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An algorithm for checking Hurwitz stability of K-symmetrizable interval matrices

by

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Abstract: The necessary and sufficient condition for stability of K-symmetrizable interval matrix is given. An algorithm for checking stability of K-symmetrizable interval matrix is proposed.

Keywords: symmetrizable matrix, interval matrix, positive definite matrix.

1. Introduction

Let A^d and A^D be real $n \times n$ matrices. Let $A^d \leq A^D$ (component-wise inequalities). A square interval matrix

$$A^I = [A^d, A^D] = \{A : A^d \leq A \leq A^D\}$$

(component-wise inequalities) is called Hurwitz stable (Schur stable) if each $A \in [A^d, A^D]$ is Hurwitz stable (Schur stable) i.e., $Re(\lambda) < 0$ ($|\lambda| < 1$) for each eigenvalue λ of A. Stability of interval matrices has been recently studied in the robust control theory because it is connected with the checking of stability of a linear time-invariant system $\dot{x}(t) = Ax(t)$ under data perturbations.

There are many sufficient conditions for stability of an interval matrix (see Rohn, 1994, 1996), but a necessary and sufficient condition formulated in terms of stability of a finite subset of matrices in A^I was given only for a symmetric interval matrix (see Rohn, 1996) and for an interval matrix with real eigenvalues (see Rohn, 1992). A necessary and sufficient condition for stability of any interval matrix has been established in Wang, Michal, Liu (1994) but it is not useful for computation. Soh (1990) proposed the necessary and sufficient condition for the Hurwitz and Schur stability of convex combination of symmetric matrices. It was done using the well-known fact that a symmetric, Hurwitz stable matrix is negative definite. Białas (1985) proposed the necessary and sufficient condition for stability of convex combination of stable matrices. In Kowynia (2001) some sufficient conditions for stability of interval matrix symmetrizable by distinct diagonal matrices have been established.

In this paper we propose a definition of the K-symmetrizable interval matrix. It is an extension of a well-known definition of symmetric interval matrices. For the K-symmetrizable interval matrix we propose the necessary and sufficient condition for Hurwitz stability. We show that for checking Hurwitz stability of such an interval matrix it is enough to check the stability of a finite subset of matrices. This is an improvement of Theorem 5 of Kowynia (2000).

Later in this paper we establish an algorithm for checking Hurwitz stability of a K-symmetrizable interval matrix. This is a generalization of an algorithm proposed in Rohn (1996).

2. Symbols and definitions

Let $A = [a_{ij}] \in R^{n\times n}$. We denote the transposition of matrix A by A^T. By $\lambda_i(A)$, $i = 1, 2, ..., n$ we denote eigenvalues of matrix A. Let I denote a unit matrix. Matrix inequalities, like $A \leq B$ or $A < B$ will be understood componentwise. Following Bellman (1960) we say that a matrix $A = [a_{ij}] \in R^{n\times n}$ is symmetrizable if and only if there exists a positive definite matrix M such that

$$MA = A^T M.$$

In such a case we say that a matrix $A = [a_{ij}] \in R^{n\times n}$ is symmetrizable by a matrix M. Following Białas (1995) and Kowynia (2000) we say that a matrix $A = [a_{ij}] \in R^{n\times n}$ is symmetrizable by a diagonal matrix if and only if there exists a matrix $K = \mathrm{diag}(k_1, k_2, ..., k_n)$, $k_i > 0$, $i = 1, 2, ..., n$ such that the matrix

$$K_0 A K_0^{-1}$$

is symmetric where $K_0 = K^{1/2} = \mathrm{diag}(\sqrt{k_1}, \sqrt{k_2}, ..., \sqrt{k_n})$.

Definition 2.1 *Let $A^I = [A^d, A^D]$ be an interval matrix. We say that $A^I = [A^d, A^D]$ is K-symmetrizable if matrices A^d, A^D are both symmetrizable by a matrix K where $K = diag(k_1, k_2, ..., k_n)$, $k_i > 0$, $i = 1, 2, ..., n$, and so matrices:*

$$K_0 A^d K_0^{-1},\ K_0 A^D K_0^{-1}$$

are symmetric, where $K_0 = K^{1/2} = diag(\sqrt{k_1}, \sqrt{k_2}, ..., \sqrt{k_n})$.

Note that an interval matrix $A^I = [A^d, A^D]$ which is K-symmetrizable, in terms of the above definition, can contain matrices which are nonsymmetrizable.

Following Rohn (1996) we will use the notation as below:

Let

$$Z = \{z = (z_1, z_2, ..., z_n) \in R^n,\ z_j \in \{-1, 1\},\ j = 1, 2, ..., n.\}. \tag{1}$$

For any interval matrix $A^I = [A^d, A^D]$ where $A^d = [(a^d)_{ij}]$, $A^D = [(a^D)_{ij}]$ and for any $z \in Z$ we define matrix $A_z = [(a_z)_{ij}]$ as follows

$$(a_z)_{ij} = \begin{cases} (a^D)_{ij}, & \text{if } z_i z_j = 1, \\ (a^d)_{ij}, & \text{if } z_i z_j = -1. \end{cases} \tag{2}$$

It is known, Rohn (1996), that a symmetric interval matrix $A^I = [A^d, A^D]$ (meaning that matrices A^d, A^D are symmetric) is Hurwitz stable if and only if matrices $A_z = [(a_z)_{ij}]$, $z \in Z$, $z = (1, z_2, ..., z_n)$ (where elements of the matrix A_z are given by the formula (2)) are Hurwitz stable.

3. Necessary and sufficient condition for stability of a K-symmetrizable interval matrix

THEOREM 3.1 *Let an interval matrix $A^I = [A^d, A^D]$ be K-symmetrizable. Then it is Hurwitz stable if and only if the symmetric interval matrix*

$$[K_0 A^d K_0^{-1}, K_0 A^D K_0^{-1}] = K_0 A^I K_0^{-1} = \{B : B = K_0 A K_0^{-1},\ A \in A^I\}$$

is Hurwitz stable.

Proof. For any matrix $A \in \mathbb{R}^{n\times n}$ we have $K_0 A K_0^{-1} = [\sqrt{\frac{k_i}{k_j}} a_{ij}]_{i,j=1}^n$. By assumption we have that matrices A^d, A^D are symmetrizable by a matrix K, so that the set

$$K_0 A^I K_0^{-1} = \{B : B = K_0 A K_0^{-1},\ A \in A^I\}$$

is the symmetric interval matrix of the form

$$[K_0 A^d K_0^{-1}, K_0 A^D K_0^{-1}].$$ ■

Now we propose a theorem which is a slight improvement over Theorem 5 of Kowynia (2000).

THEOREM 3.2 *Let an interval matrix $A^I = [A^d, A^D]$ where $A^d = [(a^d)_{ij}]$, $A^D = [(a^D)_{ij}]$, be K-symmetrizable.*

The interval matrix $A^I = [A^d, A^D]$ is Hurwitz stable if and only if matrices $A_z = [(a_z)_{ij}]$, $z \in Z$, $z = (1, z_2, ..., z_n)$ are Hurwitz stable where $(a_z)_{ij}$ are defined by the formula (2).

Proof. As the necessary condition is obvious we prove only the sufficient condition.

By assumption, matrices A^d, A^D are symmetrizable by a matrix K, so the interval matrix $K_0 A^I K_0^{-1} = [K_0 A^d K_0^{-1}, K_0 A^D K_0^{-1}]$ is a symmetric interval matrix. Hence, for this matrix we can use Theorem 1, Rohn (1996).

Let us take

$$B^d = [(b^d)_{ij}] = K_0 A^d K_0^{-1}, \;\; B^D = [(b^D)_{ij}] = K_0 A^D K_0^{-1}, \;\; \text{where } \; K_0 = K^{1/2}.$$

For such defined matrices B^d, B^D we have

$$(b^d)_{ij} = \sqrt{\frac{k_i}{k_j}}(a^d)_{ij}, \quad (b^D)_{ij} = \sqrt{\frac{k_i}{k_j}}(a^D)_{ij}.$$

For any $z \in Z$ we define matrix $B_z = [(b_z)_{ij}]$ as follows

$$(b_z)_{ij} = \begin{cases} \sqrt{\dfrac{k_i}{k_j}}(a^D)_{ij}, & \text{if } \; z_i z_j = 1, \\ \sqrt{\dfrac{k_i}{k_j}}(a^d)_{ij}, & \text{if } \; z_i z_j = -1. \end{cases}$$

From Theorem 1, Rohn (1996), we have that Hurwitz stability of matrices B_z, $z \in Z$, $z = (1, z_2, ..., z_n)$ is a necessary and sufficient condition for Hurwitz stability of an interval matrix $B^I = [B^d, B^D]$ where $B^d = K_0 A^d K_0^{-1}$, $B^D = K_0 A^D K_0^{-1}$; so a necessary and sufficient condition for Hurwitz stability of an interval matrix $A^I = [A^d, A^D]$.

Additionally, for any $z \in Z$ the matrix B_z has a form $K_0 A_z K_0^{-1}$ where $A_z = [(a_z)_{ij}]$ and $(a_z)_{ij}$ are given by the formula (2) and $K_0 = K^{1/2}$, $A^d = [(a^d)_{ij}]$, $A^D = [(a^D)_{ij}]$, $A^I = [A^d, A^D]$.

The proof is complete. ■

Example 3.1 *Let us consider an interval matrix $A^I = [A^d, A^D]$ where: $A^d = \begin{bmatrix} \frac{-7}{11} & 4 \\ \frac{1}{11} & -2 \end{bmatrix}$, $A^D = \begin{bmatrix} \frac{-1}{2} & 4\sqrt{2} \\ \frac{\sqrt{2}}{11} & -2 \end{bmatrix}$. Note that matrices A^d, A^D are both symmetrizable by the diagonal matrix $K = diag(1, 44)$. Matrices A_z, which are defined by the formula (2), have the following form:*

$$A_{(1,1)} = \begin{bmatrix} \frac{-1}{2} & 4\sqrt{2} \\ \frac{\sqrt{2}}{11} & -2 \end{bmatrix}, \quad A_{(1,-1)} = \begin{bmatrix} \frac{-1}{2} & 4 \\ \frac{1}{11} & -2 \end{bmatrix}.$$

It is easy to check that matrices $A_{(1,1)}, A_{(1,-1)}$ are Hurwitz stable. According to Theorem 3.2 we obtain that the K-symmetrizable interval matrix $A^I = [A^d, A^D]$ is Hurwitz stable.

THEOREM 3.3 *Let an interval matrix* $A^I = [A - D, A + D]$ *where* $A = [a_{ij}]$, $D = [d_{ij}] \in R^{n\times n}$ *be* K*-symmetrizable.*

The interval matrix A^I *is Hurwitz stable if the matrix* $A + \|D\|_1 I$ *is Hurwitz stable, where*

$$\|D\|_1 = \max_j \sum_i \mid d_{ij} \mid .$$

Additionally, matrix $A + \|D\|_1 I$ *is* K*-symmetrizable.*

Proof. Since an interval matrix $A^I = [A - D, A + D]$ is K-symmetrizable we obtain that matrices A and D are K-symmetrizable. So the matrix $A + \|D\|_1 I$ is K-symmetrizable and hence the matrix $K_0 A K_0^{-1} + \|D\|_1 I$ is symmetric.

To prove Hurwitz stability of an interval matrix $A^I = [A - D, A + D]$ we make use of Theorem 3.2. We prove Hurwitz stability of a matrix $A_z = [(a_z)_{ij}]$, $z \in Z$, $z = (1, z_2, ..., z_n)$, where

$$(a_z)_{ij} = \begin{cases} a_{ij} + d_{ij}, & \text{if} \quad z_i z_j = 1, \\ a_{ij} - d_{ij}, & \text{if} \quad z_i z_j = -1. \end{cases} \tag{3}$$

Let us note that for any $z \in Z$ the matrix A_z is K-symmetrizable and hence for $z \in Z$ the matrix $K_0 A_z K_0^{-1}$ is symmetric. Additionally, for $z \in Z$, the matrix A_z is Hurwitz stable if and only if the matrix $K_0 A_z K_0^{-1}$ is Hurwitz stable.

Since the matrix $K_0 D K_0^{-1}$, $K_0 = K^{1/2}$ is a symmetric matrix with nonnegative elements then for any $x = (x_1, x_2, ..., x_n) \in R^n$, $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + ... + x_n^2} = 1$ we have:

$$\langle (K_0 A_z K_0^{-1} - K_0 A K_0^{-1})x, x\rangle \le \max_{\|y\|=1} \langle K_0 D K_0^{-1} y, y\rangle =$$
$$\lambda_{max}(K_0 D K_0^{-1}) = \lambda_{max}(D) = \rho(D) \le \|D\|_1,$$

where $\rho(D)$ denotes the spectral radius of the matrix D.

From the above inequalities and from the fact that the matrix $K_0 A K_0^{-1} + \|D\|_1 I$ is negative definite we have:

$$\langle K_0 A_z K_0^{-1} x, x\rangle = \langle K_0 A K_0^{-1} x, x\rangle + \langle (K_0 A_z K_0^{-1} - K_0 A K_0^{-1})x, x\rangle \le$$
$$\langle (K_0 A K_0^{-1} + \|D\|_1 I)x, x\rangle < 0,$$

where $x = (x_1, x_2, ..., x_n) \in R^n$, $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + ... + x_n^2} = 1$, $K_0 = K^{1/2}$. So for any $z \in Z$ the maximal eigenvalue of the symmetric matrix $K_0 A_z K_0^{-1}$ is negative which means that for any $z \in Z$ the matrix $K_0 A_z K_0^{-1}$ is Hurwitz stable. ∎

THEOREM 3.4 *Let an interval matrix* $[A-D, A+D]$ *where* $A = [a_{ij}], D = [d_{ij}] \in R^{n\times n}$ *be* K*-symmetrizable.*

The interval matrix $A^I = [A-D, A+D]$ *is not Hurwitz stable if and only if there exists* $z = (1, z_2, ..., z_n) \in Z$ *such that* $\lambda_{max}(A_z) \geq 0$ *where for* $z \in Z$ *the matrix* $A_z = [(a_z)_{ij}]$ *where* $(a_z)_{ij}$ *are given by (3).*

Proof. Use Theorem 3.2 for a K-symmetrizable interval matrix $A^I = [A - D,\ A+D]$. ∎

Let an interval matrix $[A-D, A+D]$ where $A = [a_{ij}]$, $D = [d_{ij}] \in R^{n\times n}$ be K-symmetrizable.

For any $z \in Z$ we define a matrix $A_z = [(a_z)_{ij}]$ where $(a_z)_{ij}$ are given by (3).

Following Rohn (1996), for $x = (x_1, x_2, ..., x_n) \in R^n$, we define

$$sgnx = (\mathrm{sgn}(x_1), \mathrm{sgn}(x_2), ..., \mathrm{sgn}(x_n))$$

where:

$$\mathrm{sgn}(x_j) = \begin{cases} 1, & x_j \geq 0, \\ -1, & x_j < 0, \end{cases} \qquad j = 1, 2, ..., n.$$

for any $x \in R^n$; vector $sgnx \in Z$ where Z is defined by the formula (1).

Now we propose an algorithm for checking instability of a K-symmetrizable interval matrix $[A-D, A+D]$ where A, $D \in R^{n\times n}$.

Algorithm 1

Step 1
Compute $\lambda_{max}(A)$ and the related eigenvector x^1.
If $\lambda_{max}(A) \geq 0$ then the K-symmetrizable interval matrix $[A-D, A+D]$ is not Hurwitz stable. In the contrary case we put $z^1 = \mathrm{sgn}(x^1)$.

Step 2
Compute $\lambda_{max}(A_{z^1})$ and the related eigenvector x^2.
If $\lambda_{max}(A_{z^1}) \geq 0$ then the matrix A^I is not Hurwitz stable. Otherwise, we put $z^2 = \mathrm{sgn}(x^2)$.

Step 3
Compute $\lambda_{max}(A_{z^2})$ and the related eigenvector x^3. We check the sign of $\lambda_{max}(A_{z^2})$, and so on.

We proceed as described above till for some $z^k = (z_1^k, z_2^k, ..., z_n^k) \in Z$ we obtain $\lambda_{max}(A_{z^k}) \geq 0$, which means that the matrix A^I is not Hurwitz stable, or we obtain that $d_{ij} z_i^k x_i^{k+1} z_j^k x_j^{k+1} \geq 0$ for any $i, j = 1, 2, ..., n$, which means that Hurwitz stability of the K-symmetrizable matrix $A^I = [A-D, A+D]$ has not been verified.

For the above Algorithm the following holds true:

THEOREM 3.5 *Algorithm 1 generates a sequence of matrices $A_z,\ z \in Z$ with ascending eigenvalues $\lambda_{max}(A_z)$.*

Proof. We show that for $z^k, z^{k+1} \in Z$ which were generated by Algorithm 1, the following inequality holds

$$\lambda_{max}(A_{z^k}) < \lambda_{max}(A_{z^{k+1}}). \tag{4}$$

Since $z^k, z^{k+1} \in Z$ and matrices A, D are K-symmetrizable we obtain that matrices A_{z^k}, $A_{z^{k+1}}$ are K-symmetrizable. Hence matrices $K_0 A_{z^k} K_0^{-1}$, $K_0 A_{z^{k+1}} K_0^{-1}$ are symmetric and similar to matrices A_{z^k}, $A_{z^{k+1}}$. So, to prove inequality (4) we have to show that

$$\lambda_{max}(K_0 A_{z^k} K_0^{-1}) < \lambda_{max}(K_0 A_{z^{k+1}} K_0^{-1}).$$

We make use of the proof of Theorem 3, Rohn (1996).

Let us denote $S = \mathrm{diag}(z_1^k, z_2^k, ..., z_n^k)$. Using (3) we have:

$$A_{z^k} = A + SDS.$$

Let x^{k+1} be an eigenvector generated by Algorithm 1 and related to the eigenvalue $\lambda_{max}(A_{z^k})$. Then, Fiedler (1989), vector $K_0 x^{k+1}$ is an eigenvector related to the eigenvalue $\lambda_{max}(K_0 A_{z^k} K_0^{-1})$.

Let us assume that $\|K_0 x^{k+1}\|_2 = 1$. We obtain

$$\begin{aligned}
\lambda_{max}(K_0 A_{z^k} K_0^{-1}) &= \langle (K_0 A K_0^{-1} + K_0 SDS K_0^{-1}) K_0 x^{k+1}, K_0 x^{k+1} \rangle = \\
&\langle (K_0 A K_0^{-1}) K_0 x^{k+1}, K_0 x^{k+1} \rangle + \langle (S K_0 D K_0^{-1} S) K_0 x^{k+1}, K_0 x^{k+1} \rangle = \\
&\langle (K_0 A K_0^{-1}) K_0 x^{k+1}, K_0 x^{k+1} \rangle + \sum_{i,j=1}^{n} z_i^k x_i^{k+1} \sqrt{\frac{k_i}{k_j}} d_{ij} \sqrt{k_i k_j} z_j^k x_j^{k+1} < \\
&\langle (K_0 A K_0^{-1}) K_0 x^{k+1}, K_0 x^{k+1} \rangle + \sum_{i,j=1}^{n} \mid x_i^{k+1} \mid \sqrt{\frac{k_i}{k_j}} d_{ij} \sqrt{k_i k_j} \mid x_j^{k+1} \mid .
\end{aligned}$$

Let $z^{k+1} = sgn(x^{k+1})$. We put $S' = \mathrm{diag}(z_1^{k+1}, z_2^{k+1}, ..., z_n^{k+1})$. Then:

$$A_{z^{k+1}} = A + S'DS'.$$

And we have:

$$\begin{aligned}
&\langle (K_0 A K_0^{-1}) K_0 x^{k+1}, K_0 x^{k+1} \rangle + \sum_{i,j=1}^{n} \mid x_i^{k+1} \mid \sqrt{\frac{k_i}{k_j}} d_{ij} \sqrt{k_i k_j} \mid x_j^{k+1} \mid = \\
&\quad \langle (K_0 A K_0^{-1}) K_0 x^{k+1}, K_0 x^{k+1} \rangle + \sum_{i,j=1}^{n} z_i^{k+1} x_i^{k+1} \sqrt{\frac{k_i}{k_j}} d_{ij} \sqrt{k_i k_j} z_j^{k+1} x_j^{k+1} = \\
&\quad \langle (K_0 A K_0^{-1} + K_0 S' D S' K_0^{-1}) K_0 x^{k+1}, K_0 x^{k+1} \rangle \le \lambda_{max}(K_0 A_{z^{k+1}} K_0^{-1}).
\end{aligned}$$

The proof is complete. ■

Corollary 3.1 *Because of the fact that the set Z is finite we obtain that Algorithm 1, in finite number of steps, either checks that the interval K-symmetrizable matrix $A^I = [A - D, A + D]$ is not Hurwitz stable or Hurwitz stability is not verified.*

Let an interval matrix $A^I = [A^d, A^D]$ where $A^d = [(a^d)_{ij}]$, $A^D = [(a^D)_{ij}] \in R^{n\times n}$ be K-symmetrizable. Let

$$Z_0 = \{z = (z_1, z_2, ..., z_n) \in R^n,\ z_i \in \{-1, 0, 1\},\ i = 1, 2, ..., n\}.$$

For any $z \in Z_0$ we define matrices $A_z = [(a_z)_{ij}]$, $D_z = [(d_z)_{ij}]$ as follows:

$$(a_z)_{ij} = \begin{cases} \frac{1}{2}((a^d)_{ij} + (a^D)_{ij}), & \text{if } \ z_i z_j = 0,\ i \neq j, \\ (a^D)_{ij}, & \text{if } \ z_i z_j = 1 \text{ or } i = j, \\ (a^d)_{ij}, & \text{if } \ z_i z_j = -1, \end{cases} \tag{5}$$

and

$$(d_z)_{ij} = \begin{cases} \frac{1}{2}((a^D)_{ij} - (a^d)_{ij}), & \text{if } \ z_i z_j = 0,\ i \neq j, \\ 0, & \text{if } \ z_i z_j \neq 0 \text{ or } i = j, \end{cases} \tag{6}$$

$$i, j = 1, 2, ..., n.$$

Let us note that for any $z \in Z_0$, the matrices A_z, D_z are K-symmetrizable. Additionally, for any $z \in Z_0$, the matrix $D_z = [(d_z)_{ij}]$ has nonnegative elements.

Now we propose an algorithm for checking Hurwitz stability of a K-symmetrizable interval matrix $A^I = [A^d, A^D]$.

Algorithm 2

Step 1
We define a set $L \subset Z_0$ as follows:

$$L = \{z = (z_1, z_2, ..., z_n) \in Z_0 : z_1 = 1,\ z_k = 0,\ k \neq 1\}.$$

For $z = (1, 0, 0, ..., 0) \in L$ we construct matrices A_z, D_z in the way given in formulas (5), (6). Additionally, we construct a K-symmetrizable interval matrix $[A_z - D_z, A_z + D_z] \subseteq A^I$ and using Algorithm 1 we check if this matrix is not Hurwitz stable.

If instability of the interval matrix $[A_z - D_z, A_z + D_z]$ is not verified then we check Hurwitz stability of the matrix $A_z + \|D_z\|_1 I$. If this matrix is Hurwitz stable then the interval matrix $[A_z - D_z, A_z + D_z]$ is Hurwitz stable. If the matrix $A_z + \|D_z\|_1 I$ is not Hurwitz stable then we proceed as follows:

Let $i, j \in \{1, 2, ..., n\}$, $i < j$ be such that:

$$(d_z)_{ij} = \max_{k<m} (d_z)_{km}.$$

Since the matrix D_z is K-symmetrizable and has nonnegative elements and since the matrix $A_z + \|D_z\|_1 I$ is not Hurwitz stable but the matrix A_z is Hurwitz stable we obtain:

$$(d_z)_{ij} = \max_{k<m} (d_z)_{km} > 0,$$

and $z_i z_j = 0$. Then $z_i = 0$ or $z_j = 0$. If $z_i = 0$ we put $h = i$. If $z_j = 0$ we put $h = j$. So $z_h = 0$.

Now we choose $z^1, z^2 \in Z_0$ in the following way:

$$z^1 = (z_1^1, z_2^1, ..., z_h^1, ..., z_n^1) \text{ where } z_h^1 = -1,\ z_k^1 = z_k,\ k \neq h,$$
$$z^2 = (z_1^2, z_2^2, ..., z_h^2, ..., z_n^2) \text{ where } z_h^2 = 1,\ z_k^2 = z_k,\ k \neq h,$$

where $z_k, k = 1, 2, ..., n$ are coordinates of the vector: $z = (z_1, z_2, ..., z_n) = (1, 0, 0, ..., 0)$.

Step 2

We remove from the set L the vector $z = (1, 0, ..., 0)$ and put into the set L the vectors z^1, z^2. So in Step 2, the set L has a form: $L = \{z^1, z^2\}$. For the set L we repeat the procedure given in Step 1.

In each of the steps of this algorithm, which are to follow, we put into the set L only these elements that are generated from the set Z_0 and which were not considered before.

Since Z_0 is finite we obtain that in some step either $L = \emptyset$ or there exists $z^r \in L$ such that the interval matrix $[A_{z^r} - D_{z^r}, A_{z^r} + D_{z^r}]$ is not Hurwitz stable.

If the interval matrix $[A_{z^r} - D_{z^r}, A_{z^r} + D_{z^r}]$ is not Hurwitz stable then the interval matrix $A^I = [A^d, A^D]$ is not Hurwitz stable.

If in some step the set $L = \emptyset$ then the interval matrix $A^I = [A^d, A^D]$ is Hurwitz stable.

Remark 3.1 *The above Algorithm checks Hurwitz stability of an interval matrix* $A^I = [A^d, A^D]$.

Proof. Let us take any $z^0 = (z_1^0, z_2^0, ..., z_n^0) \in Z$ such that $z_1^0 = 1$. If in some step of Algorithm 2 we have $L = \emptyset$ then we know that among elements of the set Z_0 which were generated by Algorithm 2 there exists such $z \in Z_0$ that all its nonzero elements are equal to the corresponding elements of the vector z^0. Additionally, for such z, the matrix $A_z + \|D_z\|_1 I$ is Hurwitz stable. Hence the interval matrix $[A_z - D_z, A_z + D_z]$ is Hurwitz stable. But we have $A_{z^0} \in [A_z - D_z, A_z + D_z]$. So the matrix A_{z^0} is Hurwitz stable. Hence we obtain that the interval matrix A^I is Hurwitz stable. ∎

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