 1]^T. \quad (42)$$

8. Concluding remarks

The notion of internaly positive 2D model (system) has been introduced. The solution to the 2D linear model with delays has been derived by using 2D $\mathcal{Z}$ transform. The necessary and sufficient conditions for the internal positivity and for reachability of the positive 2D models have been established.

The Cayley-Hamilton theorem for 2D linear models with delays has been extended. The minimum energy control for the internally positive 2 D models with delays has been formulated and solved. The considerations are given in details for the model (8) only with d_1 and d_2 delays, but they can be easily extended for the model (1) with many delays. Using the well-known relationship between 2D models the considerations can be extended for the Fornasini-Marchesini models and the Roesser model. Extension of these considerations for continuous-time 2D models with delays is an open problem.

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