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Approximate gradient projection method with general Runge-Kutta schemes and piecewise polynomial controls for optimal control problems

by

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Abstract: This paper addresses the numerical solution of optimal control problems for systems described by ordinary differential equations with control constraints. The state equation is discretized by a general explicit Runge-Kutta scheme and the controls are approximated by functions that are piecewise polynomial, but not necessarily continuous. We then propose an approximate gradient projection method that constructs sequences of discrete controls and progressively refines the discretization. Instead of using the exact discrete cost derivative, which usually requires tedious calculations, we use here an approximate derivative of the cost functional defined by discretizing the continuous adjoint equation by the same Runge-Kutta scheme backward and the integral involved by a Newton-Cotes integration rule, both involving maximal order intermediate approximations. The main result is that strong accumulation points in L^2, if they exist, of sequences generated by this method satisfy the weak necessary conditions for optimality for the continuous problem. In the unconstrained case and under additional assumptions, we prove strong convergence in L^2 and derive an a posteriori error estimate. Finally, numerical examples are given.

Keywords: optimal control, gradient projection method, discretization, non-matching Runge-Kutta schemes, piecewise polynomial controls.

1. Introduction

The numerical solution of optimal control problems has been studied in the literature using various approaches. Discretization methods have been treated in Dontchev (1996), Dontchev, Hager and Veliov (2000), Veliov (1997), using Euler or Runge-Kutta schemes, and error estimates were derived. Abstract results on approximations to generalized solutions, error estimates, and application to finite difference schemes for control-state constrained problems have been given in Malanowski, Buskens, Maurer (1998). In Dontchev et al. (1995), second order sufficiency conditions were proved and applied to the convergence of methods based on SQP and to penalty/multiplier methods. In Dunn (1996), the convergence of the gradient projection method in an infinite dimensional setting is analyzed. Mixed discretization/optimization methods using polynomial parameterizations and matching Runge-Kutta schemes were considered in Schwartz and Polak (1996). Optimization methods for discretized nonconvex optimal control problems using relaxed controls have been studied in Chryssoverghi, Coletsos and Kokkinis (1999), and in Chryssoverghi, Coletsos and Kokkinis (2001) where non-matching trapezoidal schemes were used.

In this paper, we consider an optimal control problem for systems described by nonlinear ordinary differential equations, with a (not necessarily bounded) control constraint set. In order to solve this problem numerically, we propose an approximate gradient projection method that constructs sequences of discrete controls and progressively refines the discretization during the iterations. The state equation is discretized by an explicit Runge-Kutta scheme of maximal global order m and the controls are approximated by vector functions whose components are piecewise polynomial of degree $l \leq m-1$, but not necessarily continuous. Since the matching adjoint of the discrete state equation and the derivative of the cost functional usually involve tedious calculations of partial derivatives of composed functions, we use at each iteration an approximate cost derivative defined by discretizing the continuous adjoint equation by the same, but nonmatching, Runge-Kutta scheme backward and the integral defining the cost derivative by a Newton-Cotes integration rule with nodes equal to the $l+1$ polynomial interpolation points, both schemes involving maximal global order approximations of intermediate values of states and adjoints. Since the discrete adjoints are nonmatching here, the exact derivative of the discrete cost is not defined in the adjoint form, and one must necessarily use a progressive refining procedure, with the adjoint matching only in the limit. This approach also reduces computing time and memory and generates a single infinite sequence of controls. The main result is that strong accumulation points in L^2, if they exist, of sequences generated by this method satisfy the weak necessary conditions for optimality for the continuous problem. In the case of the coercive unconstrained problem and the discrete gradient method, we prove strong convergence in L^2 and derive an a posteriori error estimate. Finally, several numerical examples are given.

2. The continuous optimal control problem

Consider the following optimal control problem, with state equation

$$y'(t) = f(t, y(t), w(t)) \text{ in } I := [0, T], \quad y(0) = y^0,$$

where $y(t) \in \mathrm{R}^p$, constraints on the control w

$$w(t) \in U \quad \text{in } I,$$

where U is a convex, closed, but not necessarily bounded, subset of R^q, and cost functional to be minimized

$$G(w) := g(y(T)).$$

If the problem involves an additional integral cost

$$G(w) := g(y(T)) + \int_0^T \bar{g}(t, y, u)dt,$$

we can classically transform it into a problem with final cost only by adding to the system the scalar differential equation

$$\bar{y}' = \bar{g}(t, y, u), \quad \bar{y}(0) = 0,$$

and setting

$$\tilde{G}(w) := g(y(T)) + \bar{y}(T).$$

For any integer $n \geq 1$, we denote by $\|v\| := (\sum_{j=1}^{n} v_j^2)^{1/2}$ the Euclidean norm and by $|v| := \max_{1 \leq j \leq n} |v_j|$ the max norm in R^n, by $(\cdot\,,\cdot)_2$ and $\|\cdot\|_2$ the usual inner product and norm in $L^2 := L^2(I, \mathrm{R}^n)$, and by $\|\cdot\|_\infty$ the usual norms in $L^\infty := L^\infty(I, \mathrm{R}^n)$ and $C(I, \mathrm{R}^n)$ corresponding to the norm $|\cdot|$ in R^n. We define the set

$$W = \{w \in L^2(I, \mathrm{R}^q) \mid w : I \to U\},$$

endowed with the relative norm topology of L^2, and the *set of admissible controls* $W_\infty := W \cap L^\infty$, also endowed with the L^2 norm. Let B_ρ^n denote the closed ball in R^n, with center 0 and radius ρ. Setting $D_\infty := I \times \mathrm{R}^p \times \mathrm{R}^q$, we make the following general assumptions:

$f,\ f_y := \partial f/\partial y,\ f_u := \partial f/\partial u$ are continuous on D_∞ ,
$f,\ f_y,\ f_u$ are Lipschitz continuous w.r.t. (y, u) on $I \times \mathrm{R}^p \times B_\rho^q$, for every $\rho > 0$, with Lipschitz constant independent of t but depending on ρ,
$g,\ \nabla g$ are Lipschitz continuous on B_ρ^p, for every $\rho > 0$, with Lipschitz constant depending on ρ.

In the case of an additional integral cost, we suppose that

$\bar{g},\ \bar{g}_y,\ \bar{g}_u$ are Lipschitz continuous w.r.t. (y, u) on $I \times B_\rho^p \times B_\rho^q$, for every ρ, with Lipschitz constant independent of t but depending on ρ.

The above assumptions concerning $f, f_y, f_u,\ \bar{g},\ \bar{g}_y,\ \bar{g}_u$ can be relaxed to the same properties, but finitely piecewise in t, i.e. for t belonging to each interval of a partition of I into a finite number of intervals. Then, in particular, for every $w \in W_\infty$, the state equation has a unique absolutely continuous solution $y := y_w$. Moreover, for every given $b_0 \geq 0$, there exists $b_1 \geq 0$ such that $\|y_w\|_\infty \leq b_1$, for every $w \in W_\infty$, with $\|w\|_2 \leq b_0$ (or $\|w\|_\infty \leq b_0$). The following results are standard.

PROPOSITION 2.1 *Under the above assumptions, the mappings $w \mapsto y_w$, from W_∞ to $C(I, \mathrm{R}^p)$, and $w \mapsto G(w)$, from W_∞ to* R*, are continuous.*

Since optimal control problems may have no classical solutions, they are also reformulated in the relaxed form, and the relative theory and methods have been developed using this formulation (for such nonconvex problems, see Chryssoverghi, Coletsos and Kokkinis, 1999, 2001, and the references there).

PROPOSITION 2.2 *Given controls $w, w' \in W_\infty$, the directional derivative of the functional G is given by*

$$DG(w, w'-w) := \lim_{\alpha \to 0^+} \frac{G(w + \alpha(w'-w)) - G(w)}{\alpha}$$
$$= \int_I z(t)^T f_u(t, y(t), w(t))[w'(t) - w(t)]dt,$$

where $y := y_w$, and the adjoint state $z := z_w$ is defined by the equation

$$z'(t) = -f_y(t, y(t), w(t))^T z(t) \ \text{ in } I, \quad z(T) = \nabla g(y(T)).$$

Moreover, the mappings $w \mapsto z_w$, from W_∞ to $C(I, \mathrm{R}^p)$, and $(w, w') \mapsto DG(w, w'-w)$, from $(W_\infty)^2$ to R*, are continuous.*

THEOREM 2.1 *If the control $w \in W_\infty$ is optimal, then w is extremal, i.e.*

$$DG(w, w'-w) \geq 0, \ \textit{for every } w' \in W_\infty,$$

and this condition is equivalent to the weak pointwise minimum principle

$$z(t)^T f_u(t, y(t), w(t))w(t) = \min_{u \in U}[z(t)^T f_u(t, y(t), w(t))u], \quad \text{in } I.$$

Conversely, if G is convex and $w \in W_\infty$ is extremal, then w is optimal.

For example, if f is affine in (y, u) and g convex, then G is clearly convex. Also, in the case of the additional integral cost, if f is affine in (y, u), $\bar{g}$ convex in (y, u), and g convex, then G is convex.

3. Discretizations

Let $(N_n)_{n \geq 1}$ be a sequence of positive integers. We suppose that $N_n \to \infty$ as $n \to \infty$ and that for each n, either $N_{n+1} = N_n$, or $N_{n+1} = MN_n$, where $M \geq 2$

is a positive integer. For each $n \geq 1$, we define the *discretization* Δ_n by setting

$$N := N_n, \quad h_n := T/N, \quad t_{ni} = ih_n, \quad i = 0, ..., N,$$
$$I_{ni} := [t_{n,i-1}, t_{n,i}), \quad i = 1, ..., N-1, \quad I_{nN} := [t_{n,N-1}, t_{nN}].$$

For simplicity of presentation, the intervals I_{ni} are chosen here of equal length; in fact the subintervals I_{1i} of the initial discretization could be of non-equal length, in which case each discretization Δ_{n+1} would be obtained by dividing the length of the subintervals of Δ_n either by one or by M. We suppose that $h_n \leq 1$, for every n.

For given $l+1$ ($l \leq m-1$) interpolation points t^k_{ni} in each I_{ni} of the form

$$t^k_{ni} = t_{n,i-1} + h_n/l, \quad k = 0, ..., l,$$

(note that the t^k_{ni} *must* be equidistant here), define the *set of discrete admissible controls*

$$W_n = \{w_n \in W_\infty \mid \ w_n \in \Pi_l(I_{ni}) \text{ on } I_{ni},\, w_{ni}(t^k_{ni}) = w^k_{ni} \in U,$$
$$k = 0, ..., l, \quad i = 1, ..., N\},$$

where $\Pi_l(I_{ni})$ denotes the set of q-vector functions whose components are polynomials (simply called *polynomials*) of degree $\leq l$ on I_{ni}, and where it is understood that the values at the possible interpolation jump points $t^k_{ni} = t_{n,i-1}, t_{ni}$ are right/left limit values, on each I_{ni}. Consider the (vector) divided differences (abbreviated by DD in the sequel) of order $1, ..., l$, relative to the interpolation points t^k_{ni}, $k = 0, ..., l$, of a discrete control w_n, for each $i = 1, ..., N_n$. Clearly, the values w^k_{ni} of the discrete controls are bounded by some constant $\bar{b}$ independent of k, i, n (we shall say *uniformly bounded*) if and only if the corresponding piecewise (vector) Lagrange interpolation polynomials w_n are bounded by some $\bar{b}'$ independent of i, n (*uniformly bounded*). Using the Lagrange interpolation polynomial in Newton form, we can see that if the DD of order $1, ..., l' \leq l$ of the w_n (i.e. of the w^k_{ni}, $k = 0, ..., l$, for each $i = 1, ..., N_n$) are bounded by some $\bar{L}$ independent of k, i, n (*uniformly bounded*), then the piecewise (w.r.t. the $\overset{o}{I}_{ni}$) derivatives of order $1, ..., l' \leq l$ of the w_n are bounded by some $\bar{L}'$ independent of i, n (*uniformly bounded*). Conversely, if the piecewise derivatives of order $1, ..., l$ of the w_n are uniformly bounded by some $\bar{L}'$, then, by the Mean Value Theorem, the divided differences of order $1, ..., l$ of the w_n are uniformly bounded also by $\bar{L}'$. Note that we have $W_n \subset W_{n'}$, for every $n' > n$. Note also that $W_n \not\subset W$ in general, except if $l = 0$, or if $l = 1$ and $t^0_{ni} = t_{n,i-1}$, $t^1_{ni} = t_{ni}$. If the *first order* DD of $w_n \in W_n$ are bounded by some $\bar{L}$, then $|w'_n(t)| \leq \bar{L}'$ for a.a. $t \in I$, and we have

$$d(w_n(t), U) := \min_{u \in U} |w_n(t) - u| \leq \bar{L}' h_n, \quad \text{for a.a. } t \in I,$$

hence w_n belongs to the set

$$\tilde{W}_n = \{w \in L^2(I, \mathrm{R}^q) \Big| \ w : I \to \tilde{U}_n \} \supset W,$$

where the convex $\bar{L}'h_n$-neighborhood $\tilde{U}_n$ of U is defined by

$$\tilde{U}_n = \{v \in \mathrm{R}^q \big| \, d(v,U) \le \bar{L}'h_n \} \supset U.$$

In the sequel, for a given Runge-Kutta or integration scheme, we shall say that the *maximal global order* of this scheme is m if this scheme has been constructed so as to yield a global truncation error $O(h_n^m)$ for f *sufficiently* smooth (e.g. $f \in C^m$ in (t,y,u)); the *effective global order* will then be μ, $\mu \le m$, with global error $O(h_n^\mu)$, depending on the smoothness of f. The *maximal* (resp. *effective*) *local order* is then $m+1$ (resp. $\mu+1$).

Next, we discretize the state equation by an explicit Runge-Kutta scheme of maximal global order m, and with m intermediate points (not necessarily distinct), which can be written in the form

$$\phi_{ni}^j = f(\bar{t}_{ni}^j, y_{n,i-1} + h_n \sum_{s=1}^{j-1} \alpha_{is}\phi_{ni}^s, \bar{w}_{ni}^s), \quad \bar{w}_{ni}^j = w_{ni}(\bar{t}_{ni}^j), \quad j=1,...,m,$$

$$\bar{t}_{n,i}^j = t_{n,i-1} + \bar{\theta}^j h_n, \quad \text{with } \bar{\theta}^j \in [0,1], \quad j=1,...,m, \ \bar{\theta}^1 = 0, \ \bar{\theta}^m = 1,$$

$$y_{ni} = y_{n,i-1} + h_n \sum_{j=1}^{m} \beta^j \phi_{ni}^j, \quad \text{with } \sum_{j=1}^{m} \beta^j = 1, \quad \beta^j \ge 0, \quad j=1,...,m,$$

$$i = 1,...,N,$$

$$y_{n0} = y^0.$$

This scheme can be written in the general form

$$y_{ni} = y_{n,i-1} + h_n F(\bar{\mathbf{t}}_{ni}, y_{n,i-1}, \bar{\mathbf{w}}_{ni}, h_n), \quad i=1,...,N, \quad y_{n0} = y^0,$$

with $\bar{\mathbf{t}}_{ni} = (\bar{t}_{ni}^1, ..., \bar{t}_{ni}^m)$, $\bar{\mathbf{w}}_{ni} = (\bar{w}_{ni}^1, ..., \bar{w}_{ni}^m)$. Setting $\mathbf{t} = (t^1,...,t^m)$, $\mathbf{u} = (u^1,...,u^m)$, and

$$E_\infty := \{(\mathbf{t}, y, \mathbf{u}, h) | \ \mathbf{t} \in I^m, \ y \in \mathrm{R}^p, \ \mathbf{u} \in U^m, \ h \in [0,1] \},$$

we can see that the function F is continuous on E_∞, Lipschitz continuous w.r.t. $(y,\mathbf{u})$ on E_∞, and satisfies

$$F(t,...,t,y,u,...u,0) = f(t,y,u) \ \text{ in } D_\infty.$$

In the sequel, L denotes various Lipschitz constants, independent of n.

PROPOSITION 3.1 *Let $\bar{b}' \ge 0$ be given. There exists $b_2 \ge 0$ such that, for every $w_n \in W_n$, with $\|w_n\|_\infty \le \bar{b}'$, the corresponding discrete state $y_n := (y_{n0},...,y_{nN})$ satisfies $\|y_{ni}\| \le b_2$, $i=0,...,N$.*

Proof. The Lipschitz continuity of F implies that for $(\mathbf{t}, y, \mathbf{u}, h) \in E_\infty$

$$\begin{aligned} \|F(\mathbf{t}, y, \mathbf{u}, h_n)\| &\le \|F(\mathbf{t}, 0, \mathbf{u}, h_n)\| + \|F(\mathbf{t}, y, \mathbf{u}, h_n) - F(\mathbf{t}, 0, \mathbf{u}, h_n)\| \\ &\le C + L\|y\|. \end{aligned}$$

The discrete scheme yields by summation

$$y_{ni} = y^0 + \sum_{j=1}^{i} h_n F(\bar{\mathbf{t}}_{nj}, y_{n,j-1}, \bar{\mathbf{w}}_{nj}, h_n),$$

hence

$$\|y_{ni}\| \leq \left\|y^0\right\| + CT + Lh_n \sum_{j=0}^{i-1} \|y_{nj}\|.$$

It then follows form the discrete Bellman-Gronwall inequality (see Thomee, 1997) that there exists b_2 such that $\|y_{ni}\| \leq b_2$, $i = 0, ..., N$. ∎

For $w_n \in W_n$, with corresponding state y_n, we define the discrete cost

$$G_n(w_n) := g(y_{nN}),$$

the continuous piecewise affine functions

$$\hat{y}_n(t) := y_{n,i-1} + (t - t_{n,i-1}) F(\bar{\mathbf{t}}_{ni}, y_{n,i-1}, \bar{\mathbf{w}}_{ni}, h_n), \text{ for } t \in I_{ni},\ i = 1, ..., N,$$

and the piecewise constant functions

$$y_n^-(t) := y_{n,i-1},\ y_n^+(t) := y_{ni}, \quad \text{for } t \in I_{ni},\ i = 1, ..., N.$$

Let $\bar{b}' \geq 0$ be given. We set

$$b := \max(b_1, b_2),$$
$$E := \{(\mathbf{t}, y, \mathbf{u}, h) \big| \ \mathbf{t} \in I^m,\ \|y\| \leq b,\ \mathbf{u} \in U^m,\ |\mathbf{u}| \leq \bar{b}',\ h \in [0,1]\},$$
$$B := \max_E \|F(\mathbf{t}, y, \mathbf{u}, h)\|.$$

THEOREM 3.1 *(Consistency) Let* $(w_n \in W_n)$ *be a sequence with* $\|w_n\|_\infty \leq \bar{b}'$ *and first order DD of the* w_n *uniformly bounded. If* $w_n \to w$ *in* L^2*, then* $y_n^- \to y$*,* $y_n^+ \to y$*, uniformly, and* $G_n(w_n) \to G(w)$*, as* $n \to \infty$*.*

Proof. Let $\varepsilon > 0$ be given. By our assumptions, w_n is Lipschitz continuous w.r.t. t with some constant $\bar{L}'$ (independent of i and n) on each I_{ni}. Since F is uniformly continuous on the compact set E, there exists $\delta > 0$ such that

$$\|F(\mathbf{t}_1, y_1, \mathbf{u}_1, h') - F(\mathbf{t}_2, y_2, \mathbf{u}_2, h'')\| \leq \varepsilon,$$

for $|\mathbf{t}_1 - \mathbf{t}_2| \leq \delta$, $\|y_1 - y_2\| \leq \delta$, $|h' - h''| \leq \delta$ and $|\mathbf{u}_1 - \mathbf{u}_2| \leq \delta$. Now choose n such that $h_n \leq \min(\delta, \delta/B, \delta/\bar{L}')$. By construction of $\hat{y}_n$, we clearly have

$$\|\hat{y}_n(t_1) - \hat{y}_n(t_2)\| \leq B|t_1 - t_2|, \quad t_1, t_2 \in I,$$
$$\left\|\hat{y}_n(t) - y^0\right\| \leq BT, \quad \text{for } t \in I,$$

which show that $(\hat{y}_n)$ is a bounded sequence of equicontinuous functions on I. We have also

$$\|\hat{y}_n(t) - y_{n,i-1}\| \le Bh_n \le \delta, \quad \text{for } t \in I_{ni},\ i = 1, ..., N.$$

By the definition of $\hat{y}_n$, we can write

$$\hat{y}_n'(t) = f(t, \hat{y}_n(t), w_n(t)) + \alpha_n(t), \quad \text{on each } I_{ni},\ i = 1, ..., N,$$

where

$$\begin{aligned}\alpha_n(t) &:= F(\bar{\mathbf{t}}_{ni}, y_{n,i-1}, \bar{\mathbf{w}}_{ni}, h_n) - f(t, \hat{y}_n(t), w_n(t)) \\ &= F(\bar{\mathbf{t}}_{ni}, y_{n,i-1}, \bar{\mathbf{w}}_{ni}, h_n) - F(t, ..., t, \hat{y}_n(t), w_n(t), ..., w_n(t), 0),\end{aligned}$$

for $t \in I_{ni}$, $i = 1, ..., N$, and we have

$$\|\alpha_n(t)\| \le \varepsilon, \quad \text{for } t \in I_{ni},\ i = 1, ..., N.$$

Therefore $\alpha_n \to 0$ uniformly on I. By integration, we get

$$\hat{y}_n(t) = y^0 + \int_0^t [f(s, \hat{y}_n(s), w_n(s)) + \alpha_n(s)]ds.$$

By Ascoli's theorem, there exist a subsequence $(\hat{y}_n)_{n\in J}$ and a continuous function y such that $\hat{y}_n \underset{n\in J}{\to} y$ uniformly. We write

$$\begin{aligned}\hat{y}_n(t) &= y^0 + \int_0^t [f(s, \hat{y}_n(s), w_n(s)) - f(s, y(s), w_n(s))]ds \\ &+ \int_0^t [f(s, y(s), w_n(s)) - f(s, y(s), w(s))]ds \\ &+ \int_0^t f(s, y(s), w(s))ds + \int_0^t \alpha_n(s)ds.\end{aligned}$$

Since f is Lipschitz continuous w.r.t. (y, u) on D_∞ for bounded y,u, and $w_n \to w$ in L^2, hence in L^1, passing to the limit for $n \in J$ in this equation, we obtain

$$y(t) = y^0 + \int_0^t f(s, y(s), w(s))ds,$$

which shows that $y = y_w$. Since y is uniformly continuous, it follows easily that also $y_n^- \to y$, $y_n^+ \to y$. The convergence of the original sequences follows then from the uniqueness of the limit. Finally, by the continuity of g, we have

$$G_n(w_n) := g(y_{nN}) \to g(y(T)) := G(w). \qquad \blacksquare$$

Theorem 3.2 *(i) (Lipschitz continuity) Let* $(w_n \in W_n)$, $(\tilde{w}_n \in W_n)$ *be sequences with* $\|w_n\|_\infty \leq \bar{b}'$, $\|\tilde{w}_n\|_\infty \leq \bar{b}'$, *and first order DD of the* $w_n, \tilde{w}_n$ *uniformly bounded, and let* $y_n, \tilde{y}_n$ *be the corresponding discrete states. Then*

$$\max_{0\leq i\leq N} \|y_{ni} - \tilde{y}_{ni}\| \leq c\|w_n - \tilde{w}_n\|_\infty,$$

$$|G_n(w_n) - G_n(\tilde{w}_n)| \leq c\|w_n - \tilde{w}_n\|_\infty,$$

where the constant c *is independent of* n.
(ii) (Equicontinuity) Let $(u_n \in W_n)$, $(v_n \in W_n)$ *be sequences with* $\|u_n\|_\infty \leq \bar{b}'$, $\|v_n\|_\infty \leq \bar{b}'$, *and first order DD of the* w_n, v_n *uniformly bounded. For given* $\alpha \in [0, 1]$, *set* $w_{n\alpha} := u_n + \alpha(v_n - u_n)$ *and let* $y_{n\alpha}$ *be the corresponding discrete state. The discrete states* $y_{n\alpha}$ *and costs* $G_n(w_{n\alpha})$ *are Lipschitz equicontinuous w.r.t.* α

$$\max_{0\leq i\leq N} \|y_{ni\alpha} - y_{ni\tilde{\alpha}}\| \leq c'|\alpha - \tilde{\alpha}|,$$

$$|G_n(w_{n\alpha}) - G_n(w_{n\tilde{\alpha}})| \leq c''|\alpha - \tilde{\alpha}|,$$

where the constants c' *and* c'' *are independent of* n, (u_n) *and* (v_n).

Proof. (i) We write

$$y_{nj} - y_{n,j-1} = h_n F(\bar{\mathbf{t}}_{nj}, y_{n,j-1}, \bar{\mathbf{w}}_{nj}, h_n),$$

for $j = 1, ..., i$, and summing over j, we get

$$y_{ni} - y^0 = h_n \sum_{j=1}^{i} F(\bar{\mathbf{t}}_{nj}, y_{n,j-1}, \bar{\mathbf{w}}_{nj}, h_n),$$

and similarly for $\tilde{w}_n, \tilde{y}_n$. Hence

$$y_{ni} - \tilde{y}_{ni} = h_n \sum_{j=1}^{i} [F(\bar{\mathbf{t}}_{nj}, y_{n,j-1}, \bar{\mathbf{w}}_{nj}, h_n) - F(\bar{\mathbf{t}}_{nj}, \tilde{y}_{n,j-1}, \bar{\tilde{\mathbf{w}}}_{nj}, h_n)]$$

$$= h_n \sum_{j=1}^{i} [F(\bar{\mathbf{t}}_{nj}, y_{n,j-1}, \bar{\mathbf{w}}_{nj}, h_n) - F(\bar{\mathbf{t}}_{nj}, \tilde{y}_{n,j-1}, \bar{\mathbf{w}}_{nj}, h_n)]$$

$$+ h_n \sum_{j=1}^{i} [F(\bar{\mathbf{t}}_{nj}, \tilde{y}_{n,j-1}, \bar{\mathbf{w}}_{nj}, h_n) - F(\bar{\mathbf{t}}_{nj}, \tilde{y}_{n,j-1}, \bar{\tilde{\mathbf{w}}}_{nj}, h_n)],$$

therefore

$$\|y_{ni} - \tilde{y}_{ni}\| \leq h_n L \sum_{j=1}^{i} \|y_{n,j-1} - \tilde{y}_{n,j-1}\| + L h_n \sum_{j=1}^{N} \max_{1\leq k\leq m} (|\bar{w}_{nj}^k - \bar{\tilde{w}}_{nj}^k|)$$

$$\|y_{ni} - \tilde{y}_{ni}\| \leq L h_n \sum_{j=1}^{i} \|y_{n,j-1} - \tilde{y}_{n,j-1}\| + LT\|w_n - \tilde{w}_n\|_\infty.$$

It then follows from the discrete Bellman-Gronwall inequality that

$$\|y_{ni} - \tilde{y}_{ni}\| \leq c\|w_n - \tilde{w}_n\|_\infty, \;\; i = 0, ..., N,$$

and the second inequality follows from the Lipschitz continuity of g.
(ii) Since the controls u_n, v_n belong to a bounded subset of L^∞, the Lipschitz equicontinuity of $y_{n\alpha}$ and $G_n(w_{n\alpha})$ follows by setting $w_n := w_{n\alpha}$, $\tilde{w}_n := w_{n\tilde{\alpha}}$. ■

The following theorem gives the discrete error estimates relative to a fixed discrete control.

THEOREM 3.3 *For $w_n \in W_n$, with $\|w_n\|_\infty \leq \bar{b}'$ and DD of order $1, ..., \min(l, \mu)$, $\mu \leq m$, of w_n bounded by $\bar{L}$, let y_n be the corresponding discrete state and $\tilde{y}_n$ the corresponding solution of the continuous state equation. If f is sufficiently smooth (e.g. $f \in C^\mu$) w.r.t. (t, y, u) (w.r.t. (t, y) if $l = 0$), then*
(i) $\max_{0 \leq i \leq N} \|y_{ni} - \tilde{y}_n(t_{ni})\| \leq ch_n^\mu$,
(ii) $|G_n(w_n) - G(w_n)| \leq ch_n^\mu$,
(iii) $|G_n(w_n) - G_{n'}(w_n)| \leq ch_n^\mu$, $n' > n$, and $N_n \neq N_{n'}$, *where c denotes various constants, independent of n and w_n.*

Proof. The control w_n and its derivatives of order $\leq \min(l, \mu)$ are bounded by a constant independent of n and w_n. The first estimate follows from classical error estimates of the Runge-Kutta method (see e.g. Hairer, Norsett and Wanner, 1993). Next, since g is Lipschitz continuous, we get

$$|G_n(w_n) - G(w_n)| = |g(y_{nN}) - g(\tilde{y}_n(T))| \leq L|y_{nN} - \tilde{y}_n(T)| \leq ch_n^\mu.$$

Now let $n' > n$ with $N_n \neq N_{n'}$ and let $\tilde{y}_{n'}$ denote the continuous state corresponding to w_n, considered as an element of $W_{n'}$. Then obviously $\tilde{y}_n = \tilde{y}_{n'}$ and

$$\begin{aligned} &|G_n(w_n) - G_{n'}(w_n)| = \left|g(y_{nN_n}) - g(y_{n'N_{n'}})\right| \\ &\leq |g(y_{nN_n}) - g(\tilde{y}_n(T))| + \left|g(\tilde{y}_{n'}(T)) - g(y_{n'N_{n'}})\right| \leq c(h_n^\mu + h_{n'}^\mu) \leq ch_n^\mu. \end{aligned}$$

■

Remark. Note that if we suppose that f is Lipschitz continuous w.r.t. (t, y, u) on $I \times \mathrm{R}^p \times B_\rho^q$, for every $\rho > 0$, with Lipschitz constant independent of t (but possibly depending on ρ), then the inequalities in Theorem 3.3 hold with $\mu = 1$.

Next we define, for given $w_n \in W_n$, with corresponding discrete state y_n, the *approximate discrete adjoint* z_n as the solution of the initial Runge-Kutta scheme applied formally backward to the continuous time adjoint equation

$$\psi_{ni}^{j} = f_y(\bar{t}_{ni}^{j}, \bar{y}_{ni}^{j}, \bar{w}_{ni}^{j})^T (z_{n,i-1} + h_n \sum_{s=m}^{j+1} \alpha_{is} \psi_{ni}^{s}), \quad j = m, ..., 1,$$

$$z_{n,i-1} = z_{ni} + h_n \sum_{j=m}^{1} \beta^j \psi_{ni}^{j},$$

$$i = N, ..., 1,$$

$$z_{nN} = \nabla g(y_{nN}),$$

and using (instead of the exact values) *intermediate state approximations* $\bar{y}_{ni}^{j}$ at the points $\bar{t}_{ni}^{j}$, of *maximal local order* m (hence inducing, at best, a local error $O(h_n^{m+1})$ in the adjoint scheme, and a global one $O(h_n^m)$), which can be computed as linear combinations of the intermediate function evaluations ϕ_{ni}^{j} of the Runge-Kutta scheme for the state equation, with some additional function evaluations, if $m \geq 5$ (for such Runge-Kutta approximations, see Enright et al., 1986, Hairer, Norsett and Wanner, 1993, Chap. II-6, and Papakostas and Tsitouras, 1997). These evaluations require much less computations than the direct calculation of the matching adjoint of the discrete state equation, which requires the computation of Jacobians w.r.t. y of multi-stage composed functions.

Now let (as above) y_{ni}^{k}, z_{ni}^{k} be approximations also of *maximal local order* m of the state and adjoint, and w_{ni}^{k} the exact control values, at the *interpolation points* t_{ni}^{k}. For given $w_n, w_n' \in W_n$, and y_n, z_n corresponding to w_n, the *approximate discrete derivative* of the cost functional G is defined by applying formally, on each I_{ni}, some Newton-Cotes integration rule (recall that the points t_{ni}^{k} are equidistant), with nodes t_{ni}^{k} and of maximal global order m', to the continuous time cost derivative, and using in this rule the approximate values y_{ni}^{k}, z_{ni}^{k} (instead of the exact values), and the exact control interpolation values $w_{ni}^{k} = w_n(t_{ni}^{k})$

$$D_n G(w_n, w_n' - w_n) := h_n \sum_{i=1}^{N} \sum_{k=0}^{l} C_l^k (z_{ni}^{k})^T f_u(t_{ni}^{k}, y_{ni}^{k}, w_{ni}^{k})({w'}_{ni}^{k} - w_{ni}^{k}).$$

The C_l^k are the coefficients of the integration rule, with $\sum_{k=0}^{l} C_l^k = 1$, $C_l^k \geq 0$, $k = 0, ..., l$, and it is understood, in the calculation of $D_n G$, that the polynomial pieces are extended by continuity, at the two end-points, to each closed interval $\bar{I}_{ni}$, $i = 1, ..., N$. Here also the computation of Jacobians w.r.t. u of composed functions required for the calculation of the discrete cost derivative using the matching adjoint is thus avoided. In order to simplify the minimizations of the augmented Hamiltonian in the Algorithm of Section 4, we have chosen here integration nodes that coincide with the interpolation points. The procedure can often be further simplified if we can also choose the interpolation points to coincide with some of the Runge-Kutta intermediate points, since we have less function evaluations in this case.

Define the I_{ni}-piecewise constant functions

$$w_n^k(t) = w_{ni}^k \ \text{ in } I_{ni}, \ i = 1, ..., N,$$

and similarly for y_n^k, z_n^k.

LEMMA 3.1 *For $0 \le k \le l$ and any sequence $(w_n \in W_n)$ with first order DD of the w_n uniformly bounded, $w_n \to w$ if and only if $w_n^k \to w$, in L^2 strongly or weakly.*

Proof. Since the functions w_n are piecewise Lipschitz continuous with some constant $\bar{L}'$ independent of n, we have, for any sequence $(w_n \in W_n)$ as above

$$\left\| w_n - w_n^k \right\|_2 \le \sqrt{qT} \left\| w_n - w_n^k \right\|_\infty \le \sqrt{qT} \bar{L}' h_n \to 0,$$

where q is the dimension of the control space, and the lemma follows. ∎

THEOREM 3.4 *(i) (Consistency) Let $(w_n \in W_n)$, $(w_n' \in W_n)$ be sequences with $\|w_n\|_\infty \le \bar{b}'$, $\|w_n'\|_\infty \le \bar{b}'$, and first order DD of the w_n, w_n' uniformly bounded. If $w_n \to w$ and $w_n' \to w'$ in L^2 strongly, then $y_n^k \to y = y_w$, $z_n^k \to z = z_w$, uniformly, and*

$$D_n G(w_n, w_n' - w_n) \to DG(w, w' - w).$$

(ii) (Error estimates) If f, f_y are sufficiently smooth (e.g. $f, f_y \in C^\mu$) w.r.t. (t, y, u) ((t, y) if $l = 0$), and the DD of order $1, ..., \min(l, \mu)$, $\mu \le m$, of the w_n are uniformly bounded, then

$$\max_{0 \le i \le N} \|z_{ni} - \tilde{z}_n(t_{ni})\| \le c h_n^\mu,$$

where $\tilde{z}_n$ denotes the exact solution of the continuous adjoint equation corresponding to w_n and $\tilde{y}_n$. If the DD of order $1, ..., \min(l, \mu, \mu')$, $\mu \le m$, $\mu' \le m$, of the w_n, w_n' are uniformly bounded, and f, f_y, f_u are sufficiently smooth, then

$$|D_n G(w_n, w_n' - w_n) - DG(w_n, w_n' - w_n)| \le c h_n^{\bar{\mu}}, \text{ with } \bar{\mu} = \min(\mu, \mu').$$

Proof. (outline) (i) The convergence of the sequences (y_n^k), $k = 0..., l$, to y follows from the convergence $y_n^- \to y$ (Theorem 3.1) and the construction of the y_{ni}^k, which are $O(h_n)$ approximations of $y_{n,i-1}$. The convergence of the sequences $(\hat{z}_n)$, (z_n^-), (z_n^+) to z is proved similarly to Theorem 2 and using the convergence of the states. The convergence $z_n^k \to z$ follows then from the convergence $z_n^+ \to z$

and the construction of the z_n^k, which are $O(h_n)$ approximations of $z_{n,i+1}$. Next, define the I_{ni}-piecewise constant functions on I

$$u_n^k(t) := f_u(t_n^k(t), y_n^k(t), w_n^k(t))^T z_n^k(t), \quad k = 0, ..., l.$$

Since $w_n \to w$ (hence $w_n^k \to w$, by Lemma 3.1) in L^2, $t_n^k \to t$, $y_n^k \to y$, $z_n^k \to z$ uniformly, $\left\|w_n^k\right\|_\infty \le \bar{b}'$, and f_u is uniformly continuous in (t, y, u) and Lipschitz continuous in u, it follows easily that

$$u_n^k \to f_u(\,\cdot\,, y, w)^T z \;\text{ in } L^2,\; k = 0, ..., l.$$

Since also ${w'}_n^k \to w'$ in L^2 and $\sum\limits_{k=0}^{l} C_l^k = 1$, we obtain

$$D_n G(w_n, w_n' - w_n) \to DG(w, w' - w).$$

(ii) The error estimate concerning the adjoints follows from the construction of the Runge-Kutta scheme, which contains here (in the worst case) $O(h_n^\mu)$ approximations of the exact state values at the t_{ni} (Theorem 3.3) and of the intermediate exact values at the $\bar{t}_{ni}^j$ (which both induce a $O(h_n^{\mu+1})$ local error), instead of the exact values. The last estimate follows from the $O(h_n^{\mu+1})$ Runge-Kutta local errors on the state and adjoint values at the t_{ni}^k and the $O(h_n^{\mu'+1})$ local error due to integration. ■

The following control approximation result is proved in Polak (1997). Let $\bar{W}_n$ denote the set of discrete piecewise constant controls. We have $\bar{W}_n \subset W_n$ for every n.

Proposition 3.2 *For every $w \in W$ (or $w \in W_\infty$), there exists a sequence $(w_n \in \bar{W}_n)$ such that $w_n \to w$ in L^2.*

4. Approximate gradient projection method

We describe now an approximate discrete gradient projection method that progressively refines the discretization during the iterations, thus reducing computing time and memory and avoiding the tedious calculation of the exact discrete adjoint and the derivative of the cost functional for higher order schemes.

For a given constant $L' \in [0, +\infty]$, define the *projection subset of admissible controls*

$$W_n' := \{w_n \in W_n \mid \text{ the first order DD of } w_n \text{ are bounded by } L'\} \subset W_n,$$

(with $W_n' = W_n$ if $L' = +\infty$, i.e. if there are no DD constraints), and for $w_n \in W_n$, the *discrete norm*

$$\|w_n\|_{2,n}^2 := h_n \sum_{i=1}^{N} \sum_{k=0}^{l} C_l^k \left\|w_{ni}^k\right\|^2 = \sum_{k=0}^{l} C_l^k \left\|w_n^k\right\|_2^2.$$

Algorithm

Step 1. Choose an initial discretization Δ_1, an m-point Runge-Kutta scheme, an integer $l \in [0, m-1]$, an $(l+1)$-node Newton-Cotes integration rule, an integer $M \geq 2$ (if $U = \mathrm{R}^q$, or $U \neq \mathrm{R}^q$ with $l \leq 1$), or $M = l$ (if $U \neq \mathrm{R}^q$ with $l > 1$), $L' \in [0, +\infty]$, $b, c \in (0,1)$, $s \in (0,1]$ ($s \in (0, +\infty)$ if $U = \mathrm{R}^q$), $\gamma > 0$, $w_1 \in W_1'$, and set $n := 1$, $\kappa := 1$.

Step 2. Find $v_n \in W_n'$ such that

$$e_n := D_n G(w_n, v_n - w_n) + (\gamma/2)\|v_n - w_n\|_{2,n}^2$$
$$= \min_{v_n' \in W_n'} \Big[D_n G(w_n, v_n' - w_n) + (\gamma/2)\|v_n' - w_n\|_{2,n}^2 \Big],$$

and set $d_n := D_n G(w_n, v_n - w_n)$.

Step 3. (Armijo step search) Set and $\alpha^0 = s$. If the inequality

$$G_n(w_n + \alpha^l(v_n - w_n)) - G_n(w_n) \leq \alpha^l b e_n,$$

is not satisfied, set successively $\alpha^{l+1} := c\alpha^l$ and find, if it exists, the first $\alpha^l \in (0,1]$, say $\bar{\alpha}$, such that it is satisfied. [*Optional*: Else, set successively $\alpha^{l+1} := \alpha^l / c$ and find the last $\alpha^l \in (0,1]$, say $\bar{\alpha}$, such that this inequality is satisfied.]
If $\bar{\alpha}$ is *found*, set $\alpha_n := \bar{\alpha}$, $\tilde{w}_n := w_n + \alpha_n(v_n - w_n)$, $n^\kappa := n$, $\kappa := \kappa + 1$. Else, set $\tilde{w}_n := w_n$.

Step 4. Define w_{n+1} by:

(a) $U = \mathrm{R}^q$, or $U \neq \mathrm{R}^q$ with $l \leq 1$: Set $N_{n+1} = N_n$ or $N_{n+1} = M\, N_n$, according to the chosen refining procedure. In both cases, set $w_{n+1} := \tilde{w}_n$.

(b) $U \neq \mathrm{R}^q$ with $l > 1$: Set $N_{n+1} = N_n$ or $N_{n+1} = l\, N_n$ (refining procedure). If $N_{n+1} = N_n$, set $w_{n+1} := \tilde{w}_n$. If $N_{n+1} = l\, N_n$, then, for each $i = 1, ..., N_{n+1}$, compute the multi-vector of *new* interpolation values $\tilde{\mathbf{w}}_{n+1,i} := (\tilde{w}^0_{n+1,i}, ..., \tilde{w}^l_{n+1})$ of $\tilde{w}_n$ on $I_{n+1,i}$ for the discretization Δ_{n+1}, and find the projection $P_{n+1,i}\tilde{\mathbf{w}}_{n+1,i}$ of $\tilde{\mathbf{w}}_{n+1,i}$ onto U^{l+1} subject to the (linear) first order DD_{n+1} constraints (i.e. first order DD_{n+1} bounded by L'). Then define w_{n+1} as the piecewise polynomial function of degree $\leq l$ interpolating these projection values on each $I_{n+1,i}$, for the discretization Δ_{n+1}.

Step 5. Set $n := n+1$ and go to *Step 2*.

Define the *set of successful iterations* $K := (n^\kappa)_{\kappa \in \mathrm{N}}$ (see *Step 3*).

THEOREM 4.1 *We suppose that f, f_y, f_u are at least Lipschitz continuous w.r.t. (t,y,u) ((t,y) if $l = 0$). If K is finite (resp. infinite) and there exists a subsequence $(w_n)_{n \in L \subset \mathrm{N}}$ (resp. $(w_n)_{n \in L \subset K}$) that converges strongly in L^2 to some w, that is bounded in L^∞ if U is unbounded, and is such that the first order DD of the w_n, $n \in L$, are uniformly bounded if $L' = +\infty$, then w is admissible and extremal, and $e_n \underset{n \in L}{\to} 0$, $d_n \underset{n \in L}{\to} 0$, $\tilde{w}_n \underset{n \in L}{\to} w$.*

Proof. (i) By the construction of w_{n+1} in Step 4, we can see by induction that $w_n \in W'_n$ for every n. Note that w_n does not belong to the set W in general (except if $l = 0$, or if $l = 1$ and $t^0_{ni} = t_{ni}$, $t^1_{ni} = t_{n,i+1}$), but to a larger convex set $\tilde{W}_n$ (see Section 3). We shall first show that if a subsequence $(w_n)_{n\in J}$ converges strongly in L^2, is bounded in L^∞ if U is unbounded, and the first order DD of the w_n are uniformly bounded, then the corresponding subsequence $(v_n)_{n\in J}$ constructed in Step 2 converges strongly in L^2 to some v. Define the I_{ni}-piecewise constant functions

$$u^k_n(t) := w^k_n(t) - (1/\gamma) f_u(t^k_n(t), y^k_n(t), w^k_n(t))^T z^k_n(t), \quad k = 0, ..., l.$$

These functions are also uniformly bounded by our assumptions. Now, one can easily see (by completing the square) that Step 2 amounts to minimizing, for each i, the quadratic function (or each square *separately* if $L' = +\infty$)

$$\sum_{k=0}^{l} C^k_l \left\| v'^k - u^k_{ni} \right\|^2$$

w.r.t. the vector $\mathbf{v}' = (v'^0, ..., v'^l)$ on the convex set

$$\begin{aligned}\mathbf{U}_{l+1} := \{\mathbf{v} \;\; := \;\; & (v'^0, ..., v'^l) \in U^{l+1} | \\ & v'^0, ..., v'^l \text{ satisfy the first order DD constraints}\}\end{aligned}$$

i.e. to finding, for each i, the projection $\mathbf{v}_{ni} = (v^0_{ni}, ..., v^l_{ni}) = P_{l+1}\mathbf{u}_{ni}$ of the vector $\mathbf{u}_{ni} = (u^0_{ni}, ..., u^l_{ni})$ onto $\mathbf{U}_{l+1}$ w.r.t. the inner product $((\cdot\,,\cdot)) = \sum_{k=0}^{l} C^k_l(\cdot\,,\cdot)$ of $(\mathrm{R}^q)^{l+1}$, or equivalently, the projection $\mathbf{v}_n = \tilde{P}_{l+1}(\mathbf{u}_n)$ of the corresponding piecewise constant multi-vector function $\mathbf{u}_n$, with $\mathbf{u}(t) := (u^0_{ni}, ..., u^l_{ni}) \in \mathbf{U}_{l+1}$ on each I_{ni}, onto the convex set of piecewise constant multi-vector functions

$$\mathbf{W}_{l+1} = \{\mathbf{v}' \in L^2(I, \mathrm{R}^q)^{l+1} | \; \mathbf{v}'(t) := (v'^0, ..., v'^l) \in \mathbf{U}_{l+1} \text{ on each } I_{ni}\},$$

w.r.t. the inner product $((\cdot\,,\cdot))_2 = \sum_{k=0}^{l} C^k_l(\cdot\,,\cdot)_2$ of $(L^2)^{l+1}$. Since the first order DD of the w_n are uniformly bounded by our assumptions in the two cases $L' \neq +\infty$, $L' = +\infty$, similarly to the proof of Theorem 3.4, we get

$$\mathbf{u}_n \to (\underset{0}{u}, ..., \underset{l}{u}) \;\text{ in } (L^2)^{l+1} \text{ strongly, where } u = w - (1/\gamma) f_u(\cdot\,, y, w)^T z.$$

By the construction of the y^k_{ni}, z^k_{ni}, which are $O(h_n)$ approximations of $y_{n,i-1}$, $z_{n,i+1}$, respectively, since the w^k_{ni}, their first order DD, and the y^k_{ni}, z^k_{ni}, are uniformly bounded, and since f_u is assumed Lipschitz continuous in (t, y, u), we can see that the u^k_{ni} and their first order DD are also uniformly bounded. Since the sequence of (discrete) projections $(\mathbf{v}_n)_{n\in J}$ is also bounded in L^∞ and belongs to the weakly closed set W^{l+1}, by Alaoglu's Theorem there exist a subsequence $(\mathbf{v}_n)_{n\in K\subset J}$ and $\mathbf{v}\in W^{l+1}$ such that $(\mathbf{v}_n)\underset{n\in K}{\to}\mathbf{v}$ in $(L^2)^{l+1}$ weakly. Now let v_n be the piecewise vector polynomial (of degree $\le l$) interpolating the values $v_{ni}^k,\ k=0,...,l$, at the points $t_{ni}^k,\ k=0,...,l$, on each I_{ni}. The first order DD of the v_n are also uniformly bounded in the two cases $L'\neq\infty$ and $L'=\infty$, because: either (a) $(L'\neq\infty)$ $\mathbf{v}_{ni}=P_{l+1}\mathbf{u}_{ni}\in\mathbf{U}_{l+1}$ for each i, or (b) $(L'=\infty)$ $\mathbf{v}_{ni}=P_{l+1}\mathbf{u}_{ni}\in\mathbf{U}_{l+1}=U^{l+1}$, i.e. $v_{ni}^k=Pu_{ni}^k,\ k=0,...,l$, *separately*, where the projection P onto U does not augment distances. Hence, by Lemma 3.1, we have in the limit $\mathbf{v}=(\underset{0}{v},...,\underset{l}{v})$. Now let any $\phi\in W$, and let $(\phi_n\in\bar{W}_n)$ be a sequence that converges strongly to ϕ in L^2 (Proposition 3.2). Since clearly $(\underset{0}{\phi_n},...,\underset{l}{\phi_n})\in\mathbf{W}_{l+1}$, we have

$$\sum_{k=0}^{l}C_l^k(v_n^k,\phi_n)_2=\sum_{k=0}^{l}C_l^k(u_n^k,\phi_n)_2,\quad\text{with}\ \sum_{k=0}^{l}C_l^k=1,$$

and passing to the limit, for $n\in K$, we get

$$(v,\phi)_2=(u,\phi)_2.$$

Since this holds for every $\phi\in W$, we have $v=\tilde{P}u$, where $\tilde{P}$ is the projection operator onto W in L^2, and clearly $v\in L^\infty$. On the other hand, since $\mathbf{v}_n\in\mathbf{W}_{l+1}$, we have

$$\begin{aligned}|||\mathbf{v}_n|||_2^2&:=\sum_{k=0}^{l}C_l^k(v_n^k,v_n^k)_2=\sum_{k=0}^{l}C_l^k(u_n^k,v_n^k)_2\\&\underset{n\in K}{\to}\sum_{k=0}^{l}C_l^k(u,v)_2=\sum_{k=0}^{l}C_l^k(v,v)_2=\left|\left|\left|(\underset{0}{v},...,\underset{l}{v})\right|\right|\right|_2^2.\end{aligned}$$

Therefore, $\mathbf{v}_n\underset{n\in K}{\to}(\underset{0}{v},...,\underset{l}{v})$ in $(L^2)^{l+1}$ strongly, hence $v_n\to v=\tilde{P}u$ (by Lemma 3.1) in L^2 strongly, and this holds also for $n\in J$, since the limit $\tilde{P}u$ is unique.

(ii) Let n_0 be fixed and suppose that an Armijo step α_n *cannot be found for* $n\ge n_0$, i.e. the set K is finite. Since by our assumption $w_n\underset{n\in L\subset\mathrm{N}}{\to}w$, we have $v_n\underset{n\in L}{\to}v$, as in (i). Since the minimized function in Step 2 vanishes for $v_n':=w_n\in W_n'$, we have $e_n\le 0$, hence $d_n\le e_n\le 0$. By Theorem 3.4 (i)

$$\begin{aligned}&e_n\underset{n\in L}{\to}e:=DG(w,v-w)+(\gamma/2)\|v-w\|^2\le 0,\\&d_n\underset{n\in L}{\to}d:=DG(w,v-w)\le e\le 0.\end{aligned}$$

Let us show that $e=d=0$. Suppose that $e<0$. Let b',b'' be such that

$b < b' < b'' < 1$. By the definition of the directional derivative, we have

$$G(w + \alpha(v - w)) - G(w) = \alpha(d + \varepsilon_\alpha) \le b''\alpha d \le b'\alpha d_n,$$
$$\text{for } \alpha \in [0, \delta],\ n \ge n_1 \ge n_0,\ n \in L,$$

for some n_1 sufficiently large and $\delta \in (0, s]$ sufficiently small, where s is the Armijo initial step. Since $(\,\alpha \mapsto G_n(w+\alpha(v_n-w))-G_n(w)\,)_{n\in L}$ is a bounded sequence (Proposition 3.1) of (Lipschitz) equicontinuous functions (Theorem 3.2) that converges pointwise on $[0, 1]$ to the function $\alpha \mapsto G(w + \alpha(v-w)) - G(w)$ (Theorem 3.1), the convergence is uniform on $[0, 1]$, by Ascoli's Theorem. Hence

$$\begin{aligned} G_n(w_n + \alpha(v_n - w_n)) - G_n(w_n) &\le \alpha b' d_n + \eta_n \\ &= \alpha(b' d_n + \frac{\eta_n}{\alpha}) \le b\alpha d_n \le b\alpha e_n, \end{aligned}$$
$$\text{for } \alpha \in [c\delta/2, \delta],\ n \ge n_2 \ge n_1,\ n \in L,$$

which shows that the Armijo step α_n can be found for $n \ge n_2$, $n \in L$, a contradiction. Therefore, we must have $e = 0$ and $e_n \underset{n\in L}{\to} 0$. Next, by Step 2, we have

$$D_nG(w_n, v'_n - w_n) + (\gamma/2)\|v'_n - w_n\|_{2,n}^2 \ge e_n, \quad \text{for every } v'_n \in W_n.$$

Let $v' \in W_\infty$ be any control and $(v'_n \in \bar{W}_n \subset W'_n)$ a sequence converging to v' (Proposition 3.2). Passing to the limit in the above inequality, for $n \in L$, we clearly find, using also Lemma 3.1, that

$$DG(w, v' - w) + (\gamma/2)\|v' - w\|_2^2 \ge 0.$$

Replacing w' by $w + \lambda(v' - w)$, for arbitrary $\lambda \in (0, 1]$, and dividing by λ, we get

$$DG(w, v' - w) + (\lambda\gamma/2)\|v' - w\|_2^2 \ge 0,$$

therefore

$$DG(w, v' - w) \ge 0, \quad \text{for every } v' \in W_\infty.$$

Since, in particular, $d = DG(w, v-w) \ge 0$, and $d \le 0$, we get $d = 0$ and $d_n \underset{n\in L}{\to} 0$. It follows that $\|v_n - w_n\|_{2,n} \underset{n\in L}{\to} 0$, hence $\|v_n - w_n\|_2 \underset{n\in L}{\to} 0$ (by Lemma 3.1), and $w = v = \tilde{P}u \in W$. Since $w_n \underset{n\in L}{\to} w$ and $(w_n)_{n\in L}$ is bounded in L^∞, we have also $w \in L^\infty$, hence $w \in W_\infty$. Therefore w is admissible and extremal.

(iii) Suppose now that an Armijo step α_n *can be found for each* $n = n^\kappa$, $\kappa \in$ N, i.e. for every $n \in K$, where the set K is infinite. We have $\kappa \to \infty$, and by our assumption $w_n \underset{n\in L\subset K}{\to} w$. Similarly to (ii), we obtain

$$G_n(w_n + \alpha(v_n - w_n)) - G_n(w_n) \le \alpha b d_n \le \alpha b e_n,$$
$$\text{for } \alpha \in [c\delta/2, \delta],\ n \ge n_2 \ge n_1,\ n \in L,$$

which shows that for $n \geq n_2$, $n \in L$, the Armijo step satisfies $\alpha_n \geq c\delta$. Therefore (see definition of $\tilde{w}_n$ in Step 3)

$$\begin{aligned} &G_n(\tilde{w}_n) - G_n(w_n) = G_n(w_n + \alpha_n(v_n - w_n)) - G_n(w_n) \\ &\leq \alpha_n b e_n \leq c\delta b e_n \leq c\delta b e/2 := a < 0, \quad \text{for } n \geq n_3 \geq n_2,\ n \in L. \end{aligned}$$

Consider now the case where $N_{n+1} = l\,N_n$ and w_{n+1} is defined by the $N_{n+1} = l\,N_n$ discrete projections in Step 4. Since w_n, v_n, hence $\tilde{w}_n$, satisfy the first order DD_n constraints, the $\tilde{w}_n$ have uniformly bounded first derivatives. The *linear interpolant* $\bar{w}_{n+1,i} := L_{n+1,i}\tilde{w}_{n+1,i}$ of $\tilde{w}_n$ on $I_{n+1,i}$ w.r.t. the two points $t^0_{n+1,i}, t^l_{n+1,i}$ takes a multi-vector of intermediate values $\bar{\mathbf{w}} := (\bar{w}^0_{n+1,i}, ..., \bar{w}^l_{n+1})$ that *belongs* to U^{l+1} since U is convex and clearly *satisfies* the first order DD_{n+1} constraints. It yields at most a (uniform in i, n) $O(h_{n+1})$ discrete interpolation error $|\bar{\mathbf{w}}_{n+1,i} - \tilde{\mathbf{w}}_{n+1,i}|_{\infty,n+1,i}$ on $I_{n+1,i}$. By the minimum norm property of the piecewise discrete projections in Step 4, we then have

$$\begin{aligned} \|P_{l+1}\tilde{\mathbf{w}}_{n+1,i} - \tilde{\mathbf{w}}_{n+1,i}\|_{2,n+1,i} &\leq \|\bar{\mathbf{w}}_{n+1,i} - \tilde{\mathbf{w}}_{n+1,i}\|_{2,n+1,i} \\ &\leq c'|\bar{\mathbf{w}}_{n+1,i} - \tilde{\mathbf{w}}_{n+1,i}|_{\infty,n+1,i} \leq O(h_{n+1}), \end{aligned}$$

for each $i = 1, ..., N_{n+1}$, hence $|w_{n+1} - \tilde{w}_n|_{\infty,n+1} \leq O(h_{n+1})$. Since the first order DD of the $\tilde{w}_n$ (see above) and w_{n+1} (by the projections in Step 4, as above for v_n) are uniformly bounded, so are their first derivatives, and we have

$$|w_{n+1} - \tilde{w}_n|_\infty \leq O(h_{n+1}).$$

By Theorem 3.2 (i), we then get

$$|G_{n+1}(w_{n+1}) - G_{n+1}(\tilde{w}_n)| \leq |w_{n+1} - \tilde{w}_n|_\infty \leq O(h_{n+1}) = O(h_n).$$

This trivially holds also in the cases where $w_{n+1} = \tilde{w}_n$ in Step 4.
Gathering our above results and using Theorem 3.3 (iii), with $\mu = 1$, we have the three following cases:

$$\begin{aligned} &G_{n+1}(w_{n+1}) - G_n(w_n) = G_{n+1}(\tilde{w}_n) - G_n(w_n) + O(h_n) = \\ &= G_n(\tilde{w}_n) - G_n(w_n) + O(h_n) \leq a + O(h_n), \text{ for } n \geq n_3,\ n \in L \subset K, \\ &G_{n+1}(w_{n+1}) - G_n(w_n) = G_n(\tilde{w}_n) - G_n(w_n) + O(h_n) \\ &\leq \alpha_n b e_n + O(h_n) \leq O(h_n), \text{ for } n \in K, \text{ and } n < n_3 \text{ or } n \notin L, \end{aligned}$$

and since $w_{n+1} := w_n$ for $n \notin K$ (unsuccessful iterations)

$$G_{n+1}(w_{n+1}) - G_n(w_n) = G_n(w_n) - G_n(w_n) + O(h_n) = O(h_n), \text{ for } n \notin K.$$

Since $h_{n+1} = h_n$ or $h_{n+1} \leq h_n/2$ anyway, we obtain

$$\begin{aligned} G_{n+1}(w_{n+1}) - G_1(w_1) &\leq \sum_{1\leq\kappa\leq n} Ch_\kappa + \sum_{n_3\leq\kappa\leq n,\, \kappa\in L} a \\ &\leq 2Ch_1 + \sum_{n_3\leq\kappa\leq n,\, \kappa\in L} a \underset{n\in\mathrm{N}}{\rightarrow} -\infty \end{aligned}$$

which contradicts the boundedness of the sequence $(G_n(w_n) := g(y_{nN}))_{n\in \mathrm{N}}$ (Proposition 3.1). Therefore $e = 0$ and $e_n \underset{n\in L}{\to} 0$. Similarly to (ii), $d = 0$, $d_n \underset{n\in L}{\to} 0$, and w is admissible and extremal. Finally, since $\tilde{w}_n = w_n + \alpha(v_n - w_n)$, for $n \in K \supset L$, $w_n \underset{n\in L}{\to} w$, and $\|v_n - w_n\|_2 \underset{n\in L}{\to} 0$, we have also $\tilde{w}_n \underset{n\in L}{\to} w$. ■

One can easily see that Theorem 4.1 remains valid if e_n is replaced by d_n in Step 2, but e_n usually gives better results. If the limit extremal control w is Lipschitz continuous, then the first order DD constraints are usually inactive if the constant L' is chosen sufficiently large, and we can then take $L' = +\infty$ (i.e. $W'_n = W_n$), thus simplifying the projection procedures in Steps 2 and 4. The control w_{n+1} constructed by piecewise discrete projections in Step 4 yields actually a much more accurate approximation of $\tilde{w}_n$ than the piecewise linear interpolate (see the above proof), if f and the boundary of U are piecewise smooth and the control w is continuous piecewise smooth, with possibly a finite number of discontinuity points of its derivative (*folding points*), which may be either a priori known, or approximated with high accuracy (see the comments to the Numerical Examples).

In the next theorem, we prove strong convergence in L^2 and derive an a posteriori error estimate in the case of the approximate gradient method applied to the unconstrained problem, without DD constraints.

THEOREM 4.2 *We suppose that $U = \mathrm{R}^q$, that f, f_y, f_u are sufficiently smooth to guarantee the error estimates of Theorems 4 and 5, and that the linear directional derivative of G*

$$DG(u, v-u) := (G'(u), v-u)_2, \quad G'(u) := f_u^T z,$$

is Lipschitz continuous

$$\|G'(v) - G'(u)\|_2 \le L\|v-u\|_2, \quad \text{for every } u, v \in L^\infty,$$

and strongly monotone (coercive)

$$(G'(v) - G'(u), v-u)_2 \ge \beta\|v-u\|_2^2, \quad \text{for every } u, v \in L^\infty.$$

If the sequence (w_n) generated by the Algorithm is such that the w_n and the DD of order $1, ..., \min(l, \mu, \mu')$, $\mu \le m$, $\mu' \le m'$ (see Theorems 3.3 and 3.4), of the w_n are uniformly bounded, then the sequences (w_n) and $(\tilde{w}_n := w_{n+1})$ converge strongly in L^2 to the unique optimal control w, the sequences (e_n), (d_n), defined in Step 2 of the Algorithm, converge to zero, and we have the a posteriori error estimate

$$\|w_n - w\|_2^2 \le (1/\beta^2)(2\gamma|d_n| + O(h_n^{\bar{\mu}})), \quad \text{with } \bar{\mu} = \min(\mu, \mu').$$

Proof. Since here $U = \mathrm{R}^q$ and there are no DD constraints, Step 2 reduces to a projection onto R^q for each i, k separately, hence we have the *approximate values*

$$v_{ni}^k = w_{ni}^k - (1/\gamma) f_u(t_{ni}^k, y_{ni}^k, w_{ni}^k)^T z_{ni}^k, \quad k = 0, ..., l, \ i = 1, ..., N,$$
$$u_{ni}^k := v_{ni}^k - w_{ni}^k = -(1/\gamma) f_u(t_{ni}^k, y_{ni}^k, w_{ni}^k)^T z_{ni}^k, \quad k = 0, ..., l, \ i = 1, ..., N.$$

On the other hand, we have the *exact functions*

$$\tilde{v}_n(t) = w_n(t) - (1/\gamma) f_u(t, \tilde{y}_n(t), w_n(t))^T \tilde{z}_n(t),$$
$$\tilde{u}_n(t) := \tilde{v}_n(t) - w_n(t) = -(1/\gamma) f_u(t, \tilde{y}_n(t), w_n(t))^T \tilde{z}_n(t) = -(1/\gamma) G'(w_n),$$

where $\tilde{y}_n, \tilde{z}_n$ are the exact state and adjoint corresponding to w_n. Let v_n (resp. $u_n := v_n - w_n$) be the piecewise vector polynomial (of degree $\leq l$) interpolating the $O(h_n^\mu)$ approximate values v_{ni}^k (resp. $u_{ni}^k := v_{ni}^k - w_{ni}^k$), $k = 0, ..., l$, at the points t_{ni}^k, $k = 0, ..., l$, on each I_{ni}. We have

$$d_n = D_n G(w_n, u_n) = -(1/\gamma) h_n \sum_{i=1}^N \sum_{k=0}^l C_l^k \left\| u_{ni}^k \right\|^2 = -(1/\gamma) \sum_{k=0}^l C_l^k \left\| u_n^k \right\|_2^2.$$

The sequence $(\tilde{u}_n)$ and the sequences of piecewise derivatives of order $1, ...,$ $\min(l, \mu, \mu')$ of $\tilde{u}_n$ are bounded in L^∞ by our assumptions. Taking then into account the Runge-Kutta and numerical integration errors (Theorem 3.4, (ii)), we have

$$\tilde{d}_n := DG(w_n, \tilde{u}_n) = (G'(w_n), \tilde{u}_n)_2 = -(1/\gamma) \|\tilde{u}_n\|_2^2 = d_n + O(h_n^{\bar{\mu}}).$$

We shall first show that $e_n \underset{n \in \mathrm{N}}{\to} 0$. We have $d_n \leq e_n \leq 0$. Suppose that there exists $\beta > 0$ and a subsequence $(e_n)_{n \in L}$ such that $e_n \leq -\beta < 0$ for every $n \in L$. Using the Mean Value Theorem, taking into account the Runge-Kutta and interpolation errors, and setting

$$w_{n\alpha} := w_n + \alpha u_n, \ e_{n\alpha} := (1/\theta_{n\alpha})(G'(w_n + \theta_{n\alpha} \alpha u_n) - G'(w_n), \theta_{n\alpha} \alpha u_n)_2,$$

we have

$$G(w_{n\alpha}) - G(w_n) = (G'(w_n + \theta_{n\alpha} \alpha u_n), \alpha u_n)_2$$
$$= \alpha (G'(w_n), u_n)_2 + e_{n\alpha} = \alpha(\tilde{d}_n + O(h_n^{\bar{\mu}}) + O(h_n^{l+1})) + e_{n\alpha}$$
$$= \alpha(d_n + O(h_n^{\bar{\mu}}) + O(h_n^{l+1})) + e_{n\alpha} := \alpha(d_n + \eta_n) + e_{n\alpha},$$
for some $\theta_{n\alpha} \in (0,1)$ (which holds also for $\alpha = 0$, with $\theta_{n0} = 1$).

Since the sequence (u_n) is clearly bounded in L^∞, hence in L^2, and G' is Lipschitz, we have

$$|e_{n\alpha}| \leq L \theta_{n\alpha} \alpha^2 \|u_n\|_2^2 \leq L \alpha^2 \|u_n\|_2^2 \leq \alpha^2 M,$$

hence

$$G(w_{n\alpha}) - G(w_n) \leq \alpha(d_n + \eta_n + \alpha M).$$

Let b', b'' be such that $0 < b < b' < b'' < 1$. Since $d_n \leq e_n \leq -\beta < 0$, we then have, for $\alpha \in [0, \delta]$, with $\delta = (1 - b'')\beta/M$, and $n \geq n_1$, $n \in L$

$$\begin{aligned} &G(w_{n\alpha}) - G(w_n) \leq \alpha(d_n + \eta_n + \alpha M) \leq \alpha(d_n + \eta_n + \delta M) \\ &\leq \alpha(d_n + \eta_n + (1-b'')\beta) \leq \alpha(d_n + \eta_n - (1-b'')d_n) = \alpha(b''d_n + \eta_n) \leq \alpha b' d_n, \end{aligned}$$

hence, by Theorem 3.3

$$\begin{aligned} &G_n(w_{n\alpha}) - G_n(w_n) \leq \alpha b' d_n + O(h_n^{\bar{\mu}}) = \alpha[b' d_n + (1/\alpha) O(h_n^{\bar{\mu}})] \leq \alpha b d_n \leq \alpha b e_n, \\ &\text{for } \ \alpha \in [c\delta/2, \delta], \ n \geq n_2, \ n \in L. \end{aligned}$$

A contradiction follows, similarly to the proof of Theorem 4.1, (ii) and (iii), in both cases, i.e. when K is finite or infinite. Therefore $e_n \underset{n \in \mathrm{N}}{\to} e = 0$. We can easily see here that $e_n = d_n/2$, hence $d_n \underset{n \in \mathrm{N}}{\to} d = 0$.
Now, since G' is strongly monotone, we have

$$\begin{aligned} \beta \|w_n - w_{n'}\|_2^2 &\leq (G'(w_n) - G'(w_{n'}), w_n - w_{n'})_2 \\ &\leq \|G'(w_n) - G'(w_{n'})\|_2 \|w_n - w_{n'}\|_2, \end{aligned}$$

hence

$$\begin{aligned} &\beta^2 \|w_n - w_{n'}\|_2^2 \leq 2(\|G'(w_n)\|_2^2 + \|G'(w_{n'})\|_2^2) = 2\gamma(\left|\tilde{d}_n\right| + \left|\tilde{d}_{n'}\right|) \\ &= 2\gamma(|d_n| + |d_{n'}|) + O(h_n^{\bar{\mu}}) + O(h_{n'}^{\bar{\mu}}) \to 0, \quad \text{as } n, n' \to \infty, \end{aligned}$$

showing that (w_n) is a Cauchy sequence, which therefore converges to some $w \in L^2$. Since (w_n) is clearly bounded in L^∞, we have also $w \in L^\infty$, i.e. w is admissible. We then show as in the proof of Theorem 4.1 (ii) that w is extremal. Since our assumptions imply that G is strictly convex, w is the unique optimal control. Finally, keeping n fixed and passing to the limit in n', we obtain the requested posteriori error estimate. ■

For example, if f is affine w.r.t. (y, u) and the cost G is of the form

$$G(w) := g_0(y(T)) + \int_0^T [g_1(t, y(t)) + g_2(t, w(t))]dt,$$

where g_{0y} and g_{1y} are Lipschitz and monotone w.r.t. y (the vector function $\phi(y)$ is *monotone* if $(\phi(y_1) - \phi(y_2), y_1 - y_2)_2 \geq 0$, for every y_1, y_2), and g_{2u} is Lipschitz and strongly monotone w.r.t. u, then it can be shown that G' is Lipschitz and strongly monotone.

5. Numerical examples

Set $I := [0,1]$, and define the reference control $\bar{w} = (\bar{w}_1, \bar{w}_2)$, where

$$\bar{w}_1(t) := \begin{cases} 0, & t \in [0,\sigma) \\ \frac{e^{t-\sigma}-1}{e^{1-\sigma}-1}, & t \in [\sigma, 1] \end{cases} \qquad \bar{w}_2(t) := \frac{e^{1-t}-1}{e-1}, \ \ t \in I,$$

with $\sigma \in [0,1)$, and the reference state $\bar{y}(t) := (e^{-t},\ e^{-t},\ 0),\ t \in I$.

a) Consider the following optimal control problem, with state equations

$$\begin{aligned} &y_1' = -y_2 + w_1 - \bar{w}_1, \\ &y_2' = -y_1 + w_2 - \bar{w}_2, \\ &y_3' = [(y_1 - \bar{y}_1)^2 + (y_2 - \bar{y}_2)^2 + (w_1 - \bar{w}_1)^2 + (w_2 - \bar{w}_2)^2]/2, \\ &y_1(0) = y_2(0) = 1, \quad y_3(0) = 0, \end{aligned}$$

control constraint set $U := [0,1]^2$, and cost to be minimized $G(w) := y_3(1)$. Clearly, the optimal control here is $w^* = \bar{w}$, with optimal state $y^* = \bar{y}$ and cost $G(w^*) = G(\bar{w}) = 0$.

The Algorithm was applied to this example using the 4^{th} order 4-point Runge-Kutta scheme, with $\bar{\theta}^2 = 1/3$, $\bar{\theta}^3 = 2/3$, the 3/8-Newton-Cotes 4^{th} order 4-point integration rule, and piecewise cubic controls ($l = 3$) with interpolation points coinciding with the Runge-Kutta points

$$t_{ni}^k = \bar{t}_{ni}^k = t_{n,i-1},\ t_{n,i-1} + h_n/3,\ t_{n,i-1} + 2h_n/3,\ t_{ni}.$$

We used the following successive step sizes

$$h_n = 3^{-j}/60, \quad \text{for } Kj + 1 \le n \le K(j+1),\ j = 0, 1, 2,$$

with refining factor $M = l = 3$, refining period $K = 13$, first order DD constraints constant $L' = 10$, gradient projection parameter $\gamma = 0.35$, Armijo step search parameters $b = c = 0.5$, $s = 1$, option skipped in Step 3, constant initial control $(0.5, 0.5)$, and *a priori known folding point* of $\bar{w}$: $\sigma = 0.5$. The results obtained at the last iteration of each period are shown in Table 1, where

$$\varepsilon_n := \max_{1 \le i \le N} \big[\max_{0 \le k \le l} \big| \tilde{w}_{ni}^k - w^*(t_{ni}^k) \big| \big]$$

with $\tilde{w}_n$ as defined in Step 3 of the Algorithm,

$\eta_n := \max_{0 \le i \le N} (|\tilde{y}_{ni} - y^*(t_{ni})|)$, where $\tilde{y}_n$ corresponds to $\tilde{w}_n$,

$\zeta_n := G_n(\tilde{w}_n)$,

and e_n was defined in Step 2 of the Algorithm. The last control and state curves obtained are practically identical to the exact ones and are therefore not shown. It turned out that the first order DD constraints were inactive in this example, and this is due to the fact that here the first derivative of the approximated control $w^* = \bar{w}$ is bounded by $\approx 1.54 << L' = 10$. Note that this problem has actually *inactive* control constraints, though *active* in the gradient projection procedure. The known folding point $\sigma = 0.5$ is *equal* here to some point t_{ni} of the discretization for each n. The last control discrete max error found is $\approx O(h^4)$, where $\bar{h} := h_{\bar{n}} = 1/540$ is the *last* step size, i.e. at the last iteration $\bar{n}$. Note that here f (which contains $\bar{w}$) and $w^* = \bar{w}$ are smooth on each of the two intervals $[0, 0.5]$, $[0.5, 1]$.

Table 1.

n	ε_n	η_n	ζ_n	$-e_n$
13	$0.309{\cdot}10^{-3}$	$0.139{\cdot}10^{-3}$	$0.385{\cdot}10^{-7}$	$0.239{\cdot}10^{-6}$
26	$0.603{\cdot}10^{-7}$	$0.184{\cdot}10^{-7}$	$0.386{\cdot}10^{-11}$	$0.389{\cdot}10^{-14}$
39	$0.674{\cdot}10^{-11}$	$0.185{\cdot}10^{-11}$	$0.473{\cdot}10^{-13}$	$0.294{\cdot}10^{-22}$

b) The approximate gradient method, without control constraints ($U = \mathrm{R}^q$), without DD constraints, with $K = 10$, $\sigma = 0$ (no folding point of $\bar{w}$), and the rest of the parameters as in Example (a), was applied to the modified above problem and yielded the results shown in Table 2, where the last control discrete max error is also $\approx O(\bar{h}^4)$. Here f and w^* are smooth.

Table 2.

n	ε_n	η_n	ζ_n	$-e_n$
10	$0.519{\cdot}10^{-4}$	$0.512{\cdot}10^{-4}$	$0.208{\cdot}10^{-8}$	$0.548{\cdot}10^{-7}$
20	$0.107{\cdot}10^{-7}$	$0.107{\cdot}10^{-7}$	$0.386{\cdot}10^{-11}$	$0.240{\cdot}10^{-14}$
30	$0.227{\cdot}10^{-11}$	$0.229{\cdot}10^{-11}$	$0.473{\cdot}10^{-13}$	$0.105{\cdot}10^{-21}$

c) With the inactive control constraint set $[0,\, 1]^2$ replaced by the *active* one $[0.2,\, 1]^2$, $K = 9$, $\sigma = 0$ (no folding point of $\bar{w}$), and the rest of the parameters as in Example (a), we obtained the results shown in Table 3. The last control discrete max error must probably be $\approx O(\bar{h}^2)$ (comparing to $e_{\bar{n}}$). Figs. 1 and 2 show the two components of the approximate extremal control $\tilde{w}_{\bar{n}} \approx w^*$ at the last iteration. Note that since the necessary conditions for optimality are also sufficient here, the method actually approximates the optimal control w^*. Here f is smooth and w^* has three folding points, as one can see in Figs. 1 and 2.

Table 3.

n	ζ_n	$-e_n$
9	$0.470258001785180{\cdot}10^{-2}$	$0.457{\cdot}10^{-6}$
18	$0.470246757760872{\cdot}10^{-2}$	$0.534{\cdot}10^{-14}$
27	$0.470246707328355{\cdot}10^{-2}$	$0.147{\cdot}10^{-15}$

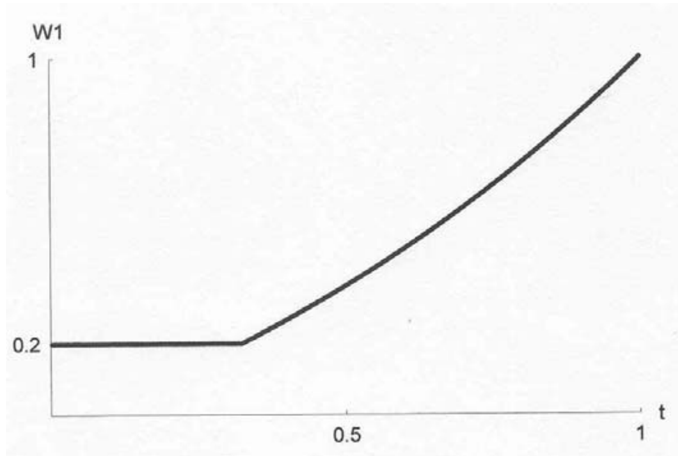

Figure 1. Example (c), Approximate optimal control, 1^{st} control component.

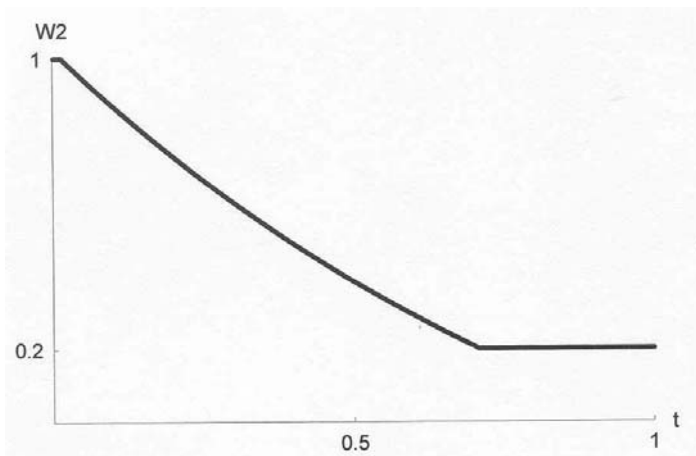

Figure 2. Example (c), Approximate optimal control, 2^{nd} control component.

In the above applications of the Algorithm, only 0-2 search iterations in Step 3 were sufficient to find the Armijo step, for each n. The above results with progressive refining were found to be of similar accuracy to those obtained with constant *last* step size $\bar{h} := h_{\bar{n}} = 1/540$, but required here about half the computing time.

We describe now a procedure for approximating with high accuracy the possible folding points of the extremal control. Consider e.g. Example (c), where f is *smooth* and the control w^* is *piecewise smooth* with a *finite number*

(3 here) of folding points $\tau_j \in I$, $j = 1, 2, 3$, where w^* reaches or leaves the boundary ∂U of U. When applying the method as above, the first order DD (resp. DD of order $\leq l = 3$) of the w_n are uniformly bounded near (resp. off) the folding points, hence the schemes used are locally 2^{nd} (resp. globally 4^{th}) order near (resp. off) these points. Hence the last states, adjoints, costs, and cost derivatives, hence w^*, are thus approximated with an $\approx O(\bar{h}^2)$ global discrete max error. Using then, for each folding point τ_j, $j = 1, 2, 3$, two computed values of the last control $w_{\bar{n}}$ in U at two consecutive discretization points t^k_{ni} nearest from the left (resp. right) its guessed value, if w^* is reaching (resp. leaving) ∂U, we linearly extrapolate these values, i.e. we find the point τ'_j near these two points where the linear interpolant w.r.t. these two points intersects ∂U; in most cases, i.e. if the trajectory $t \mapsto w_n(t)$ is not nearly tangent to ∂U, τ'_j is thus an $\approx O(\bar{h}^2)$ approximation of τ_j. Then the τ'_j can be chosen as three points t_{1i_j} of a new (uneven) initial discretization, and we can reapply the method. The control w^* is then computed with an $\approx O(\bar{h}^3)$ error, due to the $\approx O(\bar{h}^2)$ perturbations of the exact points τ_j and the finite number of these points. We can repeat the above procedure, using now three new appropriate points $t^k_{1i_j}$ and a local quadratic extrapolation, to compute $\approx O(\bar{h}^3)$ approximations τ''_j of the τ_j. Finally, a third application of the method yields an $\approx O(\bar{h}^4)$ approximation of w^*, with the method behaving now essentially as if the folding points were known, as in Example (a). This approximation procedure can be generalized to problems with sufficiently smooth f, and constraint set U with piecewise smooth boundary, defined by a finite number of equations.

A priori known non-smoothness points of f w.r.t. t should of course be chosen to be equal to some of the points t_{1i} of each first discretization used.

Now, given discrete control values computed with discrete max error $\approx O(\bar{h}^4)$, using the above schemes and piecewise cubic polynomials, and after the above procedure, if necessary, the last cubic interpolant $\tilde{w}_n$ in Step 3 yields then an $\approx O(\bar{h}^4)$ *continuous* max error. Any piecewise quadratic (resp. linear, constant) interpolant of some of these values would clearly yield an $\approx O(\bar{h}^3)$ (resp. $\approx O(\bar{h}^2)$, $\approx O(\bar{h})$) continuous max error anyway.

Suppose that the data are sufficiently smooth (i.e. f is $t-$piecewise smooth in (t, y, u), with known discontinuity points in t, and the boundary of U is piecewise smooth) and that the extremal control w^* is continuous piecewise smooth with possibly a finite number of folding points, which are either a priori exactly predictable, or approximated as above with high accuracy. Then an important factor in obtaining such essentially maximal order discrete errors at the interpolation points is the use of *maximal order* approximate state and adjoint values at *all* the intermediate Runge-Kutta *and* integration/interpolation points. On the other hand, since here the method is progressively refining, where *new intermediate values* of *interpolating* polynomials are used, say periodically, at each new refining, the use of *maximal degree* polynomials contributes also to this result.

6. Final comments

A discrete, progressively refining, gradient projection method that uses approximate adjoints and cost derivatives given by the general non-matching Runge-Kutta and integration schemes, in conjunction with piecewise polynomial discrete controls, which are not necessary continuous, has been applied to an optimal control problem involving ordinary differential equations with control constraints. The use of approximate non-matching schemes avoids the heavy calculation of the matching discrete adjoint and cost derivative, and the progressive refining reduces computing time and memory. This procedure seems to yield very accurate approximations of the extremal control and of the corresponding states, costs and cost derivatives, when this extremal control is continuous and piecewise smooth with a priori known discontinuity points of the control derivative (or even with unknown such points, after an additional approximation procedure), or piecewise smooth with a priori known discontinuity points. In the unconstrained case, we prove strong convergence in L^2 and derive an a posteriori estimate. It seems difficult to obtain an error estimate for the constrained problem, but some results could probably be obtained in this direction under additional smoothness assumptions. Progressively refining methods using exact or approximate cost derivatives can be applied to a broad class of classical or relaxed optimal control problems involving ordinary or partial differential equations, and also to state constrained problems, via penalty functionals (see Chryssoverghi, Coletsos and Kokkinis, 1999, 2001).

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