Dedicated to
Professor Czesław Olech

Control and Cybernetics

vol. **34** (2005) No. 3

Optimal synthesis via superdifferentials of value function

by

H. Frankowska

CNRS, CREA
Ecole Polytechnique
1, rue Descartes, 75005 Paris, France
e-mail: franko@shs.polytechnique.fr

Abstract: We derive a differential inclusion governing the evolution of optimal trajectories to the Mayer problem. The value function is allowed to be discontinuous. This inclusion has convex compact right-hand sides.

Keywords: Mayer problem, optimal synthesis, differential inclusion, Hamilton-Jacobi equation, semiconcave function, viability theory, viscosity solution.

1. Introduction

In this paper we address the problem of optimal synthesis governing the evolution of optimal trajectories to the Mayer problem

$$\min\,\{\varphi(x(T)) \mid \ x \text{ is a solution to } \ (2),\ x(0) = \xi_0\}, \tag{1}$$

$$x'(t) \ = \ f(x(t), u(t)), \ \ u(t) \ \in \ U. \tag{2}$$

The major problem of optimal control is to determine a set-valued map $U(t, x)$ (in general discontinuous) in such way that solutions to the control system

$$\begin{cases} x' &= f(x, u), \ \ u \in U(t, x) \\ x(0) &= \xi_0 \end{cases} \tag{3}$$

are optimal. This map is usually referred to as the optimal synthesis and in order to construct it, it is natural to apply the dynamic programming principle which in turn uses the value function.

The value function associated with Mayer's problem is defined as follows: for all $(t_0, x_0) \in [0, T] \times \mathbf{R}^n$

$$V(t_0, x_0) = \inf\{\varphi(x(T)) \mid x \text{ is a solution to } (2),\ x(t_0) = x_0\}. \tag{4}$$

When it is Lipschitz it can be used to state both the necessary and sufficient conditions for optimality (see for instance Cannarsa and Frankowska, 1991; Subbotina, 1989).

When $V \in C^1$ the synthesis problem was solved in Fleming and Rishel (1975) using the Hamilton-Jacobi equation. In Frankowska (1989b) we proposed in the case when V is merely locally Lipschitz to use the directional derivatives of V. It was namely shown that for the set-valued map

$$F(t,x) := \left\{ v \in f(x,U) \mid \frac{\partial V}{\partial(1,v)}(t,x) = 0 \right\}$$

the inclusion

$$\begin{cases} x' & \in & F(t,x) \\ x(0) & = & \xi_0 \end{cases} \tag{5}$$

characterizes optimal solutions, i.e., every solution to (5) is optimal and every optimal solution satisfies (5).

In Berkovitz (1989) a similar approach was proposed, where the author supposed in addition that F is upper semicontinuous with convex compact images. While from the results of Cannarsa and Frankowska (1991) it follows that for problems with data smooth enough F is upper semicontinuous, but, in general, beside the case when the value function is C^1 or convex, one should not expect from it to have convex values.

The aim of this paper is to associate an optimal synthesis to a lower semicontinuous value function by using its superdifferentials instead of directional derivatives. Namely we define a set-valued map G having closed convex images such that every solution to the differential inclusion

$$\begin{cases} x' & \in & G(t,x) \\ x(0) & = & \xi_0 \end{cases} \tag{6}$$

is optimal. This optimal synthesis map is defined by

$$G(t,x) = \{ v \in f(x,U) \mid \forall\, (p_t, p_x) \in \partial_+ V(t,x), \;\; p_t + \langle p_x, v \rangle = 0 \},$$

where $\partial_+ V(t,x)$ denotes the superdifferential of V at (t,x). However in general G is not upper semicontinuous. When V is semiconcave, then the map

$$\hat{G}(t,x) = \{ v \in f(x,U) \mid \exists\, (p_t, p_x) \in \partial_+ V(t,x), \;\; p_t + \langle p_x, v \rangle = 0 \},$$

is upper semicontinuous and in this case we obtain both necessary and sufficient conditions for optimality in the form of the differential inclusion

$$\begin{cases} x' & \in & \hat{G}(t,x) \\ x(0) & = & \xi_0. \end{cases} \tag{7}$$

The outline of the paper is as follows. In Section 2 we provide some preliminaries and in Section 3 we investigate superdifferentials along optimal solutions. The inclusion on the optimal synthesis is derived in Section 4.

2. Preliminaries

Consider $T > 0$, a complete separable metric space U and a continuous mapping $f : \mathbf{R}^n \times U \mapsto \mathbf{R}^n$. We associate with it the control system

$$x'(t) \;=\; f(x(t), u(t)), \;\; u(t) \;\in\; U. \tag{8}$$

Let an extended function $\varphi : \mathbf{R}^n \mapsto \mathbf{R} \cup \{+\infty\}$ and $\xi_0 \in \mathbf{R}^n$ be given. Consider the minimization problem, called *free end point Mayer problem*:

$$\min \{\varphi(x(T)) \mid \; x \text{ is a solution to } \; (8), \; x(0) = \xi_0\}. \tag{9}$$

The value function associated with this problem is defined as: for all $(t_0, x_0) \in [0, T] \times \mathbf{R}^n$

$$V(t_0, x_0) = \inf\{\varphi(x(T)) \mid x \text{ is a solution to } (8), \; x(t_0) = x_0\}. \tag{10}$$

Under standard assumptions V is lower semicontinuous.

Theorem 2.1 *Assume that φ is lower semicontinuous and*

$$\left\{\begin{array}{ll} i) & \forall\, R > 0, \; \exists\, c_R > 0 \text{ such that} \\ & \forall\, u \in U, \; f(\cdot, u) \text{ is } c_R - \text{Lipschitz on } B_R(0) \\ & \\ ii) & \exists\, k > 0 \text{ such that} \\ & \forall\, x \in \mathbf{R}^n, \;\; \sup_{u \in U} \|f(x, u)\| \;\leq\; k(1 + \|x\|) \\ & \\ iii) & \forall\, x \in \mathbf{R}^n, \;\; f(x, U) \text{ is closed and convex.} \end{array}\right. \tag{11}$$

Then V is lower semicontinuous and for all $(t_0, x_0) \in [0, T] \times \mathbf{R}^n$ the infimum in (10) is attained.

This result is well known. Its proof can be deduced, for instance, from Ioffe (1977), Olech (1976).

Throughout the whole paper we set

$$\forall\, t \notin [0, T], \;\; \forall\, x \in \mathbf{R}^n, \;\; V(t, x) = -\infty.$$

Definition 2.1 *Let $g : \mathbb{R}^n \mapsto \mathbf{R} \cup \{\pm\infty\}$ be an extended function, $v \in \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$ be such that $g(x_0) \neq \pm\infty$.*

We define the contingent epiderivative *of g at x_0 in the direction v by*

$$D_\uparrow g(x_0)(v) \;=\; \liminf_{h \to 0+, \; v' \to v} \frac{g(x_0 + hv') - g(x_0)}{h}$$

and the contingent hypoderivative *of g at x_0 in the direction v by*

$$D_\downarrow g(x_0)(v) \;=\; \limsup_{h \to 0+, \; v' \to v} \frac{g(x_0 + hv') - g(x_0)}{h}.$$

Let $x_0 \in \mathbb{R}^n$ *be such that* $g(x_0) \neq \pm\infty$*. The* superdifferential *of* g *at* x_0 *is the closed convex set defined by:*

$$\partial_+ g(x_0) = \left\{ p \in \mathbf{R}^n \mid \limsup_{x \to x_0} \frac{g(x) - g(x_0) - \langle p, x - x_0 \rangle}{\|x - x_0\|} \le 0 \right\}.$$

The superdifferentials may also be characterized using *contingent hypoderivatives*:

PROPOSITION 2.1 (Frankowska, 1989b) *Let* $g : \mathbf{R}^n \mapsto \mathbf{R} \cup \{\pm\infty\}$ *be an extended function. Then*

$$\partial_+ g(x_0) = \{ p \in \mathbf{R}^n \mid \ \forall\, v \in \mathbf{R}^n, \ D_\downarrow g(x_0)(v) \le \langle p, v \rangle \}.$$

The Hamiltonian associated to our control problem is defined by

$$\forall\, x,\, p \in \mathbf{R}^n, \ \ H(x,p) = \sup_{u \in U} \langle p, \ f(x,u) \rangle\,.$$

It is well known that under adequate assumptions the value function solves the Hamilton-Jacobi equation

$$-\frac{\partial V}{\partial t}(t,x) \ + \ H\left(t, \ x, \ -\frac{\partial V}{\partial x}(t,x)\right) \ = \ 0, \quad V(T,\cdot) \ = \ \varphi(\cdot) \tag{12}$$

in the generalized (viscosity) sense. In this work we shall use discontinuous subsolutions of this equation.

DEFINITION 2.2 *An extended function* $W : [0,T] \times \mathbf{R}^n \to \mathbf{R} \cup \{-\infty\}$ *is called a* viscosity subsolution *to (12) if for all* $(t,x) \in \mathrm{Dom}(W), \ t < T$ *we have*

$$\forall\, (p_t, p_x) \ \in \ \partial_+ W(t,x), \quad -p_t \ + \ H(x, -p_x) \ \le \ 0.$$

PROPOSITION 2.2 *Assume that (11) holds true. Then the value function is a viscosity subsolution to (12).*

Proof. This result can be deduced from Frankowska (1989b). We sketch its proof for the sake of completeness. Fix any $(t,x) \in \mathrm{Dom}(V), \ t < T$ and let $u \in U$. Consider a solution to our control system with this fixed control u starting at (t,x). Since V is nondecreasing along solutions of the control system, we get $D_\downarrow V(t,x)(1, f(x,u)) \ge 0$. Applying Proposition 2.1 and using the fact that $u \in U$ is arbitrary we derive the desired inequality. ■

DEFINITION 2.3 *Let* K *be a closed subset of* $\mathbf{R}^n$*. The contingent cone to* K *at* $x \in K$ *is defined by*

$$T_K(x) = \left\{ v \in \mathbf{R}^n \mid \liminf_{h \to 0+} \frac{dist(x + hv, K)}{h} = 0 \right\}.$$

DEFINITION 2.4 *Let* $z : [0,T] \mapsto \mathbf{R}^n$. *The contingent derivative* $Dz(t)(1)$ *of* z *at* $t \in [0,T[$ *in the direction* 1 *is defined by*

$$Dz(t)(1) = \text{ Limsup}_{h\to 0+}\frac{z(t+h)-z(t)}{h}$$

and $Dz(t)(-1)$ *of* z *at* $t \in]0,T]$ *in the direction* -1 *by*

$$Dz(t)(-1) = \text{ Limsup}_{h\to 0+}\frac{z(t-h)-z(t)}{h},$$

where Limsup denotes the Painlevé-Kuratowski upper limit.

We shall also use in the sequel the following simple fact:

PROPOSITION 2.3 *Consider a nondecreasing lower semicontinuous function* $\psi : [0,T] \mapsto \mathbf{R}$ *and assume that for some* $t \in [0,T[$, $D_{\downarrow}\psi(t)(1) < \infty$. *Then* ψ *is continuous at* t. *If in addition for some* $m \geq 0$ *and for all* $t \in [0,T[$, $\liminf_{s\to t+} \frac{\psi(s)-\psi(t)}{s-t} \leq m$, *then* ψ *is* $m-$*Lipschitz.*

Proof. Indeed since ψ is nondecreasing and lower semicontinuous it is left continuous. On the other hand $\limsup_{t'\to t+} \frac{\psi(t')-\psi(t)}{t'-t} < \infty$ and therefore ψ is continuous from the right. For all $t \geq T$ define $\psi(t) = \psi(T)$. To prove the second statement we consider the viability problem

$$\begin{cases} t'(s) = 1 \\ x'(s) = m \\ (t(s), x(s)) \in K := \text{ Epigraph}(\psi) \end{cases}$$

and observe that for all $(t,y) \in K$, $(1,m)$ belongs to the contingent cone $T_K(t,y)$. By the viability theorem from Aubin (1991) for all $t_0 \in [0,T]$ and all $s \leq T - t_0$ we have $(t_0+s, \psi(t_0)+ms \in K$. Thus, for all $0 \leq t_0 \leq t_1 \leq T$

$$\psi(t_0) + m(t_1 - t_0) \geq \psi(t_1) \geq \psi(t_0)$$

which completes the proof. ■

PROPOSITION 2.4 *Assume that* φ *is continuous,* $\partial_+\varphi(x) \neq \emptyset$ *for all* $x \in \mathbb{R}^n$, *that (11) holds true and that* f *is differentiable with respect to* x. *Consider a solution* z *to the control system (8) defined on* $[t_0,T]$. *Then* $t \mapsto V(t,z(t))$ *is continuous.*

Proof. Set $\psi(t) = V(t,z(t))$. Since ψ is nondecreasing and lower semicontinuous it is left continuous. We only check its right continuity at t_0. Let $(x(\cdot),u(\cdot))$ solve the minimization problem (10), i.e. $V(t_0,x_0) = \varphi(x(T))$ and x_h be solutions to the systems

$$x' = f(x,u(t)), \;\; x_h(t_0+h) = z(t_0+h).$$

If $m(t_0) \in Dz(t_0)(1)$ then for some $h_i \to 0+$, $(z(t_0+h_i) - z(t_0))/h_i$ converge to $m(t_0)$. Thus for every $p_T \in \partial_+\varphi(x(T))$ and the solution w to the system

$$w' = \frac{\partial f}{\partial x}(x(t), u(t))w, \quad w(t_0) = m(t_0)$$

we have

$$D_\uparrow \psi(t_0)(1) \leq \limsup_{i\to\infty} \frac{\varphi(x_{h_i}(T)) - \varphi(x(T))}{h_i} \leq \langle p_T, w(T)\rangle .$$

By Proposition 2.3, ψ is continuous from the right at t_0. ∎

Definition 2.5 *Consider a convex subset K of $\mathbf{R}^n$. A function $g : K \mapsto \mathbf{R}$ is called* semiconcave *if there exists $\omega : \mathbf{R}_+ \times \mathbf{R}_+ \mapsto \mathbf{R}_+$ such that*

$$\forall\, r \leq R,\ \forall\, s \leq S,\ \ \omega(r,s) \leq \omega(R,S)\ \ \&\ \ \lim_{s\to 0+} \omega(R,s) \ = \ 0 \tag{13}$$

and for every $R > 0$, $\lambda \in [0,1]$ and all $x, y \in K \cap RB$

$$\lambda g(x) + (1-\lambda) g(y) \leq g(\lambda x + (1-\lambda)y) + \lambda(1-\lambda)\|x-y\|\ \omega(R, \|x-y\|).$$

We say that g is semiconcave at x_0 if there exists a neighborhood of x_0 in K such that the restriction of g to it is semiconcave. We call the above function ω a modulus of semiconcavity of g.

Every concave function $g : K \mapsto \mathbf{R}$ is semiconcave (with ω equal to zero). From Cannarsa and Sinestrari (2004) we also know that every semiconcave function is locally Lipschitz and has nonempty superdifferentials.

Theorem 2.2 *Let $K \subset \mathbf{R}^n$ be a convex set, $x_0 \in K$ and let a function $g : K \mapsto \mathbf{R}$ be semiconcave at x_0. Then for every $v \in T_K(x_0)$*

$$\liminf_{\substack{v' \to v,\, h \to 0+ \\ x' \to_K x_0,\ x' + hv' \in K}} \frac{g(x'+hv') - g(x')}{h} = \lim_{\substack{v' \to v,\, h \to 0+ \\ x_0 + hv' \in K}} \frac{g(x_0+hv') - g(x_0)}{h}.$$

Furthermore, setting $g = -\infty$ outside of K,

$$\mathrm{Limsup}_{x \to_{\mathrm{Int}(K)} x_0}\ \partial_+ g(x) \ \subset\ \partial_+ g(x_0).$$

Proof. Since g is semiconcave, by Cannarsa and Sinestrari (2004) it is locally Lipschitz. It is enough to consider the case $\|v\| \leq 1$. Fix such v and let $\delta > 0$ be so that g is semiconcave on $K \cap B_{2\delta}(x_0)$ with semiconcavity modulus $\omega(\cdot) := \omega(2\delta, \cdot)$. Let $x \in K \cap B_\delta(x_0)$. Then for all $0 < h_1 \leq h_2 \leq \delta$ such that $x + h_2 v \in K$ we have

$$g(x + h_1 v) - g(x) \;=\; g\left(\frac{h_1}{h_2}(x + h_2 v) \;+\; \left(1 - \frac{h_1}{h_2}\right) x\right) \;-\; g(x)$$
$$\geq\; \frac{h_1}{h_2} g(x + h_2 v) \;-\; \frac{h_1}{h_2} g(x) -\; h_1 \left(1 - \frac{h_1}{h_2}\right) \|v\|\, \omega(h_2 \|v\|).$$

Consequently,

$$\frac{g(x + h_1 v) - g(x)}{h_1} \geq \frac{g(x + h_2 v) - g(x)}{h_2} - \left(1 - \frac{h_1}{h_2}\right) \omega(h_2 \|v\|)$$

and we proved that for every $x \in K \cap B_\delta(x_0)$ and all $0 < h' \leq h \leq \delta$,

$$\frac{g(x + h'v) - g(x)}{h'} \;\geq\; \frac{g(x + hv) - g(x)}{h} - \omega(h \|v\|). \tag{14}$$

Thus for every $0 < h \leq \delta$

$$\liminf_{\substack{h' \to 0+ \\ v' \to v \\ x + h'v' \in K}} \frac{g(x + h'v') - g(x)}{h'} \geq \frac{g(x + hv) - g(x)}{h} - \omega(h \|v\|).$$

Taking lim sup in the right-hand side of the above inequality when $x = x_0$, we deduce that

$$\lim_{\substack{h \to 0+, v' \to v \\ x_0 + hv' \in K}} \frac{g(x_0 + hv') - g(x_0)}{h}$$

does exist. Fix $\varepsilon > 0$ and $0 < \lambda < \delta$. From the Lipschitz continuity of g it follows that there exists $0 < \alpha < \delta$ such that for all $x \in K \cap B_\alpha(x_0)$ and $v' \in B_\alpha(v)$

$$\frac{g(x_0 + \lambda v) - g(x_0)}{\lambda} \;\leq\; \frac{g(x + \lambda v') - g(x)}{\lambda} \;+\; \varepsilon$$

whenever $x_0 + \lambda v \in K$ and $x + \lambda v' \in K$. Thus, using (14), we obtain that for all sufficiently small $\alpha > 0$,

$$\frac{g(x_0 + \lambda v) - g(x_0)}{\lambda}$$
$$\leq \inf_{\substack{x \in K \cap B_\alpha(x_0) \\ h \in]0, \lambda], v' \in B_\alpha(v) \\ x + hv' \in K}} \frac{g(x + hv') - g(x)}{h} \;+\; \omega(\lambda \|v'\|) \;+\; \varepsilon.$$

Letting ε, α and λ converge to zero we end the proof of the first statement. In particular this yields that for all $x_0 \in Int(K)$, $\partial_+ g(x_0)$ is the generalized gradient (see Cannarsa and Sinestrari, 2004) of g at x_0. Thus

$$\partial_+ g(x_0) = \overline{co}\ \mathrm{Limsup}_{x\to x_0} \{\nabla g(x)\}\,. \tag{15}$$

To prove the last statement we set $g = -\infty$ outside of K. Consider a sequence $x_m \in \mathrm{Int}(K)$ converging to x_0 and a sequence $p_m \in \partial_+ g(x_m)$ converging to some p. We have to show that $p \in \partial_+ g(x_0)$. From (15) and the Carathéodory theorem, we deduce that there exist $\lambda_i^m \geq 0$ and $x_i^m \in \mathrm{Int}(K)$ converging to x_0 when $m \to \infty$ such that g is differentiable at x_i^m, and for all i the sequence $\nabla g(x_i^m)$ converges to some p_i when $m \to \infty$, and for every m, $\sum_{i=0}^n \lambda_i^m = 1$,

$$\lim_{m\to\infty} \left(\sum_{i=0}^n \lambda_i^m \nabla g(x_i^m) \right) \; = \; p.$$

Taking a subsequence and keeping the same notations, we may assume that $(\lambda_0^m, ..., \lambda_n^m)$ converge to some $(\lambda_0, ..., \lambda_n)$. Thus $p = \sum_{i=0}^n \lambda_i p_i$. Since $\partial_+ g(x_0)$ is convex, the above yields that it is enough to prove the last statement only in the case when g is differentiable at x_m. Fix $v \in T_K(x_0)$ and consider $h_m \to 0+$ such that $x_m + h_m v \in K$ and

$$\frac{g(x_m + h_m v) - g(x_m)}{h_m} \; \leq \; \langle \nabla g(x_m),\ v \rangle \; + \; \frac{1}{m}.$$

This and the first claim imply that

$$\lim_{v'\to v,\ h\to 0+} \frac{g(x_0 + hv') - g(x_0)}{h} \; \leq \; \langle p, v \rangle\,.$$

Hence from Proposition 2.1 we deduce that $p \in \partial_+ g(x_0)$. ■

We provide next a sufficient condition for semi-concavity of the value function on $[0,T] \times \mathbf{R}^n$. Let us assume that

$$\left\{ \begin{array}{l} \exists\, \omega : \mathbf{R}_+ \times \mathbf{R}_+ \mapsto \mathbf{R}_+ \text{ such that (13) holds true and} \\ \\ \forall\, \lambda \in [0,1],\ R > 0,\ x_0, x_1 \in B_R(0),\ t \in [0,T],\ u \in U \\ \|\lambda f(t,x_0,u) + (1-\lambda) f(t,x_1,u) - f(t,x_\lambda,u)\| \\ \leq\ \lambda(1-\lambda)\, \|x_1 - x_0\|\, \omega(R, \|x_1 - x_0\|), \\ \text{where } \ x_\lambda = \lambda x_0 + (1-\lambda) x_1 \\ \\ \varphi : \mathbf{R}^n \mapsto \mathbf{R} \ \text{ is semiconcave.} \end{array} \right. \tag{16}$$

Remark 2.1 1) *Assumptions (16) hold true, in particular, when φ is continuously differentiable and f is continuously differentiable with respect to x*

uniformly in (t,u). *More precisely, if we assume that there exists a function* $\omega : \mathbf{R}_+ \times \mathbf{R}_+ \mapsto \mathbf{R}_+$ *satisfying (13) such that*

$$\left\| \frac{\partial f}{\partial x}(t,x_1,u) - \frac{\partial f}{\partial x}(t,x_2,u) \right\| \leq \omega(R, \|x_1 - x_2\|)$$

for all $t \in [0,T]$, $u \in U$ *and* $x_1, x_2 \in B_R(0)$.

2) *Vice versa, Theorem 2.2 implies that, if* f *satisfies (16), then* f *is continuously differentiable with respect to* x.

THEOREM 2.3 *Assume (11) and (16). Then the value function is semi-concave on* $[0,T] \times \mathbf{R}^n$.

See Cannarsa and Frankowska (1991) for the proof.

3. Superdifferentials along optimal solutions

In this section we extend several results from Cannarsa and Frankowska (1991) to the case of non Lipschitz value function.

THEOREM 3.1 *Assume (11), that* φ *is lower semicontinuous and let* $(t_0, x_0) \in [0,T] \times \mathbf{R}^n$. *Consider a solution* $z : [t_0, T] \mapsto \mathbf{R}^n$ *to control system (8) with* $z(t_0) = x_0$ *and let* $\overline{u}$ *be a corresponding control. If for every* $t \in]t_0, T[$, *there exists* $p(t) \in \mathbf{R}^n$ *and*

$$m(t) \in \operatorname{Limsup}_{h \to 0+} \frac{z(t+h) - z(t)}{h}$$

such that

$$\left\{ \begin{array}{rcl} \langle p(t), m(t) \rangle & = & H(z(t), p(t)) \\ (H(z(t), p(t)),\ -p(t)) & \in & \partial_+ V(t, z(t)) \end{array} \right. \tag{17}$$

and $\lim_{t \to t_0+} V(t, z(t)) = V(t_0, x_0)$, *then* z *is optimal for problem (10).*

Proof. The map $\psi(t) = V(t, z(t))$ is nondecreasing and lower semicontinuous. Let $t \in]t_0, T[$ and $m(t)$ be as above. Consider $h_i \to 0+$ such that $(z(t+h_i) - z(t))/h_i \to m(t)$ Then,

$$\begin{array}{l} 0 \ = \ \langle (H(z(t),p(t)), -p(t)), (1, m(t)) \rangle \geq D_\downarrow V(t, z(t))(1, m(t)) \\ \geq \ \limsup_{i \to \infty} \frac{V(t+h_i, z(t+h_i)) - V(t, z(t))}{h_i} \geq \liminf_{h \to 0+} \frac{\psi(t+h) - \psi(t)}{h}. \end{array}$$

Hence

$$\forall\, t \in]t_0, T[,\ (1,0) \in T_{\mathrm{Epi}(\psi)}(t, \psi(t)).$$

By Aubin and Frankowska (1990)

$$\forall\, t \in]t_0, T[,\ \ \forall\, y \geq \psi(t),\ \ (1,0) \in T_{\mathrm{Epi}(\psi)}(t, y).$$

Set $\psi(t) = \psi(T)$ for all $t > T$. The viability theorem from Aubin (1991) yields that for every $t_1 > t_0$ the problem

$$\begin{cases} s' &=& 1, \;\; s(0) = t_1 \\ y' &=& 0, \;\; y(0) = \psi(t_1) \\ (s, y(s)) &\in& \mathrm{Epi}(\psi) \end{cases}$$

has a solution defined on $[t_1, T]$. Thus, for all $t \in [t_1, T]$, $\psi(t) \leq \psi(t_1)$. Consequently, $\psi \equiv \psi(t_1)$ on $[t_1, T]$. We deduce that the map $t \mapsto V(t, z(t))$ is constant on $]t_0, T]$. By our assumptions ψ is continuous at t_0 and therefore ψ is constant on $[t_0, T]$. So z is optimal. ■

THEOREM 3.2 *Assume (11), that φ is lower semicontinuous, and that f is differentiable with respect to x. Consider an optimal trajectory/control pair $(z, \overline{u})$ to control system (8) with $z(t_0) = x_0$. Then for every $p_T \in \partial_+\varphi(z(T))$ the solution $p : [t_0, T] \mapsto \mathbf{R}^n$ to the adjoint system*

$$-p'(t) \;=\; \left(\frac{\partial f}{\partial x}(z(t), \overline{u}(t))\right)^{\star} p(t), \;\; p(T) \;=\; -p_T \tag{18}$$

satisfies the maximum principle for all $t \in [t_0, T]$:

$$\forall\, m(t) \in \; \mathrm{Limsup}_{s \to t} \frac{z(s) - z(t)}{s - t}, \; \langle p(t), m(t) \rangle = H(z(t), p(t)) \tag{19}$$

and the generalized transversality conditions

$$(H(z(t), p(t)), -p(t)) \;\in\; \partial_+ V(t, z(t)) \;\text{ for every }\; t \in]t_0, T[\,. \tag{20}$$

REMARK 3.1 *This result is a refinement of Pontryagin's maximum principle and transversality conditions of the type obtained in Cannarsa and Frankowska (1991), Subbotina (1989). Also the the fact that any $p_T \in \partial_+\varphi(z(T))$ can be taken as the final state for the adjoint variable $p(\cdot)$ is in the same spirit as the maximum principle in Cannarsa, Frankowska and Sinestrari (2000), Theorem 3.1. We underline that conditions (19) and (20) hold true everywhere instead of "for almost all t". Such refinement is needed to study the optimal synthesis problem.*

Proof. The proof is analogous to the one given in Cannarsa, Frankowska and Sinestrari (2000). Fix $s_i \to t$ and $m(t) \in \; \mathrm{Limsup}_{s \to t} \frac{z(s)-z(t)}{s-t}$ such that

$$\frac{z(s_i) - z(t)}{s_i - t} \to m(t).$$

By the mean value theorem from Aubin and Cellina (1984), $m(t) \in \overline{\mathrm{co}} f(z(t), U)$. It is not restrictive to assume that $s_i - t$ are either all positive or all negative.

<u>Case 1.</u> $s_i < t$ for all $i \geq 1$. Fix any $u \in U$ and consider a solution x_i to the system

$$x' = f(x,u), \;\; x(s_i) = z(s_i).$$

Then $x_i(t) = z(s_i) + (t - s_i)f(z(t),u) + o(t - s_i)$. Thus

$$x_i(t) - z(t) = (t - s_i)(f(z(t),u) - m(t)) + o(t - s_i).$$

Then, setting

$$R(t) := \{x(t) \mid x \text{ solves (8) on } [t_0,T],\; x(t_0) = x_0\}$$

(the reachable set of (8) from (t_0, x_0) at time t, we deduce that $f(z(t),u) - m(t) \in T_{R(t)}(z(t))$. By Frankowska (1989a), $\langle p(t), f(z(t),u) - m(t)\rangle \leq 0$ and (19) follows because $u \in U$ is arbitrary.

<u>Case 2</u>. $s_i > t$ for all $i \geq 1$. Then, by Frankowska (1989a),

$$z(t) + (s_i - t)\,\overline{\text{co}}\, f(z(t),U) \subset R(s_i) + o(s_i - t)B.$$

Thus $\overline{\text{co}}\, f(z(t),U) - m(t)$ is contained in the set $W_z(t)$ of variations introduced in Frankowska (1989a). By Frankowska (1989a), $p(t) \in W_z(t)^-$, implying, (19).

Fix $t \in]t_0,T[,\; v \in \mathbf{R}^n,\; \alpha \in \mathbf{R}$. Then there exist $h_i \to 0+,\; \alpha_i \to \alpha,\; v_i \to v$ such that

$$D_\downarrow V(t,z(t))(\alpha,v) = \lim_{i\to\infty} \left(V(t + \alpha_i h_i, z(t) + h_i v_i) - V(t,z(t))\right)/h_i.$$

Taking a subsequence and using the same notations we may assume that $\overline{m}(t) := \lim_{i\to\infty} \frac{z(t+h_i\alpha_i)-z(t)}{h_i} \in \alpha \text{ Limsup}_{s\to t} \frac{z(s)-z(t)}{s-t}$. Set $m(t) = \overline{m}(t)/\alpha$ if $\alpha \neq 0$ and $m(t) = 0$ otherwise. Then for some $y_i \to v$

$$D_\downarrow V(t,z(t))(\alpha,v) =$$
$$= \lim\nolimits_{i\to\infty} \left(V(t + \alpha_i h_i, z(t + \alpha_i h_i) + h_i(y_i - \alpha m(t))) - V(t,z(t))\right)/h.$$

Consider the solution $w(\cdot)$ to the linear system

$$\begin{cases} w'(s) &= \frac{\partial f}{\partial x}(z(s),\overline{u}(s))w(s), \quad s \in [t,T] \\ w(t) &= v - \alpha m(t). \end{cases}$$

Then $w(T) = X(T)X(t)^{-1}(v - \alpha m(t))$, where $X(\cdot)$ is the fundamental solution to

$$\begin{cases} X'(s) &= \frac{\partial f}{\partial x}(z(s),\overline{u}(s))X(s), \quad s \in [t,T] \\ X(t) &= \text{Id}. \end{cases}$$

Let x_i solve

$$\begin{cases} x' = f(x, \overline{u}(s)), \quad s \in [t, T] \\ x(t+\alpha_i h_i) = z(t+\alpha_i h_i) + h_i(y_i - m(t)). \end{cases}$$

By the variational equation the difference quotients $(x_i - z)/h_i$ converge uniformly to w. Consequently,

$$\begin{aligned}
& D_{\downarrow}V(t, z(t))(\alpha, v) = \\
& = \lim_{i\to\infty} \left(V(t+\alpha_i h_i, x_i(t+\alpha_i h_i)) \ - \ V(t, z(t))\right)/h_i \\
& \leq \ \limsup_{i\to\infty} \left(\varphi(x_i(T)) - \varphi(z(T))\right)/h_i \ \leq \ \langle p_T, w(T)\rangle \\
& = \left\langle p_T, X(T)X(t)^{-1}(v - \alpha m(t))\right\rangle = \left\langle (X(t)^\star)^{-1} X(T)^\star p_T, v - \alpha m(t)\right\rangle \\
& = \ \langle -p(t), v - \alpha m(t)\rangle \ = \ \langle -p(t), v\rangle \ + \ \alpha \langle p(t), m(t)\rangle \\
& = \ \alpha H(z(t), p(t)) \ + \ \langle -p(t), v\rangle
\end{aligned}$$

and (20) follows from Proposition 2.1. ■

By Proposition 2.2 the value function satisfies the inequality

$$\forall\, t < T, \ \ \forall\, (p_t, p_x) \in \partial_+ V(t, x), \ \ -p_t + H(x, -p_x) \leq 0 \tag{21}$$

i.e., it is a subsolution of the Hamilton-Jacobi-Bellman equation. Actually, the equality holds true along optimal solutions. This fact was first observed in Frankowska (1989b) and then generalized in Cannarsa and Frankowska (1991), but in both papers the result was given for almost all t. We provide its improvement with the proof similar to that in Frankowska (1989b).

Proposition 3.1 *Assume (11) and let z be an optimal solution to problem (10). Then for every $t \in]t_0, T[$,*

$$\forall\, (p_t, p_x) \in \partial_+ V(t, z(t)), \ \ -p_t \ + \ H(z(t), -p_x) \ = \ 0 \tag{22}$$

and

$$\forall\, m(t) \in Dz(t)(-1), \ \ \langle -p_x, m(t)\rangle = H(z(t), -p_x).$$

Proof. Observe first that z is Lipschitz. Fix $t_0 < t < T$. Then for a sequence $h_i \to 0+$ and some $v \in \overline{co}\, f(z(t), U)$

$$\lim_{i\to\infty} \frac{z(t-h_i) - z(t)}{h_i} \ = \ -v.$$

Fix $(p_t, p_x) \in \partial_+ V(t, z(t))$. Then for all $(p_t, p_x) \in \partial_+ V(t, z(t))$

$$0 \geq \limsup_{s\to t-} \frac{V(s, z(s)) - V(t, z(t)) - p_t(s-t) - \langle p_x, z(s) - z(t)\rangle}{|s-t| + \|z(s) - z(t)\|}.$$

Since $V(\cdot, z(\cdot))$ is constant on $[0,T]$, the above estimate yields $0 \geq p_t + \langle p_x, v\rangle$. Thus,$-p_t + H(z(t), -p_x) \geq -p_t + \langle -p_x, v\rangle \geq 0$. Since the opposite inequality is satisfied, we get the result. ■

PROPOSITION 3.2 *Under all assumptions of Theorem 3.2, if z is an optimal solution to problem (10) and $\partial_+\varphi(z(t)) \neq 0$, then for all $t \in]t_0, T[$,*

$$\forall\, m(t) \in Dz(t)(1), \quad D_\downarrow V(t, z(t))(1, m(t)) = 0.$$

Proof. Fix $m(t) \in Dz(t)(1)$ and let $h_i \to 0+$ be such that $(z(t+h_i) - z(t))/h_i \to m(t)$. By Theorem 3.2, for all $t \in]t_0, T]$

$$H(z(t), p(t)) = \langle p(t), m(t)\rangle\,.$$

Set $\psi(t) = V(t, z(t))$. By Theorem 3.2 for all $t \in [t_0, T[$

$$\begin{aligned} 0 = \limsup_{i\to\infty} \frac{\psi(t+h_i) - \psi(t)}{h_i} &\leq D_\downarrow V(t, z(t))(1, m(t)) \\ &\leq H(z(t), p(t)) - \langle p(t), m(t)\rangle = 0. \end{aligned}$$

■

THEOREM 3.3 *We impose all the assumptions of Theorem 3.2. Then z, satisfying $\partial_+\varphi(z(t)) \neq 0$ is optimal for the problem (10) if and only if for all $t \in]t_0, T[$*

$$\forall\, m(t) \in Dz(t)(1), \quad D_\downarrow V(t, z(t))(1, m(t)) = 0.$$

and $\lim_{t\to t_0} V(t, z(t)) = V(t_0, z(t_0))$.

Proof. From Proposition 3.2 we get the necessity. Assume next that a trajectory z of (8) satisfies the above equality. To prove sufficiency, we deduce from Proposition 2.3 that $\psi(t) := V(t, z(t))$ is continuous. Then for some $h_i \to 0+$,

$$D_\downarrow \psi(t)(1) = \lim_{i\to\infty} \frac{\psi(t+h_i) - \psi(t)}{h_i}.$$

It is not restrictive to assume that $\frac{z(t+h_i)-z(t)}{h_i}$ converge to some $m(t) \in Dz(t)(1)$. Then

$$D_\downarrow \psi(t)(1) \leq D_\downarrow V(t, z(t))(1, m(t)) = 0.$$

Applying the viability theorem exactly in the same way as in the proof of Theorem 3.1 we get $\psi(T) \leq \psi(t_0)$. Thus ψ is constant and the proof follows. ■

COROLLARY 3.1 *Under all assumptions of Theorem 3.2 if z is optimal for the problem (10) and $\partial_+\varphi(z(t)) \neq 0$, then for every $t \in]t_0, T[$ and all $(p_t, p_x) \in \partial_+ V(t, z(t))$*

$$\forall\, u \in U, \ \forall\, m(t) \in Dz(t)(1), \quad D_\downarrow V(t, z(t))(1, m(t)) \leq p_t + \langle p_x, f(x, u)\rangle\,.$$

Proof. We apply Theorem 3.3 and use (21). ■

4. Optimal synthesis

We introduce the set-valued maps

$$U(t,x) = \{u \in U \mid \forall\, (p_t, p_x) \in \partial_+ V(t,x),\ \ p_t + \langle p_x, f(x,u)\rangle = 0\}\,,$$

$$G(t,x) = f(x, U(t,x))$$

and consider the new control system

$$\left\{ \begin{array}{rcl} x' & = & f(x,u), \ \ u \in U(t,x) \\ x(t_0) & = & x_0 \end{array} \right. \tag{23}$$

and the corresponding differential inclusion

$$\left\{ \begin{array}{rcl} x' & \in & G(t,x) \\ x(t_0) & = & x_0. \end{array} \right. \tag{24}$$

Since in general $U(\cdot,\cdot)$ is discontinuous it does not have continuous selections. To give meaning to solutions to (23) we say that an absolutely continuous $x : [t_0, T] \mapsto \mathbf{R}^n$ solves (23) if for almost all $t \in [t_0, T],\ x'(t) \in f(t, U(t, x(t)))$, i.e., if x solves the differential inclusion (24).

PROPOSITION 4.1 *Assume (11). Then the set-valued map G has closed convex, possibly empty, images.*

From Proposition 3.1 and Theorem 3.2 we deduce the following results.

THEOREM 4.1 *Assume that φ is lower semicontinuous, (11) holds true and f is differentiable with respect to x. If z is optimal for problem (10) and $\partial_+\varphi(z(t)) \neq 0$, then it solves (23) and there exists $M > 0$ such that*

$$\forall\, t \in]t_0, T[,\ \ \inf_{(p_t,p_x)\in\partial_+ V(t,z(t))} \|p_x\| \le M. \tag{25}$$

Our next aim is to show that the converse statement holds true as well.

THEOREM 4.2 *Assume that φ is lower semicontinuous and (11) holds true. If for some $M > 0$ there exists a solution z to (23), (25) satisfying $z(t_0) = x_0$ and $\lim_{t\to t_0} V(s, z(s))) = V(t_0, x_0)$. Then z is optimal for problem (10).*

Proof. Set $\psi(t) = V(t, z(t))$. Then ψ is nondecreasing, lower semicontinuous and for all $t \in]t_0, T[$ and all $m(t) \in Dz(t)(1)$, $(p_t, p_x) \in \partial_+ V(t,x)$ we have

$$\begin{aligned} D_\uparrow \psi(t)(1) \le D_\downarrow V(t, z(t))(1, m(t)) &\le p_t + \langle p_x, m(t)\rangle \\ &= H(z(t), -p_x) + \langle p_x, m(t)\rangle\,. \end{aligned}$$

This and Proposition 2.3 imply that ψ is Lipschitz. From the above inequality and definition of $U(t,x)$ we deduce that $\psi'(t) \le 0$ almost everywhere. Thus, ψ is also nonincreasing. Consequently, it is constant and z is optimal. ■

THEOREM 4.3 *Assume that* φ *is continuous,* $\partial_+\varphi(x) \neq \emptyset$ *for all* $x \in \mathbb{R}^n$*, that (11) holds true and that* f *is differentiable with respect to* x*. Then every solution* z *to (23) is optimal.*

Proof. By Proposition 2.4, $\psi(t) = V(t, z(t))$ is continuous nondecreasing. By the proof of Proposition 2.4

$$\sup_{t\in[t_0,T[} D_\uparrow\psi(t)(1) < \infty.$$

From Proposition 2.3 we deduce that ψ is Lipschitz. By the same arguments as in the proof of Theorem 4.2 we obtain $\psi'(t) \le 0$ for almost all t. Therefore ψ is constant. ∎

The natural question arises whether it is possible to restrict the attention only to (23). The answer is positive if instead of absolutely continuous solutions to (23) we consider its contingent solutions.

DEFINITION 4.1 *We say that a continuous mapping* $z : [t_0, T] \mapsto \mathbf{R}^n$ *is a contingent solution to (23) if*

$$\forall\, t \in]t_0, T[, \quad Dz(t)(1) \cap f(z(t), U(t, z(t))) \neq \emptyset.$$

PROPOSITION 4.2 *Assume (11). If* z *is a contingent solution to (23), then* z *is Lipschitz.*

Proof. Let $\gamma = \sup\{\|f(z(t), u)\|\, u \in U\}$. Consider $t_1 > t_0$ and the viability problem

$$\left\{\begin{array}{lcll} s' & = & 1, & s(0) = t_1 \\ x' & \in & \gamma B, & x(0) = z(t_1) \\ \forall\, t & \ge & 0, & (s(t), x(t)) \in \mathrm{Graph}(z). \end{array}\right. \tag{26}$$

Its solution $(s(t), x(t))$ is $(\gamma + 1)-$Lipschitz. Since $x(s) = z(t_1 + s)$, we deduce that z is also $(\gamma+1)-$Lipschitz on $[t_1, T]$. But $t_1 > t_0$ being arbitrary, the proof follows. ∎

THEOREM 4.4 *Assume that* φ *is lower semicontinuous and that (11) holds true. If* z *is a contingent solution to (23) and* $\lim_{t\to t_0+} V(t, z(t)) = V(t_0, x_0)$*, then* z *is optimal.*

Proof. Let ψ be defined as in the proof of Theorem 4.2. Then for any $t \in]t_0, T[$ and $m(t) \in D\psi(t)(1) \cap f(z(t), U(t, z(t)))$,

$$D_\uparrow\psi(t)(1) \le D_\downarrow V(t, z(t))(1, m(t)) \le p_t + \langle p_x, m(t)\rangle = 0.$$

Applying the same arguments as in the proof of Theorem 4.2 we conclude. ∎

Proposition 3.1 yields the following result.

THEOREM 4.5 *Assume that (11) holds true. If z is optimal, then for all $t \in]t_0, T[$, $z(t) \in \mathrm{Dom}(G(t,\cdot))$.*

We do not know if every optimal trajectory z is a contingent solution to (24). It is possible, however, to increase the feedback map G by loosing its convexity but gaining a necessary and sufficient condition for optimality.

We introduce the set-valued maps

$$\hat{U}(t,x) = \{u \in U \mid \exists \ (p_t, p_x) \in \partial_+ V(t,x), \ \ p_t + \langle p_x, f(x,u)\rangle = 0\}$$

and

$$\hat{G}(t,x) = f(x, \hat{U}(t,x)).$$

Consider the new control system

$$\begin{cases} x' &= f(x,u), \ \ u \in \hat{U}(t,x) \\ x(t_0) &= x_0. \end{cases} \tag{27}$$

and the differential inclusion

$$\begin{cases} x' &\in \hat{G}(t,x) \\ x(t_0) &= x_0\,. \end{cases} \tag{28}$$

Solutions to (27) are understood as absolutely continuous solutions to (28).

Theorem 3.2 and the very same proof as the one of Theorem 4.4 imply the following necessary and sufficient condition for optimality.

THEOREM 4.6 *Assume (11), that φ is continuous, $\partial_+\varphi(x) \neq \emptyset$ for all $x \in \mathbb{R}^n$ and that f is differentiable with respect to x. A trajectory z of (8) is optimal if and only if it is a contingent solution to (28) and* $\lim_{t\to t_0+} V(t,z(t)) = V(t_0,x_0)$.

When in addition the value function V is semiconcave, then the set-valued map $\hat{G}$ is upper semicontinuous.

THEOREM 4.7 *Assume that (11) and (16) hold true. Then the set-valued map G is upper semicontinuous on* $[0,T] \times \mathbf{R}^n$.

Proof. By Theorem 2.3 V is semiconcave on $[0,T] \times \mathbf{R}^n$. Thus it is locally Lipschitz on $[0,T] \times \mathbf{R}^n$ and have nonempty superdifferentials. Fix $(t_0,x_0) \in [0,T] \times \mathbf{R}^n$.

By Aubin and Frankowska (1990) it is enough to show that graph(G) is closed. Consider $(t_i,x_i) \to (t_0,x_0)$ and $u_i \in U(t_i,x_i)$ such that $f(x_i,u_i) \to f(x_0,u)$. Let $(p_t^i, p_x^i) \in \partial_+ V(t_i,x_i)$ be such that $p_t^i = \left\langle p_x^i, -f(x_i,u_i)\right\rangle = H(x_i, -p_x^i)$. Taking a subsequence and keeping the same notations we may assume that $(p_t^i,p_x^i) \to (p_t,p_x)$. By Theorem 2.2, $(p_t,p_x) \in \partial_+V(t_0,x_0)$. Since the Hamiltonian is continuous, $p_t = \langle p_x, -f(x_0,u)\rangle = H(x_0,-p_x)$. ■

Notice that all bounded upper semicontinuous map with closed images do have closed graph. This property is very useful in numerical approximations of solutions to differential inclusions.

References

AUBIN, J.-P. (1991) *Viability Theory.* Birkhäuser, Boston, Basel, Berlin.

AUBIN, J.-P. and CELLINA, A. (1984) *Differential Inclusions.* Springer-Verlag, Grundlehren der Math. Wiss. 264.

AUBIN, J.-P. and FRANKOWSKA, H. (1990) *Set-Valued Analysis.* Birkhäuser, Boston.

BARDI, M. and CAPUZZO DOLCETTA, I. (1997) *Optimal Control and Viscosity Solutions of Hamilton–Jacobi Equations.* Birkhäuser, Boston.

BERKOVITZ, L.D. (1989) Optimal feedback controls. *SIAM J. Control Optimiz.* **27**, 991-1006.

CANNARSA, P. and FRANKOWSKA, H. (1991) Some characterizations of optimal trajectories in control theory. *SIAM J. on Control and Optimization* **29**, 1322-1347.

CANNARSA, P., FRANKOWSKA, H. and SINESTRARI, C. (2000) Optimality conditions and synthesis for the minimum time problem. *J. Set-Valued Analysis* **8**, 127-148.

CANNARSA, P. and SINESTRARI, C. (2004) *Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control.* Birkhaüser, Boston.

FLEMING, W.H. and RISHEL, R.W. (1975) *Deterministic and Stochastic Optimal Control.* Springer-Verlag, New York.

FRANKOWSKA, H. (1989a) Contingent cones to reachable sets of control systems. *SIAM J. on Control and Optimization* **27**, 170-198.

FRANKOWSKA, H. (1989b) Optimal trajectories associated to a solution of contingent Hamilton-Jacobi equations. *Applied Mathematics and Optimization* **19**, 291-311.

IOFFE, A.D. (1977) On lower semicontinuity of integral functionals. *SIAM J. Control Optim.* **15**, 521-521 and 991-1000.

OLECH, C. (1976) Weak lower semicontinuity of integral functionals. *J. Optim. Theory Appl.* **19**, 3-16.

SUBBOTINA, N.N. (1989) The maximum principle and the superdifferential of the value function. *Problems of Control and Information Theory* **18**, 151-160.