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Control zeros and maximum-accuracy/maximum-speed control of LTI MIMO discrete-time systems

by

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Abstract: Based on new definitions of "control zeros" and minimum phase property for possibly nonsquare LTI MIMO discrete-time systems, generalizations of perfect regulation and perfect filtering are presented both for polynomial matrix and state space models. Consequently, general equivalence results are announced for multi-step and single-step optimal controls as well as for maximum-accuracy and maximum-speed controls for LTI MIMO discrete-time systems. The latter is made visible after the introduction of a new category of time-optimal control, namely infimum-time control. The equivalence conditions refer to the system's right-invertibility and the (newly-defined) minimum phase behavior, which demonstrates the usefulness of the new approach to zeros of multivariable systems.

Keywords: multivariable systems, system zeros, minimum phase systems, cheap control, perfect regulation, perfect filtering, time-optimal control.

1. Introduction

Both in robust control (Chen, Lin, Liu, 2002; Hermann, Spurgeon, Edwards, 2001; Kaczorek, 1999; Latawiec, 1998; Latawiec, Rojek, 2000; Liu, Chen, Lin, 2001; Middleton, 1991; Newman, 1990; Pao, Franklin, 1993; Seron, Braslavsky, Goodwin, 1997; Yaniv, Gutman, Chepovetsky, 1999) and, in particular, high-accuracy control (Åström, Wittenmark, 1989; Borisson, 1979; Chen, Lin, Liu, 2000, 2002; Clements, Anderson, 1978; Davison, 1983; Francis; 1979; Glizer, 1999; Kaczorek, 1999; Kimura, 1981; Kwakernaak, Sivan, 1972; Latawiec, 1978; Latawiec, Bańka, Tokarzewski, 1999; Latawiec, Hunek, 2002; Latawiec, Hunek, Łukaniszyn, 2004; Latawiec et al., 2003, Latawiec, Korytowski, Bańka, 2001; Latawiec, Korytowski, Rojek, 2001; Latawiec, Rojek, 2000; Lin et al., 1996; Liu, Chen, Lin, 2001; Marro, Prattichizzo, Zatoni, 2002; Middleton, 1991; Middleton, Braslavsky, 2000; Oloomi, Sawan, 1997; Saberi, Sanutti, 1987; Scherzinger, Davison, 1985) of linear multivariable systems, the problem of performance limitation (Goodwin, Seron, 1997; Havre, Skogestad, 2001; Middleton, 1991; Seron, Braslavsky, Goodwin, 1997; Yaniv, Gutman, Chepovetsky, 1999) is inevitably related to the minimum/nonminimum phase behavior of a system under control, that is to the notion of zeros of multivariable systems (Kaczorek, 1999; Latawiec, 1998; Latawiec, Bańka, Tokarzewski, 1999; Tokarzewski, 2002).

The elusive concept of zeros for *nonsquare* MIMO systems has raised polemics and discussions (Latawiec, 1998; Latawiec, Bańka, Tokarzewski, 1999), leading to some misinterpretations in design of both high-accuracy and robust controllers. The reason was an unclear notion of the minimum/nonminimum phase property for such systems. In fact, nonsquare systems generically (or "commonly", or "typically", or "almost always") have no transmission zeros and yet they may exhibit the nonminimum phase behavior (Latawiec, 1998; Latawiec, Bańka, Tokarzewski, 1999; Latawiec, Hunek, 2002; Latawiec, Hunek, Łukaniszyn, 2004; Latawiec et al., 2003). Confusingly, some authors happened to consider control problems for nonsquare minimum/nonminimum phase systems, relating the property to transmission/invariant zeros (Davison, 1983; Francis, 1979), which has been shown incorrect, in general (Latawiec, 1998; Latawiec, Bańka, Tokarzewski, 1999; Latawiec, Hunek, 2002; Latawiec, Hunek, Łukaniszyn, 2004; Latawiec et al., 2003).

A new concept of "control zeros", followed by a general redefinition of minimum/nonminimum phase systems, has been presented to fill the gap for nonsquare systems under perfect/output-zeroing/inverse-model/minimum-variance control (Latawiec, 1998; Latawiec, Bańka, Tokarzewski, 1999; Latawiec, Hunek, 2002; Latawiec, Hunek, Łukaniszyn, 2004; Latawiec et al., 2003). The concept has been utilized in the development of a new, simple but robust, multivariable predictive control strategy for nonminimum phase systems (Latawiec, 1998; Latawiec, Rojek, 2000). However, the new results, related to the new definitions of control zeros and minimum phase systems, appear not to be well known to the control community. Papers are still published, in which the classical, improper, transmission/invariant-zeros-based definition of minimum phase systems is employed to analyze and design control systems for nonsquare plants (Chen, Lin, Liu, 2002; Goldsmith, 2002; Havre, Skogestad, 2001; Hermann, Spurgeon, Edwards, 2001; Jeong, Choi, 2002; Liu, Chen, Lin, 2001; Saberi, Han, Stoorvogel, 2002). In spite of a very elegant, general mathematical framework presented in those papers, their results are unfortunately valid for square plants only.

This paper makes yet another attempt at rectifying some important issues related to (finite) multivariable zeros and minimum phase systems, with obvious implications to maximum-accuracy and maximum-speed controls for nonsquare discrete-time systems. The new results are presented in as simple way as possible in order to make them available to control engineers as well. As a matter of fact, the minimum/nonminimum phase behavior is essential when designing (e.g.) process control systems for (not only) nonsquare plants and yet so many sophisticated control designers cannot avoid the trap related to the misdefined minimum phase property. And control problems for nonsquare plants have increasingly been approached in e.g. control engineering practice, to mention only a few early 2002 editions of CEP (Bolek, Sasiadek, Wiszniewski, 2002; Leskens, Van Kessel, Van den Hof, 2002; Stenlund, Medvedev, 2002; Zhu, Butoyi, 2002).

In this paper, extending the results of Latawiec, Korytowski, Bańka (2001) and Latawiec, Korytowski, Rojek (2001), it is shown that the new definitions of control zeros and minimum phase systems contribute to nonsquare-system generalization of the familiar concepts of perfect regulation and perfect filtering. Consequently, the definitions conduce to formulation of new, general equivalence results on multi-step and single-step controls for LTI MIMO discrete-time systems. After the introduction of the notion of infimum-time control, it is also demonstrated how maximum-accuracy and maximum-speed controls can be equivalent to each other. All the equivalence results, in addition to the two definitions, are new contributions to the analysis and synthesis of control systems for nonsquare LTI MIMO discrete-time systems.

The remainder of the paper is organized as follows. The new definitions of control zeros and minimum/nonminimum phase systems, relevant to the perfect regulation problems considered, are recalled in Section 2. In Section 3, the maximum-accuracy control issues are discussed for systems governed by a polynomial matrix model. The zero-related perfect regulation and output zeroing problems are also considered in Section 4 for state space-modeled systems, and a general equivalence result for multi-step and single-step linear optimal controls is given. A dual, perfect filtering problem is briefly generalized in Section 5. In Section 6, a new category of time-optimal control, namely infimum-time control, is introduced and the equivalence results on maximum-accuracy and maximum-speed controls are given for systems modeled both by matrix polynomial and state space descriptions. New results of the paper are summarized in conclusions of Section 7.

2. Control zeros and nonminimum phase systems

Based on the output zeroing problem, which appears to be more general than the transmission blocking one (Latawiec, 1998; Latawiec, Bańka, Tokarzewski, 1999), a new definition of *control zeros* for possibly nonsquare MIMO systems has been introduced and the minimum/nonminimum phase property has been redefined in Latawiec (1998) and Latawiec, Bańka, Tokarzewski (1999). Here we present a slightly modified, more formally justified definition of control zeros. For clarity, we refer to the specific case of full normal rank transfer-function matrices. A general, non-full normal rank case has been considered in Latawiec (1998) and Latawiec, Bańka, Tokarzewski (1999).

Denote by $\mathbf{R}^{p\times r}(z)$ the set of all $p \times r$ matrices with entries in the field $\mathbf{R}(z)$ of rational functions in the complex variable z with real coefficients and let $\mathbf{R}^{p\times r}[z]$ be the set of all $p \times r$ matrices with entries in the ring $\mathbf{R}[z]$ of polynomials in z with real coefficients. Also denote by $\mathbf{\Gamma}$ the set of all rectangular, full normal rank matrices whose elements are rational functions in z with real coefficients. Consider an LTI discrete-time n_u-input n_y-output system governed by the transfer-function matrix $G \in \mathbf{R}^{n_y \times n_u}(z)$ of full normal rank $n_y \leq n_u$, having the left coprime matrix fraction description (MFD) form $\underline{A}^{-1}(z)\underline{B}(z) = z^{-d}A^{-1}(z^{-1})B(z^{-1})$, with $\underline{A} \in \mathbf{R}^{n_y \times n_y}[z]$, $\underline{B} \in \mathbf{R}^{n_y \times n_u}[z]$ and d being the time delay.

"Symmetrical" considerations can be made for the transfer-function matrix G of full normal rank $n_u \leq n_y$, having its right coprime MFD (Latawiec, 1998; Latawiec, Bańka, Tokarzewski, 1999), but the related issues are not relevant to the perfect regulation problems considered. For every $n_y \times n_u$ matrix $G \in \mathbf{\Gamma}$, its inverse is defined as an $n_u \times n_y$ matrix $G^+ \in \mathbf{\Gamma}$, where G^+ is (any) right or left inverse of G, to be denoted by G^R and G^L, respectively. Of particular importance is the case to follow when G^+ is the (unique) T-minimum norm right inverse or T-least squares left inverse, to be denoted by $G_0^{R_1} = z^d B_0^{R_1} A = z^d B^T(BB^T)^{-1}A$ and $G_0^{L_1} = z^d A B_0^{L_1} = z^d A(B^T B)^{-1}B^T$, respectively.

Another important case to follow is when G^+ is the (unique) inverse $G_0^{R_2} = z^d B_0^{R_2} A = z^d\{I + (b_0)_0^R[B - b_0]\}^{-1}(b_0)_0^R A$ or $G_0^{L_2} = z^d A B_0^{L_2} = z^d A\{I + (b_0)_0^L[B - b_0]\}^{-1}(b_0)_0^L$, where $(b_0)_0^R$ and $(b_0)_0^L$ are respectively the minimum norm right and least squares left inverses of b_0, the leading coefficient (matrix) of $B(z^{-1})$. In the sequel, we exploit the specific case of right-invertible systems, which is relevant to the perfect regulation problem considered.

Definition 2.1 Let an n_u-input n_y-output LTI discrete-time system be described by the transfer-function matrix $G \in \mathbf{\Gamma}$ of full normal rank $n_y \leq n_u$, having the left coprime MFD form $\underline{A}^{-1}(z)\underline{B}(z) = z^{-d}A^{-1}(z^{-1})B(z^{-1})$, with $\underline{A} \in \mathbf{R}^{n_y \times n_y}[z]$, $\underline{B} \in \mathbf{R}^{n_y \times n_u}[z]$ and d being the time delay. Let c be a function $\mathbf{\Gamma} \to \mathbf{\Gamma}$ such that $c(G) = G^R = z^d B^R A$ is a specific right inverse of G, for every $G \in \mathbf{\Gamma}$. The *c-inverse system* is defined as the system with the *transfer-function matrix* $c(G)$. In particular, the c-inverse system involving either the $G_0^{R_1}$ or $G_0^{R_2}$ inverses of G will be called the c_0-inverse system.

Remark 2.1 Depending on the context, the original system described by the full normal rank transfer-function matrix $G \in \mathbf{\Gamma}$ will be referred to as c-invertible or c_0-invertible, the latter being related to its c_0-inverse system.

Definition 2.2 The complex number ζ is called a *control zero* of the LTI discrete-time system described by the full normal rank transfer-function matrix $G \in \mathbf{\Gamma}$ iff ζ is a pole of the transfer-function matrix $c(G)$ of the c_0-inverse system.

Remark 2.2 The new definition of control zeros is embedded in the concept of output-zeroing/inverse-model/minimum-variance/perfect control (Latawiec, 1998; Latawiec, Bańka, Tokarzewski, 1999), the *stability* of which is related to the properties of a (generalized) inverse of $G(z)$. (It is interesting to note that operating with minimum norm right or least squares left inverses, involving *conjugate* transposes and associated with output *transform* zeroing control, has been conjectured to end up with transmission zeros, Latawiec et al., 2003).

Definition 2.3 (Latawiec, 1998; Latawiec, Bańka, Tokarzewski, 1999; Latawiec, Hunek, 2002; Hunek, 2003) An LTI system described by the full normal rank transfer-function matrix is called *minimum phase* iff its c_0-inverse system is asymptotically stable; otherwise the system is called *nonminimum phase*.

Remark 2.3 It is obvious that the notion of a c-inverse system is nonunique for *nonsquare* systems. Of a plethora of c-inverse systems available, including those related to "squaring" the original system down, we distinguish such zeros which are associated with a c_0-inverse system of full dimensions $n_u \times n_y$ (otherwise some input(s) or output(s) could be removed). The interest in c_0-inverse systems is additionally justified by the following

Example 2.1 Consider a nonsquare discrete-time system $y(t) + a_1 y(t-1) + a_2 y(t-2) = b_0 u(t-1) + b_1 u(t-2) + b_2 u(t-3)$, with the polynomial matrix $B(q^{-1}) = b_0 + b_1 q^{-1} + b_2 q^{-2}$ being right-invertible, where q^{-1} is the backward shift operator. Equating the 1-step output predictor to zero (or, in general, to an output reference) we obtain

$$b_0 u(t) + b_1 u(t-1) + b_2 u(t-2) - a_1 y(t) - a_2 y(t-1) = 0 \tag{E1}$$

On the one hand, equation (E1) immediately leads to the perfect regulation (or, in the stochastic case, minimum variance regulation) law

$$u(t) = (b_0 + b_1 q^{-1} + b_2 q^{-2})^R [a_1 y(t) + a_2 y(t-1)] \tag{E2}$$

But on the other hand, assuming that b_0 is of full rank, equation (E1) can be given the alternative form $u(t) = (b_0)^R [y_{ref}(t+1) - b_1 u(t-1) - b_2 u(t-2) + a_1 y(t) + a_2 y(t-1)]$, which can be rewritten as

$$u(t) = [I + (b_0)^R (b_1 q^{-1} + b_2 q^{-2})]^{-1} (b_0)^R [a_1 y(t) + a_2 y(t-1)] \tag{E3}$$

Although both control laws (E2) and (E3) are derived from the *same* predictor as in (E1), it is rather surprising that these laws are different in general and this is because $B^{R_1}(q^{-1}) = (b_0 + b_1 q^{-1} + b_2 q^{-2})^R \neq [I + (b_0)^R (b_1 q^{-1} + b_2 q^{-2})]^{-1} (b_0)^R = B^{R_2}(q^{-1})$, in general. The difference results from the specific properties of right inverses for *polynomial matrices*.

Since the two control laws can clearly be considered *time-domain* equations, our interest in the $B_0^{R_1}(q^{-1})$ and $B_0^{R_2}(q^{-1})$ inverses, including regular rather than conjugate transposes, is obvious. The control zeros related to equation (E2) and generated by the inverse $B_0^{R_1}(q^{-1})$ are the "classical" control zeros introduced in Latawiec (1998), Latawiec, Bańka, Tokarzewski (1999), which will be referred to as type 1 control zeros. Another control zeros, related to equation (E3) and different in general from type 1 control zeros, are generated by the inverse $B_0^{R_2}(q^{-1})$ and called type 2 control zeros. Type 2 control zeros have been demonstrated to have more attractive properties than their type 1 counterparts; see Latawiec, Hunek, Łukaniszyn (2004) for numerical calculations of both types of control zeros for specific minimum phase and nonminimum phase systems (having no transmission zeros!).

The following two theorems provide means for calculation of control zeros type 1 and type 2.

THEOREM 2.1 *(Latawiec, 1998; Latawiec, Bańka, Tokarzewski, 1999). Consider an LTI discrete-time system governed by the transfer-function matrix* $G \in \boldsymbol{R}^{n_y \times n_u}(z)$ *of full normal rank* $n_y \leq n_u$*, having the left coprime MFD form* $\underline{A}^{-1}(z)\underline{B}(z) = z^{-d}A^{-1}(z^{-1})B(z^{-1})$*, with* $\underline{A} \in \boldsymbol{R}^{n_y \times n_y}[z]$*,* $\underline{B} \in \boldsymbol{R}^{n_y \times n_u}[z]$ *and* d *being the time delay. Let its* c_o*-inverse system involve the* T*-minimum norm right inverse* $G_0^{R_1} \in \boldsymbol{R}^{n_u \times n_y}(z)$ *and let* $\{\beta_i,\ i = 1,\dots,M\}$ *be the set of all square submatrices of dimension* n_y *of the polynomial matrix* $B(z^{-1})$ *(or* $\underline{B}(z)$*), with* $M = n_u!/[n_y!(n_u - n_y)!]$*. Then the system is minimum phase if all zeros of* $\sum_{i=1}^{M}\ [\det(\beta_i)]^2$*, that is all* type 1 control zeros*, lie inside the unit disc.*

A dual result can be obtained for $n_y \geq n_u$ and a system described by the right coprime MFD form (Latawiec, 1998; Latawiec, Bańka, Tokarzewski, 1999).

THEOREM 2.2 *Consider an LTI discrete-time system governed by the transfer-function matrix* $G \in \boldsymbol{R}^{n_y \times n_u}(z)$ *of full normal rank* $n_y \leq n_u$*, having the left coprime MFD form* $\underline{A}^{-1}(z)\underline{B}(z) = z^{-d}A^{-1}(z^{-1})B(z^{-1})$*, with* $\underline{A} \in \boldsymbol{R}^{n_y \times n_y}[z]$*,* $\underline{B} \in \boldsymbol{R}^{n_y \times n_u}[z]$ *and* d *being the time delay. Let its* c_o*-inverse system involve the right inverse* $G_0^{R_2} \in \boldsymbol{R}^{n_u \times n_y}(z)$*. Then the system is minimum phase if all zeros of* $\det\{I + (b_0)_0^R[B(z^{-1}) - b_0]\} = 0$*, that is all* type 2 control zeros*, lie inside the unit disc.*

Proof will follow immediately from the perfect regulation stability result of the next section.

The above output zeroing control-related definitions present a (nonsquare) MIMO system generalization of those holding for SISO systems. Specifically, control zeros of both type 1 and type 2 for square invertible MIMO (including SISO) systems coincide with transmission zeros. Moreover, transmission zeros, *if any*, of nonsquare c_o-invertible systems make a *subset* of control zeros of both type 1 (Latawiec, 1998; Latawiec, Bańka, Tokarzewski, 1999) and type 2 (Latawiec, Hunek, Łukaniszyn, 2004).

REMARK 2.4 The above definitions can be easily extended to include a state space model and its possible decoupling zeros, in addition to control zeros. Redefining (new, say c-)invariant zeros as those including control zeros and decoupling zeros, the whole analysis to follow could be repeated for the state space description, with so defined invariant zeros substituted for control zeros. We will continue with control zeros (and controllable and observable systems) however, as the aforementioned controversies on multivariable zeros concern transmission zeros exclusively (decoupling zeros are free from any misinterpretations).

3. Output zeroing control for matrix polynomial-modeled systems

Consider an n_u-input n_y-output LTI discrete-time system governed by the matrix polynomial model (being a special case of the ARX model)

$$A(q^{-1})y(t) = q^{-d}B(q^{-1})u(t) \tag{1}$$

where $\underline{A} \in \mathbf{R}^{n_y \times n_y}[z]$, with $\underline{A}(0) = I$, and $\underline{B} \in \mathbf{R}^{n_y \times n_u}[z]$ are left-coprime, with $\underline{A}(z) = z^n A(z^{-1})$ and $\underline{B}(z) = z^m B(z^{-1})$, and n and m are the orders of the respective matrix polynomials. Unless necessary, we will not distinguish between $A(z^{-1})$ and $\underline{A}(z)$, nor between $B(z^{-1})$ and $\underline{B}(z)$, especially that $\underline{A}^{-1}(z)\underline{B}(z) = z^{m-n}A^{-1}(z^{-1})B(z^{-1}) = G(z)$, with $n - m = d$.

THEOREM 3.1 *Let an LTI system be described by the model (1), with* $\underline{A} \in \boldsymbol{R}^{n_y \times n_y}[z]$ *and* $\underline{B} \in \boldsymbol{R}^{n_y \times n_u}[z]$ *being left-coprime. Then the minimum-norm regulation (MNR) law, minimizing the (single-step) performance index* $\|y(t+d)\|^2 = y^T(t+d)y(t+d)$*, is of the form*

$$u(t) = -[F(q^{-1})B(q^{-1})]^{\#}H(q^{-1})y(t) \tag{2}$$

where $t \in \{0, 1, 2, \dots,\}$*,* $(.)^{\#}$ *is a generalized inverse of* $(.)$ *and the appropriate polynomial matrices* $F \in \boldsymbol{R}^{n_y \times n_y}[z]$ *and* $H \in \boldsymbol{R}^{n_y \times n_y}[z]$ *(of order d-1 and n-1, respectively) are computed from the matrix polynomial identity* $I = F(z^{-1})A(z^{-1}) + z^{-d}H(z^{-1})$.

Proof. Premultiplying both sides of equation (1) by $F(q^{-1})$ and using the above polynomial identity one can easily arrive at the familiar output predictor $y(t+d) = Hy(t) + FBu(t)$ (which is precisely the same here as the deterministic part of the output predictor for the stochastic square MIMO case, Borisson, 1979, Latawiec, 1998). The result follows (asymptotically) from the minimization of the performance index. ■

REMARK 3.1 Notice that MNR is a special case of minimum variance control (MVC). Also note that the MNR and zero-reference (white-noise) MVC laws are identical. Of course, MNR (or, for stochastic systems, MVC) provides the *maximum achievable accuracy* to the control system. Notice that, in general, the minimum value of the performance index under the control law (2) does not have to necessarily be zero. The result below provides the condition under which the performance index under the control law (2) can reach its absolute minimum of zero.

COROLLARY 3.1 *(Latawiec, 1998) Let an LTI system be described by the model (1), with* $\underline{A} \in \boldsymbol{R}^{n_y \times n_y}[z]$ *and* $\underline{B} \in \boldsymbol{R}^{n_y \times n_u}[z]$ *being left-coprime. Assume that the MNR law (2) stabilizes the control system. Then MNR provides output-zeroing control (OZC), i.e.* $y(t+d)=0$*, if* $\operatorname{rank}_{R[z]} \underline{B}(z) = n_y \le n_u$*, that is if* $\underline{B}(z)$ *is right-invertible.*

REMARK 3.2 Note that the right-invertibility condition on $\underline{B}(z)$ is only sufficient. In fact, zeroing of $y(t+d)$ may also occur in a "nongeneric" case of the range of H contained in the range of FB.

REMARK 3.3 Note that OZC provides an upper bound for the maximum achievable accuracy of MNR, or the zero control inaccuracy. This is usually referred to as *perfect regulation*.

Taking account of the right-invertibility condition, the OZC/perfect regulation law will thus be recalled in the form

$$u(t) = -B^R(q^{-1})F^{-1}(q^{-1})H(q^{-1})y(t) \tag{3}$$

with a special attention paid to the right inverses $B_0^{R_1}(q^{-1})$ and $B_0^{R_2}(q^{-1})$, generating control zeros type 1 and type 2, respectively.

REMARK 3.4 It is interesting to note that

$$\begin{aligned}[F(q^{-1})B(q^{-1})]_0^{R_2} &= \{I + (b_0)_0^R[F(q^{-1})B(q^{-1}) - b_0]\}^{-1}(b_0)_0^R \\ &= B_0^{R_2}(q^{-1})F^{-1}(q^{-1}).\end{aligned}$$

This property copies the "classical" one:

$$[F(q^{-1})B(q^{-1})]_0^{R_1} = B_0^{R_1}(q^{-1})F^{-1}(q^{-1}).$$

(Note: The leading coefficient of $F(q^{-1})$ is the identity matrix.)

Let us now examine the issue of stability of the OZC/perfect regulation system.

THEOREM 3.2 *(Latawiec, 1998). Let an LTI system be described by the model (1), with $\underline{A} \in \boldsymbol{R}^{n_y \times n_y}[z]$ and $\underline{B} \in \boldsymbol{R}^{n_y \times n_u}[z]$ being left coprime, and* rank $_{R[z]}\,\underline{B}(z) = n_y \leq n_u$. *Then the OZC/perfect regulation law (3) (identical with the zero-reference MVC one) is asymptotically stable iff $\underline{B}(z)$ is stably (right-)invertible, i.e. iff the system is minimum phase.*

REMARK 3.5 The minimum phase property must be understood precisely in the sense of the new Definition 2.3 (and, generically, not any other, in particular not the specific definition based on transmission zeros (Davison, 1983; Francis, 1979). In fact, control zeros lying outside or on the unit circle can make the OZC/MVC/perfect regulation system unstable (Latawiec, 1998; Latawiec, Bańka, Tokarzewski, 1999).

REMARK 3.6 For example, a two-input single-output c_0-invertible system described by the transfer function matrix $[z-2 \quad z-1]/(z-3)(2z-5)$ has *no transmission zeros* but 1) is *nonminimum phase* and so 2) its OZC/perfect regulation (3) (or in the stochastic case, minimum variance control) is *unstable* under both $B_0^{R_1}(q^{-1})$ and $B_0^{R_2}(q^{-1})$, which can be easily induced from its *control zeros type 1* equal to $1.5\pm 0.5i$ or *control zeros type 2* equal to $0.75\pm 1.3229i$. Latawiec, Hunek, Łukaniszyn (2004) present less obvious examples, with more favorable properties of type 2 control zeros indicated.

Remark 3.7 Notice that the above perfect regulation conditions, as well as the forthcoming ones for the state space description, are independent of the structure of infinite zeros. In fact, output zeroing controller or perfect regulator has, in general, nothing to do with a high-gain controller. Well, except for the case when a system is ("close" to) nonminimum phase, precisely in the sense of our Definition 2.3.

4. Perfect regulation/output zeroing control of state space systems

Consider the problem of minimization of the (multi-step) control performance index

$$J_\rho(x_o, u) = \sum_{t=0}^{\infty} [\|y(t+1)\|^2 + \rho \|u(t)\|^2] \quad \rho > 0 \tag{4}$$

subject to

$$x(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0 \tag{5a}$$

$$y(t) = Cx(t) \tag{5b}$$

where $u(t) \in \mathbf{R}^{n_u}$, $y(t) \in \mathbf{R}^{n_y}$, $x(t) \in \mathbf{R}^{n}$ and A, B, C are the parameter matrices of appropriate dimensions. To skip over possible trivial solutions we assume that x_o and A are nonzero.

Let the system be controllable and observable and let the control sequence $u \in \mathbf{U}$, where $\mathbf{U}$ is the class of bounded controls for which

$$\text{i. } \lim_{t\uparrow\infty} x(t) = 0 \tag{6a}$$

$$\text{ii. } \sum_{t=0}^{\infty} \|u(t)\|^2 < \infty \tag{6b}$$

The familiar optimal control law is

$$u(t) = -(\rho I + B^T SB)^{-1} B^T SAx(t) \tag{7}$$

where $S = S^T \geq 0$ is a solution to the algebraic matrix Riccati equation

$$S = R + A^T SA - A^T SB(\rho I + B^T SB)^{-1} B^T SA \tag{8}$$

with $R = C^T C$.

Remark 4.1 For due comparison of matrix polynomial and state space descriptions, controllability and observability of a system is assumed throughout the paper. In this way we limit ourselves to the consideration of control zeros only. Of course, it would suffice for state space-modeled systems to be assumed stabilizable and detectable, in which case decoupling zeros would be additionally involved. However, it would not take us any further (compare Remark 2.3).

The so-called "cheap" optimal control (Bikdash, 1993; Francis, 1979; Glizer; 1999; Latawiec, 1998; Marro, Prattichizzo, Zatoni, 2002; Oloomi, Sawan, 1997; Saberi, Sanuti, 1987; Scherzinger, Davison, 1985) is obtained as $\rho \downarrow 0$ and is a particular case of singular optimal control (Clements, Anderson, 1978; Silverman, 1981). Such a type of optimal control plays a role in cases where unweighted controls are admissible (and stable, of course). There appears to be two main problems that are of practical interest when exploiting the cheap optimal control: 1) when is it possible for the control performance index (4) to go to zero (perfect regulation), and 2) when is a perfect regulation system asymptotically stable. Here we present new general solutions to these problems, apparently associated with the issue of the control zeros.

The state-space perfect regulation problem for linear continuous-time systems has been first solved by Kwakernaak and Sivan (1972) and generalized by Francis (1979). Later elegant contributions of Kimura (1981), Scherzinger and Davison (1985), Lin et al. (1996) and Chen, Liu, Lin (2000), just like that of Francis (1979), have all been unfortunately related to the improper definition of minimum phase systems based on transmission/invariant zeros (Davison, 1983; Francis, 1979). Therefore, their results are valid for square systems only. A state-space approach based on the new definition of the minimum/nonminimum phase property is presented below to solve the problem in a general way for possibly nonsquare discrete-time systems.

Lemma 4.1 *A sufficient condition for the performance index (4) to go to its absolute minimum as $\rho \downarrow 0$, that is*

$$\lim_{\rho \downarrow 0} \min_{u} J_\rho(x_o,\ u) = J_0(x_o) \tag{9}$$

is the existence a certain control sequence $u \in \boldsymbol{U}$.

Proof. Recall that the minimal value of the performance index is nonincreasing with decreasing ρ (compare Clements, Anderson, 1978, Silverman, 1976) so that the absolute minimum (whether zero or not) of the cost functional can be obtained as $\rho \downarrow 0$. Now, as $\rho \downarrow 0$ the second term of the right-hand side of (4) vanishes if $u \in \mathbf{U}$. ■

Lemma 4.2 *(limiting solution to the matrix Riccati equation, Latawiec, 1998; Latawiec, Korytowski, Rojek, 2001). Let CB be right-invertible. Then the limiting solution to (8) as $\rho \downarrow 0$ is $S_o = R$.*

REMARK 4.2 It is essential that employing the generalized inverse concept in the perfect regulation problem enables putting in the limit $\rho = 0$ (not only for SISO systems where the regular inverse of $b^T S_o b$ exists, except for trivial cases).

Lemmas 4.1 and 4.2 lead to the following

THEOREM 4.1 *Consider a system described by equations (5) to be controlled according to the performance index (4). $J_\rho(x_o, u)$ tends to 0 as $\rho \downarrow 0$ for every x_o under the feedback control law*

$$u(t) = -(CB)^{\#} CAx(t) \tag{10}$$

if the control sequence $u \in \boldsymbol{U}$ and (CB) is right-invertible.

Proof. From Lemmas 4.1 and 4.2 it follows that, under $u \in \mathbf{U}$, a *much simpler* task of *zeroing the output* of a system (while maintaining the system stable) can now be equivalently considered instead of the LQ-oriented perfect regulation problem (Clements, Anderson, 1978; Silverman, 1976). Now, pursuing the zero value of the cost functional as $\rho \downarrow 0$ we demand that $y(t+1) = CAx(t) + CBu(t) = 0$. The best (minimum norm least squares) approximate solution for $u(t)$ is derived as in (10), which drives the output $y(t+1) = [I - CB(CB)^{\#}]CAx(t)$ to zero if CB is right-invertible, i.e. if rank $(CB) = n_y \leq n_u$. ■

REMARK 4.3 Note that the right-invertibility condition is only sufficient. In fact, canceling $y(t+1)$ may also independently occur in some “nongeneric” cases, e.g. if the range of CA is contained in the range of CB; but this range condition is not necessary because of the behavior of $x(t)$ which might be stuck on a low dimensional subspace.

COROLLARY 4.1 *Perfect regulation can generically be achieved if $n_u \geq n_y$.*

Proof. Immediate. ■

We will further refer to the state space-formulated perfect regulation law

$$u(t) = -(CB)^R CAx(t)\,. \tag{11}$$

REMARK 4.4 As $\rho \downarrow 0$, the solution $S_o = R$, along with the associated CB right-invertibility condition, converts the *multi-step* (MS) optimal control law (7) directly to (10), which minimizes the *single-step* (SS) control performance index $\|y(t+1)\|^2$, or zeroes the output according to the requirement $y(t+1) = CAx(t) + CBu(t) = 0$. Thus, under the (stabilizing) control law (11) we have $\lim_{\rho \downarrow 0} \min_u \sum_{t=0}^{\infty} [\|y(t+1)\|^2 + \rho \|u(t)\|^2] = \min_u \|y(t+1)\|^2 = 0$.

COROLLARY 4.2 *Let the matrix polynomial model (1) be obtained from the state space description (5) of a system. Then the (stabilizing) perfect regulation laws (3) and (11) are equivalent.*

Proof. Referring to the transfer function matrix $G = C(zI - A)^{-1}B = z^{-1}C(I - z^{-1}A)^{-1}$ of a state space-described system, the model (5) can be rewritten in the form of equation (1), with $A(q^{-1}) = \det(I - q^{-1}A)$, $B(q^{-1}) = C\,\mathrm{adj}(I - q^{-1}A)B$ and d=1. Recalling that the considered perfect regulation laws are designed to satisfy the output zeroing requirement $y(t+1) = H(q^{-1})y(t) + F(q^{-1})B(q^{-1})u(t) = 0 = CAx(t) + CBu(t)$, the result follows. ■

The problem of when is a perfect regulation system asymptotically stable will be approached using the transfer-function matrix representation.

THEOREM 4.2 *Let the irreducible transfer-function matrix* $C(zI-A)^{-1}B = G \in \boldsymbol{R}^{n_y \times n_u}(z)$ *of a system (5) be of full normal rank* $n_y \leq n_u$*. Then* $J_\rho(x_o,u)$ *tends to 0 as* $\rho \downarrow 0$ *(or* ρ*=0) for every* x_o *under the control law (11) and the control system is asymptotically stable iff the system under control is minimum phase.*

Proof. By virtue of Theorems 3.2 and 4.1 and Corollary 4.2 we can state that the requirement on $J_\rho(x_o,u)$ to tend to 0 as $\rho \downarrow 0$ for every x_o is equivalent to the existence of a stable right-inverse of $G(z)$ (or $B(z)$), which occurs, in view of Definition 2.3, for minimum phase systems only. Now it suffices to note that the control law (11), with $G(z)$ stably right-invertible, belongs to the set **U** and the result follows. ■

REMARK 4.5 Again, the minimum phase condition must be understood precisely in the sense of Definition 2.3 (and, generically, not any other), with the immediate relationship to control zeros (and, generically, not any other multivariable zeros).

REMARK 4.6 Alternatively, when the input (11) is applied to equation (5a), the solution takes the form $x(t) = \{[I - B(CB)^R C]A\}^t x_o$, $t = 1, 2, \ldots$, and equation (11) assumes the form $u(t) = -(CB)^R CA\{[I - B(CB)^R C]A\}^t x_o$. Thus, $u(t)$ defined by (11) satisfies equation (6a) if and only if the matrix $[I - B(CB)^R C]A$ has all its eigenvalues inside the unit disc, the condition to be commented in final remarks.

REMARK 4.7 Possible "nongeneric" solutions for both state-space and transfer function approaches, omitting the right-invertibility condition, have not so far been taken into account, leading to less general results than the above presented. In addition, making use of an improper definition of the minimum phase property, based on transmission rather than control zeros, has made all the previous results valid in the particular, square-system cases only.

The above results lead immediately to the following important

THEOREM 4.3 *(equivalence of multi-step and single-step optimal controls, Latawiec, 1998, Latawiec, Tokarzewski, Bańka, 1999). Let a system be described by the state space model (5) and assume that the considered MS and SS control systems are asymptotically stable. Then the limiting MS linear optimal control law (7) and the SS one of equation (10) are equivalent if* $\mathrm{rank}\,(CB) = n_y \leq n_u$.

REMARK 4.8 Note that the above condition is only sufficient and it covers the generic cases only.

REMARK 4.9 The equivalence of the limiting MS and SS controls is no longer valid if CB is not right-invertible (generically, again), i.e. for 1) $n_y > n_u$ and 2) $n_y \leq n_u$ and $rank(CB) < n_y$. In fact, in these cases there is $S_o \neq R$.

Resulting directly from Theorem 4.3 is

COROLLARY 4.3 *Generically, the considered limiting MS and SS controls are equivalent for stably right-invertible systems, i.e. for right-invertible minimum phase systems.*

REMARK 4.10 This result is valid for the external model description as well. Also see Åström, Wittenmark (1989), Borisson (1979) for some special cases (SISO and square MIMO systems).

We can summarize the above results and state quite generally (including *nongeneric* solutions) as follows:

COROLLARY 4.4 *Let a system be described by the state space model (5). Then the considered limiting MS and SS controls are equivalent in all cases where asymptotically stable perfect regulation is achievable.*

Let us now briefly extend the above results to

Control-delayed systems (Latawiec, 1998)

Let a linear discrete-time system be governed by

$$x(t+1) = Ax(t) + Bu(t-d+1) \qquad x(0) = x_o \tag{12a}$$

$$y(t) = Cx(t) \tag{12b}$$

and the control performance index to be minimized:

$$J_\rho(x_o, u) = \sum_{t=0}^{\infty} \left[\|y(t+d)\|^2 + \rho \|u(t)\|^2 \right] \quad \rho \geq 0. \tag{13}$$

To skip over trivial cases we assume that $x_o \neq 0$ and A is non-nilpotent.

Now, the optimal control law, minimizing (13) as $\rho \downarrow 0$ *or* $\rho = 0$, is

$$\begin{aligned} u(t) &= -(CB)^{\#} CAx(t+d-1|t) \\ &= -(CB)^{\#} CA[A^{d-1}x(t) + \sum_{i=1}^{d-1} A^{i-1}Bu(t-i)] \end{aligned} \tag{14}$$

and, with obvious modifications to $G(z) = z^{-d+1}C(zI-A)^{-1}B$ and the output zeroing requirement $y(t+d) = 0$, *all* the above results concerning perfect regulation for nondelayed (or rather single-step delayed) systems are valid.

Remark 4.11 Note that under the assumption that the first nonzero input is $u(0)$ we have in case of perfect regulation of the system (12)

$$\sum_{t=0}^{\infty} \|y(t)\|^2 = \|x_o\|^2_{\underline{S}} + \sum_{t=0}^{\infty} \|y(t+d)\|^2 = \|x_o\|^2_{\underline{S}} \tag{15}$$

where $\underline{S} = \sum_{i=0}^{d-1} (A^i)^T R A^i$.

5. Perfect filtering

A dual, perfect filtering problem has also been approached without consideration of control zeros (Braslavsky et al., 1999; Goodwin, Seron, 1997; Seron, Braslavsky, Goodwin, 1997), thus making the obtained results valid for squares systems only. Here we briefly present general perfect filtering generalizations for nonsquare LTI MIMO discrete-time systems.

Consider a stochastic system modeled by

$$x(t+1) = Ax(t) + Bu(t) + v_1(t), \quad x_o = x(0) \tag{16a}$$

$$y(t) = Cx(t) + v_2(t) \tag{16b}$$

where $v_1$and v_2 are zero-mean, Gaussian, white noise disturbances with intensities $R_1 = WW^T \geq 0$ and $\sigma R_2 \geq 0$, respectively, and σ is a nonnegative scalar factor.

It is well known that using the optimal filter/predictor

$$\hat{x}(t+1) = A\hat{x}(t) + Bu(t) + K[y(t) - C\hat{x}(t)] \tag{17}$$

we obtain

$$\lim_{t\uparrow\infty} E[\|e(t)\|^2] = \operatorname{tr} Q \tag{18}$$

where $K = AQC^T(\sigma R_2 + CQC^T)^{\#}$ is the stationary Kalman filter gain, $e(t)$ is the state estimation error and $Q = Q^T \geq 0$ is the solution of the equation

$$Q = R_1 + AQA^T - AQC^T(\sigma R_2 + CQC^T)^{\#} CQA^T\,. \tag{19}$$

Assuming that A is nonsingular and there are no pole-zero cancellations in the transfer-function matrix $G_f(z) = C(zI - A)^{-1}W$ we can directly utilize the control/filtering duality concept and state that

$$\lim_{\sigma\downarrow 0} Q = R_1\,. \tag{20}$$

It is apparent that, unlike for continuous-time systems, the estimation error cannot be in general reduced to zero by decreasing the measurement noise to zero.

Similarly, we can state that the "perfect" filtering requirement

$$\lim_{\substack{t\uparrow\infty \\ \sigma\downarrow 0}} E[\|e(t)\|^2] = \text{tr } R_1 \tag{21}$$

is equivalent to the existence of a stable left-inverse of the transfer-function matrix $G_f(z)$ (compare Francis, 1979). In particular, its minimum phase property is necessary for perfect filtering. Moreover, this is also critical for the system detectability and so numerical stability/convergence of the perfect filter.

Equally, the system detectability and left-invertibility of the CW matrix can be required for perfect filtering.

When the matrix A is singular the statements related with (20) and (21) are no longer valid. In such cases perfect filtering problems are more complex than perfect regulation ones and no general solutions are available yet.

6. Infimum-time control

Although rather seldom exploited in control practice (due to the lack of robustness), perfect regulation/control or, in the stochastic case, minimum variance control have above been shown to constitute a reference for maximally achievable accuracy of optimal controllers. The new maximum-accuracy results have been derived on a basis of both matrix polynomial and state space models. In particular, the matrix polynomial approach has in a natural way been based on calculation of a (deterministic) predictor of the controlled output. The d-step output predictor $y(t+d)$, d being the plant delay, was designed to reach the setpoint (in particular, zero), which thus constituted the maximum achievable speed of the control system (except for some trivial cases). This has raised the question of possible relationship of such maximum-speed control with familiar time-optimal control. The relationship is here illustrated on a basis of a new-introduced concept of "infimum-time control". Consequently, general conditions for equivalence of maximum-accuracy and maximum-speed controls are presented, the conditions being apparently related to our control zeros.

Definition 6.1 Discrete-time control is defined as *infimum-time output control* iff the system output is driven to the origin in *at most* d time steps, where d is the time delay.

The above definition also covers trivial cases of zero initial conditions and nilpotent matrix A, for which the origin can be reached in less than d time steps. We will further concentrate on the nontrivial cases of nonzero initial conditions and non-nilpotent matrix A. In such cases, the infimum possible time for the output of *a time-delayed system* to reach zero is just the delay d. Infimum-time control thus constitutes an upper bound for maximum achievable speed of a control system. Now, referring to the perfect regulation result of Theorem 3.2 we have immediately

THEOREM 6.1 *Let an LTI system be described by the model (1), with* $\underline{A} \in \boldsymbol{R}^{n_y \times n_y}[z]$ *and* $\underline{B} \in \boldsymbol{R}^{n_y \times n_u}[z]$ *left-coprime, and let* $\underline{B}(z)$ *be stably right-invertible (that is the system be minimum phase and right-invertible). Then the perfect regulation law (3) provides infimum-time output control.*

Proof. Under the above assumptions, the perfect regulation law (3) drives the output predictor $y(t+d) = Hy(t) + FBu(t)$, $t \in \{0, 1, 2, ..., \}$ (see Theorem 3.1), to zero within at most d steps. Accounting for Definition 6.1, the result follows. ■

REMARK 6.1 Again, the minimum phase requirement must be understood precisely in the sense of the new Definition 2.3 and not any other, in particular not that of Davison (1983)and Francis (1979). Otherwise, unstable modes related to undetected control zeros lying outside or on the unit circle might not be taken into account.

The above theorem is important in that it specifies the general conditions for the equivalence of (stabilizing) maximum-accuracy and maximum-speed controls for systems described by the model (1).

The infimum-time control problem will be state-space approached now. Familiar, state space-formulated deadbeat or time-optimal control consists in the construction of a control sequence to drive the discrete system state to the origin in a minimum number of time steps. For the state equation as in (5a), this minimum number $\nu \leq n$ is referred to as the reachability or controllability index. Various approaches to solve the discrete time-optimal control problem have been reviewed by O'Reilly (1981). Here we present an extension of the state space-formulated time-optimal control problem by recalling our new category of *infimum-time control* and we demonstrate its close relationship to perfect regulation. The reason for distinguishing the infimum-time control category from a class of time-optimal controls is fourfold. Firstly, the minimum number ν is by no means the infimum number of time steps, in which it is possible for the system state to reach the origin (even though generically it is). Secondly, the notion of minimum-time control, which is often used equivalently to time-optimal control, is by no means related to the infimum number of time steps to drive the system state to zero (even though generically it is). Thirdly, although infimum-time control is considered a special case of time-optimal control, many available time-optimal control algorithms (O'Reilly, 1981) could not be rearranged so that they would include the infimum-time control solution as a special case. Last not least, unlike perfect regulation marking an upper bound for the achievable accuracy of optimal control, there has been no such category introduced for time-optimal control, which would constitute an upper bound for the achievable control speed.

We consider a general, control-delayed, controllable and observable system governed by the model (12), under the standard assumptions on B and C being of full rank.

Definition 6.2 Discrete-time control is defined as *infimum-time state control* iff the system state, governed by (12a), is driven to the origin in at most d time steps.

Remark 6.2 Again, infimum-time state control provides an upper bound for the maximum achievable speed of the state controller.

Lemma 6.1 *Let the matrices CB and C be right- and left-invertible, respectively. Then $B(CB)^R C = I$.*

Proof. From right invertibility of CB it follows that C is right invertible. Since C is also left invertible, C is square and nonsingular. Now, the relation $CB(CB)^R = I$ can be premultiplied by C^{-1} and postmultiplied by C. ■

Theorem 6.2 *Let the system state be described by the model (12a), with C (regularly) invertible. Then the (stabilizing) control law (14) provides infimum-time state control if the matrix CB is right-invertible.*

Proof. Assuming that $u(t) = 0$ for $t < 0$ and making use of Lemma 6.1 it is easy to verify that $u(0) \neq 0$ and $u(t) = 0$, $t = 1, ..., d-1$, as well as $x(t) \neq 0$, $t = 1, ..., d-1$, and $x(t) = 0$ for $t > d-1$ if CB is right-invertible. In fact, $u(0) = -(CB)^R CA\, A^{d-1} x_o$ and for $t > 0$ we have $u(t) = -(CB)^R CA\,\{[I - B(CB)^R C]A\}^t A^{d-1} x_o$ and $x(t+d-1) = \{[I - B(CB)^R C]A\}^t A^{d-1} x_o$. ■

Remark 6.3 Notice that possible trivial solutions to the infimum-time state control problem, involving nilpotent (or zero) matrix A and/or zero initial condition, are also taken into account, both in Definition 6.2 and Theorem 6.2. Also notice that, in the nongeneric case considered (apart from trivial solutions), the infimum d can be lower than the reachability index ν.

Apparently, Theorem 6.2 is quite similar to the perfect regulation result of Theorem 4.1 (with the control law as in the above equation (14)). However, the additional requirement on invertibility of C makes infimum-time state control much more restrictive than infimum-time output control. In fact, the latter does not require the C invertibility to hold, even in the state space formulation (obviously, the condition is nonexistent in case of the external, matrix polynomial model). Furthermore, the perfect/infimum-time regulation law (14) is asymptotically stable if the system is minimum phase, with the conditions of Theorem 4.2 still retained.

Remark 6.4 Note that, again, the right-invertibility condition is only sufficient.

Remark 6.5 Note that a nondelay (or rather unit-delay) system, governed by equation (12a) with d=1, can be reachable in *one* time step.

Remark 6.6 Since the matrix C in the state space description is generically *not* square and invertible, the infimum-time *state* control is generically *not* achievable. This is in contrast to the case of infimum-time *output* control, where C

is not required to be invertible. This can be seen from the next result, being similar to Theorem 4.2.

THEOREM 6.3 *Let a minimum phase LTI MIMO system be described by the model (12), with B and C being of full rank, and let CB be right-invertible. Then the perfect regulation law (14) provides infimum-time output control.*

Proof. Immediate. ■

REMARK 6.7 All the above results can be readily extended to nonzero-reference and stochastic cases. One example involving e.g. the ARX model is related to familiar minimum variance control (Borisson, 1979).

REMARK 6.8 Notice that the immediate solution to the state-zeroing problem, resulting from equating $x(t+d) = Ax(t+d-1) + Bu(t) = 0$ (or, in particular, $x(t+1) = Ax(t) + Bu(t) = 0$), would generically be not feasible since B is, generically, not right-invertible.

Let us also indicate an interesting relationship of infimum-time control with a specific case of the limiting Riccati equation solution employed in a deadbeat control scheme by Leden (1976, 1977).

THEOREM 6.4 *Let the matrix Riccati recursion associated with the deadbeat control problem for the system (5) be given by*

$$S_t = R + A^T[S_{t+1} - S_{t+1}B(B^T S_{t+1} B)^{\#} B^T S_{t+1}]A,$$

with $t = N-1, ..., 0$ and $S_N = R$, and let CB be right-invertible. Then the solution $S_0 = S_N$ gives rise to infimum-time output control and, with C being invertible, to infimum-time state control.

Proof. By factorizing $S_{t+1} = M_{t+1}^T M_{t+1}$ we arrive at $S_t = R + (M_{t+1}A)^T[I - M_{t+1}B(M_{t+1}B)^{\#}]M_{t+1}A$. Since $M_N = C$ the result follows. ■

Infimum-time control presents some extension to time-optimal control, which accounts for nongeneric cases. However, it is worth mentioning that infimum-time control is of some theoretical significance only. In fact, it lacks robustness, even as compared with ν-reachable time-optimal control, which is often in practice replaced by still more robust, "approximate" time-optimal controls (Marro, Prattichizzo, Zatoni, 2002; Newman, 1990, Pao, 1994; Pao, Franklin, 1993). Nevertheless, a time-optimal cheap control problem has been effectively approached by Bikdash (1993).

It is worth emphasizing once more that the conditions for infimum-time state control to achieve are much more restrictive than those for infimum-time output control, in particular in terms of invertibility of C. Finally, it is worth mentioning about a certain curiosity in the analysis of the state space-modeled perfect/infimum-time regulation system under the control law as in, e.g., equation (4), with CB right-invertible. The closed-loop stability depends on eigenvalues of the matrix $A_{cl} = [I - B(CB)^R C]A$. The stability condition based on eigenvalues of that matrix can in no way be related to the (previously defined) minimum phase property of a system, even for a square system, also single-input single-output one. (This has caused the necessity to analyze the closed-loop stability on the basis of the matrix polynomial model instead, see Corollary 4.2 and Theorem 4.2). Moreover, assuming that the C matrix is invertible we arrive at the zero matrix A_{cl}, which makes the stability analysis impossible. No such problems appear in case of the external model analyzed in Section 3. It is an obvious drawback to operate, in this specific control case, with the state space description, for which the minimum phase property cannot be clearly evaluated (unlike for the matrix polynomial model).

7. Conclusions

The newly-defined "control zeros" for possibly nonsquare MIMO systems adequately characterize the stabilizing potential of output-zeroing/perfect/minimum-variance control, in terms of its sensitivity to the new-redefined nonminimum phase behavior of a plant to be controlled. Control zeros and inverse systems give rise to new, simple results concerning generalization and extension of perfect regulation/filtering to the case of nonsquare systems. Also, they contribute to the determination of the equivalence of single-step and multi-step optimal controls, providing the maximally achievable control accuracy. The conditions of the minimum phase behavior and right-invertibility of certain matrices play a crucial role in maximum-accuracy control. Pursuing the tracks of the right-invertibility condition on the one hand and the relationship of the state/output predictor with time-optimal control on the other, we have introduced a new category of time-optimal control, called infimum-time control. The infimum-time controller provides the shortest achievable time for driving the system state/output to the origin. We have also shown how this maximum-speed controller can be equivalent to the maximum-accuracy controller, i.e. perfect regulator. It is essential that the stability of an infimum-time or perfect control system is conditioned by the minimum phase behavior of a MIMO plant, involving the control zeros (and, generically, not any other type of multivariable zeros) right in the same way as "regular" zeros of SISO systems. Perfect regulation and infimum-time control mark an upper bound for the maximally achievable accuracy and speed of a control system. We have also pointed out that when analyzing the specific perfect/infimum-time control problem it is more advantageous to operate with external, rather than state space models. Current research on control zeros and related problems is focused on extension of the above contributions to continuous-time systems (Hunek, 2003; Latawiec, Hunek, 2002) and on control-oriented identification of a complex industrial plant (Stanisawski, 2004).

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