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Self-adjoint extensions of differential operators and exterior topological derivatives in shape optimization

by

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Abstract: Self-adjoint extensions are constructed for a family of boundary value problems in domains with a thin ligament and an asymptotic analysis of a L_q-continuous functional is performed. The results can be used in numerical methods of shape and topology optimization of integral functionals for elliptic equations. At some stage of optimization process the singular perturbation of geometrical domain by an addition of thin ligament can be replaced by its approximation defined for the appropriate self-adjoint extension of the elliptic operator. In this way the topology variation of current geometrical domain can be determined and used e.g., in the level-set type methods of shape optimization.

Keywords: self-adjoint extension, shape optimization, exterior topological derivative, ligament.

1. Introduction

Topology optimization is a new field of research in shape optimization. We refer the reader to e.g., Allaire, Jouve, Toader (2004), Bendsoe, Sigmund (2003), Burger, Hackl, Ring (2004), Eschenauer, Kobelev, Schumacher (1994), Garreau, Guillaume, Masmoudi (2001), Novotny et al. (2003), Sokołowski, Żochowski (2001, 2003a), for recent developments in the field, using in particular the topological derivatives introduced in Sokołowski, Żochowski (1999), see also Nazarov, Sokołowski (2003) for the complete mathematical background. In the mathematical analysis of such optimization problems a special role is assigned to the shape functional in the form of the energy of weak solutions, the so-called compliance in solid mechanics (Lewiński, Sokołowski, 2003). On the other hand, for some nonlinear problems, e.g. the variational inequalities, the knowledge of the first term of asymptotics of the energy functional, can be used to construct the outer approximations of solutions and as a result to obtain the topological derivatives of shape functionals (Sokołowski, Żochowski, 2003b). The topology variation in shape optimization means in particular the introduction of small holes, or cavities in the geometrical domains, and leads to the singular perturbations of the associated boundary value problems. Therefore, we have to deal with geometrical domains depending on a small parameter which measures the size of such holes or cavities. From the numerical point of view such a construction is not satisfactory since it involves mesh generation for singularly perturbed domains. To overcome the difficulty and define the boundary value problem in a fixed domain we could construct the so-called self-adjoint extensions of elliptic operators (Nazarov, 1996). In the present paper such a construction is performed for the exterior topological variation of boundary value problems for the Laplacian.

Our analysis is applicable to the energy functional, and to a more general class of shape functionals which is defined in Section 6. The choice of the shape functional is limited by the results of asymptotic analysis performed for the boundary value problem under considerations, see (6) below for the precise conditions. In particular, our analysis does not cover the case of shape functional depending on the gradient of solutions.

Outline of the paper is the following. In Section 2 the boundary value problem in a domain with thin ligament is introduced and the shape functional is defined. In Section 3 the potential applications of our results are described. In Section 4 the limit problems are defined and the asymptotic approximations of solutions to problem in singularly perturbed domain are constructed. Self-adjoint extensions are obtained in Section 5. Finally, the first terms of asymptotics for shape functionals are determined in Section 6.

2. Statement of the problem

Let Ω be a domain on the plane $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$ and the compact closure $\overline{\Omega} = \partial\Omega \cup \Omega$. Suppose also that Γ is a simple, closed as a subset of $\mathbb{R}^2$, and smooth contour which intersects $\partial\Omega$ at the two points $P^{\pm}$ at nonzero angles and has endpoints inside Ω, see Fig. 1. In a neighborhood of Γ we introduce the natural curvilinear coordinates (ν, τ), where τ is the arc length along Γ and $|\nu|$ is the distance to Γ. We assume that P^+ and P^- correspond to the values $\tau = l$ and $\tau = -l$, respectively; here $2l$ is the length of the curve $\Gamma \setminus \overline{\Omega}$. Taking some freedom, we now identify a point and its coordinate on Γ. Considering functions $H_{\pm} \in C^{\infty}(\overline{\Gamma})$ such that $H := H_+ + H_- > 0$ we define the curvilinear strip

$$\Lambda_h = \{x : \tau \in \Gamma,\ \zeta := h^{-1}\nu \in \omega(\tau) := \big(-H_-(\tau), H_+(\tau)\big)\} \tag{1}$$

and the domain

$$\Omega(h) = \Omega \cup \Lambda_h, \tag{2}$$

depending on a small geometric parameter $h \in (0, h_0]$ (we fix the upper bound $h_0 > 0$ so that the ends of strip (1) are contained in Ω for $h \leqslant h_0$). The part of Λ_h outside Ω is called the *ligament* and denoted by $\Lambda(h) = \Lambda_h \setminus \overline{\Omega}$.

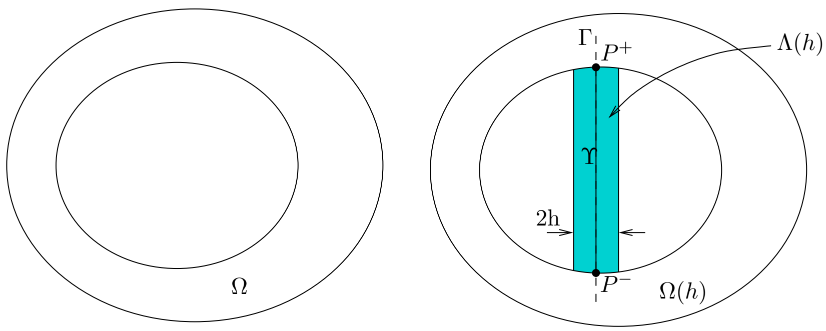

Figure 1. The geometrical domain $\Omega(h) = \Omega \cup \Lambda(h)$

We consider the following mixed boundary-value problem for the Poisson equation in the singularly perturbed domain (2), which implies a junction of sets with various limit dimensions:

$$\begin{aligned} &-\Delta_x u(h,x) = f(h,x),\ x \in \Omega(h),\\ &\partial_n u(h,x) = 0,\ x \in \partial\Omega(h) \setminus \overline{\Sigma},\\ &u(h,x) = 0,\ x \in \Sigma, \end{aligned} \tag{3}$$

where ∂_n is the derivative along the outward normal. Problem (3) has a unique solution $u(h,\cdot) \in \mathring{H}^1(\Omega(h);\Sigma)$ for every right-hand side $f(h,\cdot) \in L_2(\Omega(h))$. For the simplicity of the presentation, we assume that Σ is a connected component of the boundary $\partial\Omega$ and $\overline{\Sigma} \cap \overline{\Lambda}_h = \emptyset$ for $h \in (0, h_0]$. Thus, the solution $u(h,\cdot)$ is of class H^2 everywhere, possibly except at four angular points at which $\partial\Lambda_h$ intersects $\partial\Omega$. In fact, we should speak of a family of problems and the corresponding family of solutions parameterized by the *relative thickness* $h \in (0, h_0]$ of the ligament $\Lambda(h)$. Nevertheless, in construction of the self-adjoint extension we can assume that the parameter $h > 0$ is small but fixed and speak of the only one problem and its solution.

The asymptotic behavior of a solution $u(h,x)$ is determined in particular by the dependence of the right-hand side of equation $(3)_1$ on the parameter h. Suppose that

$$f(h,x) = \widetilde{f}(h,x) + \begin{cases} f_\Omega(x), & x \in \Omega; \\ f_\Lambda(\tau), & x \in \Lambda(h). \end{cases} \tag{4}$$

Here f_Ω and f_Λ are some given functions on the *body* Ω and on the *ligament axis* $\Upsilon = (-l,l) \ni \tau$, and $\widetilde{f}$ is a small remainder which can be ignored in asymptotic analysis. We state the exact conditions on f_Ω, f_Λ and $\widetilde{f}$ in Section 3. Note that all conditions are satisfied in the most reasonable situation: the function f in $(3)_1$ is independent of the parameter h and is smooth in a neighborhood of $\overline{\Omega}$ containing the ligament $\overline{\Lambda(h)}$.

Solutions of problem (3) are used in order to evaluate the functional

$$\mathcal{J}(u;h) = \int_{\Omega(h)} J(x;u(h,x))dx\,, \tag{5}$$

which is supposed to verify (compare with Nazarov, Sokołowski, 2005) the L_q-continuity condition: for some $q \in [1,\infty)$ and any $u, v \in L_q(\Omega(h))$ the inequality holds

$$|\mathcal{J}(u;h) - \mathcal{J}(v;h)| \leqslant c||u-v; L_q(\Omega(h))||(||u; L_q(\Omega(h))||^{q-1} + ||v; L_q(\Omega(h))||^{q-1})\,, \tag{6}$$

where the constant c is independent of the functions u, v and the parameter $h \in (0,h_0]$. In particular, the assumption (5) with $q = 1$ covers the case of the energy functional investigated in Nazarov, Sokołowski (2004).

3. Objectives and methods of shape and topology optimization

With problem (3) and functional (5) a shape optimization problem can be associated. The minimization or maximization of such a functional over a family of admissible domains involves the so-called interior and exterior topological derivatives. For the interior topological derivatives of shape functionals we have an almost complete mathematical theory at our disposal. In contrary, the exterior topological derivatives are now the subject of intensive studies for model problems.

The basic question, associated with the investigations of shape optimization problems over the family of singularly perturbed domains $\{\Omega(h)\}_{h\in(0,h_0]}$, concerns model simplification. Namely, whether the boundary value problems defined on domains $\Omega(h)$ can be approximated by a boundary value problems posed on the fixed geometrical domain. Usually, such an approximation leads to regular or singular perturbations of the differential operator with coefficients depending on a small parameter h. Such a modification, required by numerical methods, should preserve the main properties of the shape functional under consideration. For the interior topological derivatives of shape functionals the positive answer to such a question is provided in Nazarov (1996), Nazarov, Sokołowski (2003, 2005), on the basis of self-adjoint extensions of differential operators. The notion is proposed in Berezin, Faddeev (1961) and investigated in Karpeshina, Pavlov (1986), Pavlov (1987), Kamotski, Nazarov (1998) and others (see also the monograph of Albeverio, Kurasov, 2000, where the technique is used in some other applied problems).

The variable domain (2) can be replaced by the *hybrid* domain

$$\Omega^{\star} = \Omega \cup \overline{\Upsilon}\,, \tag{7}$$

which consists of the two-dimensional domain Ω, the one dimensional curve Υ (the ligament axis) and two tips $P^{\pm}$ of the curve Υ. The well posed, self-adjoint boundary value problems on hybrid domains, which include transmission conditions at the joining points, are investigated in Nazarov (2004). In this framework, generalized Green's formulae are applied and spaces with separated asymptotics are used (we refer to Nazarov, Plamenevskii, 1992, 1994, and to Nazarov, 1996, 1999, respectively, and to others). The problem formulations proposed in Nazarov (2004) in particular ensure its reformulations as minimization problems, applicability of a symmetric Green's formulae, the zero index and some other properties as expected for the usual self-adjoint problems in geometrical domains. The self-adjoint extensions are not investigated in Nazarov (2004) in all generality, therefore in the paper we provide the necessary results on the topic applied to the particular boundary value problem. Actually, a formulation of a problem on the hybrid domain (7) is given, with the associated unbounded self-adjoint operator $\mathfrak{A}$ defined in the space $\mathbb{L} = L_2(\Omega) \times L_2(\Upsilon)$. In addition, the solutions of the abstract equation

$$\mathfrak{A}\{\mathbf{v}, \mathbf{w}\} = \{\mathbf{f}_\Omega, \mathbf{f}_\Lambda\} \in L_2(\Omega) \times L_2(\Upsilon) \tag{8}$$

can be considered as approximations of solutions to the boundary value problem (3) in $\Omega(h)$, and used for the asymptotic formula of the functional (5).

4. Limit problems and asymptotic solutions

The first limit problem results from (3) by passing from $h > 0$ to $h = 0$, i.e., by elimination of the ligament from (2) and the remainder $\widetilde{f}$ from $(4)_1$,

$$\begin{aligned} &-\Delta_x v_0(x) = f_\Omega(x),\ x \in \Omega, \\ &\partial_n v_0(x) = 0,\ x \in \partial\Omega \setminus \overline{\Sigma};\, v_0(x) = 0,\ x \in \Sigma\,. \end{aligned} \tag{9}$$

If, for example, $f_\Omega \in L_2(\Omega)$, then there exists a unique weak solution $v_0 \in \mathring{H}^1(\Omega(h); \Sigma)$ to problem (9), the symbol $\mathring{H}$ means that $v_0 = 0$ on Σ. Since

$\Sigma \subset \partial\Omega$ is a connected component, there are no collision points on the boundary; the solution v_0 of the mixed boundary-value problem belongs to the Sobolev space $H^2(\Omega)$ and

$$||v_0; H^2(\Omega)|| \leqslant c||f; L^2(\Omega)|| .$$

It is not excluded that $\Sigma = \emptyset$, but then the solution v_0 is not unique.

A particular singular solution to problem (9) is of a further use. To specify it and to describe the behaviour of a solution near the points $P^{\pm}$, it is convenient to employ Kondratiev spaces (see Kondrat'ev, 1967 and also Maz'ya, Plamenevskii, 1978, and, for example Nazarov, Plamenevskii, 1994), which are defined as completion of the linear space $C_0^\infty(\overline{\Omega} \setminus P^{\pm})$ in the weighted norm

$$||v; V_\beta^{l,q}(\Omega)|| = \sum_{k=0}^{l} \left(||d_P^{\beta-l+k} \nabla_x^k v; L_q(\Omega)||^q \right)^{\frac{1}{q}} . \tag{10}$$

Here $l \in \mathbb{N}_0 = \{0, 1, 2, ...\}$, $q \in (1, +\infty)$ and $\beta \in \mathbb{R}$ are the indices of regularity, integrability, and weights, respectively, $d_P(x) = \min\{\mathrm{dist}(x, P^{\pm})\}$ being the weight factor.

It is known (see e.g., the introductory chapter 2 in Nazarov, Plamenevskii, 1994), that in the case of

$$f \in V_\beta^{0,q}(\Omega), \qquad 1 - \frac{2}{q} > \beta > -\frac{2}{q} , \tag{11}$$

there exists a solution v_0 to problem (9), which admits the representation

$$v_0(x) = \sum_{\pm} \chi_\Omega(r_\pm) \Big\{ v_0(P^\pm) + (s - s^\pm)\partial_s v_0(P^\pm) \Big\} + \widetilde{v}_0(x), \tag{12}$$

and the estimate

$$||\widetilde{v}_0; V_\beta^{2,q}(\Omega)|| + \sum_{\pm} \left\{ |v_0(P^\pm)| + |\partial_s v_0(P^\pm)| \right\} \leqslant ||f; V_\beta^{0,q}(\Omega)|| . \tag{13}$$

Here s is the arc length on $\partial\Omega$, $s^\pm$ are the coordinates of $P^\pm \in \partial\Omega$, and χ_Ω is a cut-off function in $C_0^\infty(\mathbb{R})$ which equals to one for $r < r_0/2$ and zero for $r > r_0$. Finally, $r_\pm = \mathrm{dist}(x, P^\pm)$ and the radius r_0 is sufficiently small.

REMARK 4.1 **1)** *Assume that* $q \geqslant 2, \beta \in (-\frac{2}{q}, 0]$. *Then the function*

$$f \in V_\beta^{0,q}(\Omega) \tag{14}$$

belongs to the intersection $L_2(\Omega) \cap V_\beta^{0,q}(\Omega)$. *Furthermore, both solutions to problem* (9) *introduced already coincide, namely, the weak (generalized) solution and the solution given by* (12).

2) *Inclusion* (14) *means that* $d_P^{\beta} f \in L_q(\Omega)$*, i.e., the restriction* (7) *in the paper Nazarov, Sokołowski (2004)*

$$d_P^{\mu} f \in L_2(\Omega) \quad \text{for some } \mu \in (0,1) \tag{15}$$

is satisfied. In view of the Hölder inequality the relation

$$||d_P^{-\mu} f; L_2(\Omega)|| \leqslant c||d_P^{\beta} f; L_q(\Omega)||\,, \qquad q \geqslant 2, \tag{16}$$

is satisfied for any $\mu < 1 - \beta - \frac{2}{q}$*, whence for* $\beta > -\frac{2}{q}$ *condition* (15) *is always verified.*
3) *The restrictions imposed on the weight indices in* (11) *heuristically can be seen by the following argument: In accordance with the Taylor expansion* (12) *the space* $V_{\beta}^{2,q}(\Omega)$ *includes functions, decaying as* $O(r_{\pm}^2)$ *in the vicinity of points* $P^{\pm}$*, however, it does not include functions with the linear behaviour in the vicinity of these points.*

In Nazarov, Sokołowski (2004) the following asymptotic ansatz for the restriction of the solutions to (3) to the ligament $\Lambda(h)$ is accepted:

$$u(h,x) \sim w_0(\tau) + hw_1(\zeta,\tau) + h^2 w_2(\zeta,\tau) + \dots \tag{17}$$

(compare with Dzhavadov, 1968; Mazja, Nazarov, Plamenevskii, 1991, chapter 15; Nazarov, 2002, chapter 1, and others). Here w_j are the functions to be determined and τ and $\zeta = h^{-1}\nu$ are the slow longitudinal and fast transversal variables on the ligament. The third term in (17) can be obtained by solving a family of problems defined on sections $\omega(\tau)$ of the ligament, see (1), parametrized by the variable $\tau \in \Upsilon = (-l, l)$:

$$\begin{aligned} &-\partial_\zeta^2 w_2(\zeta,\tau) = f_\Lambda(\tau) + \partial_\tau^2 w_0(\tau),\ \zeta \in \omega(\tau), \\ &\pm\partial_\zeta w_2(\pm H_\pm(\tau),\tau) = H'_\pm(\tau)\partial_\tau w_0(\tau)\,. \end{aligned} \tag{18}$$

Problem (18) admits a unique solution if and only if the right-hand sides verify the compability conditions which take the form of ordinary differential equations for the first term w_0 (we refer the reader to Nazarov, Sokołowski, 2004, Section 3, for all the details)

$$-\partial_\tau(H(\tau)\partial_\tau w_0(\tau)) = H(\tau) f_\Lambda(\tau),\ \tau \in \Upsilon\,. \tag{19}$$

We point out that a solution of the Neumann problem (18) is defined up to an additive term $w_2^0(\tau)$, which does not depend on the variable ζ. Actually, the term $h^2 w_2(\zeta,\tau)$ has no influence on the construction of global asymptotic approximation. The function w_1 is not fixed since it can be arbitrary in the framework of the required asymptotic precision (compare with Section 5 in Nazarov, Sokołowski, 2004). What is important, is that in the final Theorem 4.1 on asymptotics, we are free from all amphibologies.

The ordinary differential equations (19) is supplemented with the Dirichlet boundary conditions

$$w_0(\pm l) = v_0(P^{\pm}), \tag{20}$$

resulting from the comparison of ansatz (17) with the elementary ansatz $u(h,x) \sim v_0(x)$. If

$$f_\Lambda \in L_q(\Upsilon)\,, \tag{21}$$

there is a unique solution $w_0 \in H^{2,q}(\Upsilon)$ to problem (19), (20) and, by (13), there holds the estimate

$$||w_0; H^{2,q}(\Upsilon)|| \leqslant c\left(||f_\Lambda; L_q(\Upsilon)|| + ||f_\Omega; V_\beta^{0,q}(\Omega)||\right). \tag{22}$$

Besides that, by the elementary embedding theorems for $q \geqslant 2$ (assumption in Nazarov, Sokołowski, 2004)

$$w_0(\tau) = \sum_{\pm} \chi_\Lambda(\tau \mp l)\Big\{v_0(P^{\pm}) + (\tau \mp l)\partial_\tau w_0(\pm l)\Big\} + \widetilde{w}_0(\tau), \tag{23}$$

with the term $|\partial_\tau w_0(\pm l)|$ smaller than the right-hand side of (22) and $\widetilde{w}_0$ a function which vanishes along with its derivatives at $\tau = \pm l$. In (23) $\chi_\Lambda \in C^\infty(\mathbb{R})$ is a cut-off function equal to one near $t = 0$ and zero for $t > l/2$.

In Nazarov, Sokołowski (2004) it is shown that condition (21) can be obtained by investigation of a boundary layer phenomenon, which appears in the zones of junction of the ligament to the domain Ω, and is described by the fast variables $\xi^{\pm} = h^{-1}(x - P)$ (see Section 4 in Nazarov, Sokołowski, 2004). Furthermore, in the framework of the method of compound asymptotic expansions the decay condition for the boundary layer at a distance from the points $P^{\pm}$ (or the procedure of matching ansätze in the framework of the method of matched asymptotic expansions) leads to appearance of logarithmic singularities of the term v_1 from the ansatz

$$u(h,x) = v_0(x) + hv_1(x) + \ldots, \tag{24}$$

accepted in the set $\overline{\Omega}$ outside of small neighborhoods of the points $P^{\pm}$. Actually, v_1 is a solution to the problem

$$\begin{aligned} &-\Delta_x v_1(x) = 0,\ x \in \Omega, \quad v_1(x) = 0,\ x \in \Sigma, \\ &\partial_n v_1(x) = \sum_{\pm} \mp H(\pm l)\partial_\tau w_0(\pm l)\delta(s - s^{\pm}),\ x \in \partial\Omega \setminus \overline{\Sigma}, \end{aligned} \tag{25}$$

which admits the representation

$$v_1(x) = \sum_{\pm} \chi_\Omega(r_\pm)\Big\{\pm \frac{1}{\pi} H(\pm l)\partial_\tau w_0(\pm l)\ln r_\pm + c^{\pm}\Big\} + \widetilde{v}_1(x)\,. \tag{26}$$

Here δ is the Dirac function, $-\pi^{-1}\ln r_\pm$ is the Poisson kernel, $c^\pm$ are constants depending on H, w_0 and Ω, while $\widetilde{v}_1 \in H^1(\Omega)$ is the regular part with $\widetilde{v}_1(P^\pm) = 0$.

It is not difficult to conclude that $v_1 \in L_q(\Omega)$ for any $q \geqslant 1$. Besides that, $v_1 \in V^{2,q}_{\beta_1}(\Omega)$ and $\widetilde{v}_1 \in V^{2,q}_{\widetilde{\beta}_1}(\Omega)$, where

$$\beta_1 \in \left(2 - \frac{2}{q},\ 3 - \frac{2}{q}\right), \qquad \widetilde{\beta}_1 \in \left(1 - \frac{2}{q},\ 2 - \frac{2}{q}\right). \tag{27}$$

In addition, the norms of $v_1 \in V^{2,q}_{\beta_1}(\Omega)$ and $\widetilde{v}_1 \in V^{2,q}_{\widetilde{\beta}_1}(\Omega)$ are bounded by the quantity $c(|\partial_\tau w_0(l)|+|\partial_\tau w_0(-l)|)$ for the indices listed in (27), thus, are bounded by the right-hand side of inequality (22).

Using the Green's functions $G_\pm$ of problem (9) with the singularities at the points $P^\pm$, the formula holds

$$v_1(x) = \sum_\pm \mp H(\pm l)\partial_\tau w_0(\pm l) G_\pm\,. \tag{28}$$

We note that $G_\pm$ is a solution of the homogeneous problem (9) which admits the representation

$$G_\pm(x) = \chi_\Omega(r_\pm)\left\{-\frac{1}{\pi}\ln r_\pm + G^0_\pm(P^\pm)\right\} + \chi_\Omega(r_\mp)G^0_\pm(P^\mp) + \widetilde{G}_\pm(x) \tag{29}$$

with the remainders $\widetilde{G}_\pm \in V^{2,q}_{\widetilde{\beta}_1}(\Omega)$ (we indicate that boundary condition $(9)_2$ at the point $P_\pm \in \partial\Omega \setminus \Sigma$ is not verified for the function $G_\pm$). Now, the values of $c^\pm$ from (26) are given by

$$c^\pm = H(-l)\partial_\tau w_0(-l)G^0_-(P^\pm) - H(l)\partial_\tau w_0(l)G^0_+(P^\pm)\,. \tag{30}$$

It is known, and can be easily proved that the 2×2-matrix

$$\mathbf{G} = \begin{bmatrix} G^0_+(P^+) & G^0_+(P^-) \\ G^0_-(P^+) & G^0_-(P^-) \end{bmatrix} \tag{31}$$

is symmetric.

The solutions given above for the limit problems, (9), (25), and (19), (20), as well as solutions of the boundary layer type, which are not described here, can be used in order to construct the global asymptotic approximation $\mathcal{U}(h,x)$ of the solution $u(h,x)$ to problem (3) in the domain with ligament. The related construction, rather complex, is, however, necessary for the further analysis of functional (5), and therefore is presented here. The restrictions of the function $\mathcal{U}$ to the body Ω and the ligament $\Lambda(h)$ look as follows

$$\begin{aligned} \mathcal{U}(h,x) &= \mathcal{V}(h,x) + h\textstyle\sum_\pm \chi_\Omega(r_\pm) Z^{\,\pm}(h,x), \quad x \in \Omega, \\ \mathcal{U}(h,x) &= \mathcal{W}(h,x) + h\textstyle\sum_\pm \chi_\Lambda(\tau \mp l) Z^{\,\pm}(h,x), \quad x \in \Lambda(h)\,. \end{aligned} \tag{32}$$

Detailed description of the structure of (32) is given in Section 5 of Nazarov, Sokołowski (2004), here, we explain the notation only. The main terms in the right-hand sides of (32) are given by the formulae

$$\begin{aligned}
&\mathcal{V}(h,x) = \mathcal{X}_\Omega(h,x)\left\{\widetilde{v}_0(x) + h\widetilde{v}_1(x)\right\} + \\
&\; + \sum_{\pm} \chi_\Omega(r_\pm)\left\{v_0(P^\pm) + h\left(c^\pm + \pi^{-1}H(\pm l)\partial_\tau w_0(\pm l)\ln h\right) + \right.\\
&\; \left. + \left(1 - \chi_\Omega(h^{-1}r_\pm)\right)\left[(s-s_\pm)\partial_s v_0(P^\pm) + h\pi^{-1}H(\pm l)\partial_\tau w_0(\pm l)(\ln r_\pm - \ln h)\right]\right\}, \\
&\mathcal{W}(h,x) = \mathcal{X}_\Lambda(h,x)\left\{\widetilde{w}_0(\tau) + h\widetilde{w}_1(\tau)\right\} + \\
&\; + \sum_{\pm} \chi_\Lambda(\tau \mp l)\left\{w_0(\pm l) + hw_1(\pm l) + \left(1 - \chi_\Lambda(h^{-1}(\tau \mp l))\right)(\tau \mp l)\partial_\tau w_0(\pm l)\right\}.
\end{aligned} \tag{33}$$

Here the cut-off functions are used

$$\mathcal{X}_\Omega(h,x) = 1 - \sum_{\pm} \chi_\Omega(h^{-1}r_\pm), \quad \mathcal{X}_\Lambda(h,x) = 1 - \sum_{\pm} \chi_\Lambda(h^{-1}(\tau \mp l)), \tag{34}$$

and $\widetilde{v}_0$, $\widetilde{v}_1$ and $\widetilde{w}_0$ are the remainders in the representations (12), (26) and (23), the function w_1 is arbitrary, e.g., a linear function of the variable τ, which verifies the condition

$$\begin{aligned}
&w_1(\pm l) = a_1^\pm \partial_s v_0(P^\pm) \pm \pi^{-1}H(\pm l)\partial_\tau w_0(\pm l)\left[a_0^\pm + \ln h\right] + b_1^\pm, \\
&w_1(\tau) = \widetilde{w}_1(\tau) + \sum_{\pm} \chi_\Lambda(\tau \mp l) w_1(\pm l).
\end{aligned}$$

Finally, $Z_\pm$ are the terms of boundary layer type, which have been mentioned several times. We need only the estimates presented in Section 4 of Nazarov, Sokołowski (2004)

$$\begin{aligned}
&|Z_\pm(h,x)| \leqslant C_\Omega h(h + r_\pm)^{-1}, \quad \text{in } \Omega, \\
&|Z_\pm(h,\tau)| \leqslant C_\Lambda \exp\left[-\delta_H h^{-1}(l - |\tau|)\right], \quad \text{in } \Lambda(h).
\end{aligned} \tag{35}$$

Here C_Ω, C_Λ and δ_H are certain positive constants.

In Nazarov, Sokołowski, (2004) an estimate is derived for the asymptotic remainder $\mathcal{R} = u - \mathcal{U}$ under the following restrictions on the terms in decomposition (4) and the right-hand $f(h,x)$ in problem (3):

$$\begin{aligned}
&\widetilde{N}_\Omega := h^{-1-\mu}(1 + |\ln h|)^{-1} || d_P(1 + |\ln d_P|)\widetilde{f}; L_2(\Omega)||, \\
&\int_{-hH_-(\tau)}^{hH_+(\tau)} \widetilde{f}_\perp(h,\nu,\tau)\, d\nu = 0, \quad \tau \in \left(-l - 2h\lambda, l + 2h\lambda\right); \\
&\widetilde{f}(h,x) = \widetilde{f}_0(h,x) + \widetilde{f}_\perp(h,\nu,\tau), \\
&\widetilde{N}_\Lambda := (1 + |\ln h|)^{-1}\left\{h^{-3/2}||\widetilde{f}_0; L_2(\Lambda(h))|| + h^{-1/2}||\widetilde{f}_\perp; L_2(\Lambda(h))||\right\},
\end{aligned} \tag{36}$$

with the terms $\widetilde{N}_\Omega$ and $\widetilde{N}_\Lambda$ of order $1 = h^0$, and the number λ chosen in such a way that the tips of the curve $\{x \in \Gamma \ : \ |\tau| < l + 2h\lambda\}$ are inside of Ω.

Theorem 4.1 *Assume that the function f verifies the conditions* (4), (21) *with* $q = 2$, $(36)_2$ *and* (15) *with* $\mu = -1/2$. *Then the solution* $u \in \mathring{H}^1(\Omega(h); \Sigma)$ *of problem* (3) *and its asymptotic approximation* (33) *are related by the inequality*

$$||d(u - \mathcal{U}); L_2(\Omega)|| + ||\nabla_x(u - \mathcal{U}); L_2(\Omega)|| \leqslant C\, h^{3/2}\mathcal{N}(1 + |\ln h|)^2 \tag{37}$$

in which d denotes the weight factor

$$d(h, x) = \begin{cases} d_P(x)^{-1}(1 + |\ln d_P(x)|)^{-1}, & x \in \Omega; \\ (h + d_P(x))^{-1}(1 + |\ln h|)^{-1}, & x \in \Lambda(h)\,, \end{cases} \tag{38}$$

the constant C is independent of either the parameter $h \in (0, h_0]$ or the components in decomposition (4),

$$\mathcal{N} = \mathcal{N}_\Omega + \widetilde{\mathcal{N}}_\Omega + \mathcal{N}_\Lambda + \widetilde{\mathcal{N}}_\Lambda\,, \tag{39}$$

the quantities $\widetilde{\mathcal{N}}_\Omega$ and $\widetilde{\mathcal{N}}_\Lambda$ are introduced in (36), *$\mathcal{N}_\Omega$ is the left-hand side of* (16) *and finally* $\mathcal{N}_\Lambda = ||f_\Lambda; L_2(\Upsilon)||$.

Since, owing to (38), $d(h, x) > c_\Omega$ for $x \in \Omega$ and $d(h, x) \geqslant c_\Lambda(1 + |\ln h|)^{-1}$ for $x \in \Lambda(h)$, where c_Ω, c_Λ are positive constants, we have

Corollary 4.1 *Under the assumptions of Theorem* 4.1 *the following inequalities hold*

$$\begin{aligned} &||u - \mathcal{U}; H^1(\Omega)|| \leqslant C h^{3/2}\mathcal{N}(1 + |\ln h|)^2\,, \\ &||u - \mathcal{U}; H^1(\Lambda(h))|| \leqslant C h^{3/2}\mathcal{N}(1 + |\ln h|)^3\,. \end{aligned} \tag{40}$$

5. Self-adjoint operators defined in limit and hybrid domains

Let us consider the unbounded operators $\mathcal{A}^0_\Omega$ in $L_2(\Omega)$ and $\mathcal{A}^0_\Upsilon$ in $L_2(\Upsilon)$, defined by the differential expressions $-\Delta_x$ and $-\partial_\tau H(\tau)\partial_\tau$, respectively, with the domains of definition

$$\begin{aligned} &\mathcal{D}(\mathcal{A}^0_\Omega) = \{v \in C^\infty_0(\overline{\Omega} \setminus P^\pm) : \quad v = 0 \quad \text{on} \quad \Sigma\,, \partial_n v = 0 \quad \text{on} \quad \partial\Omega \setminus \overline{\Sigma}\}\,, \\ &\mathcal{D}(\mathcal{A}^0_\Upsilon) = C^\infty_0(\Upsilon)\,. \end{aligned} \tag{41}$$

In other words, the functions $v \in \mathcal{D}(\mathcal{A}^0_\Omega)$ are smooth in $\overline{\Omega}$, verify boundary conditions $(9)_2$ and vanish in the vicinity of the points $P^\pm$. A function $w \in \mathcal{D}(\mathcal{A}^0_\Upsilon)$ is a smooth function, which vanishes in the vicinity of the ends of the segment $\Upsilon = (-l, l)$. It can be shown that the operators $\mathcal{A}^0_\Omega$ and $\mathcal{A}^0_\Upsilon$ admit the symmetric closures $\overline{\mathcal{A}}^0_\Omega$ and $\overline{\mathcal{A}}^0_\Upsilon$ with the domains

$$\mathcal{D}(\overline{\mathcal{A}}^0_\Omega) = \{v \in H^2(\Omega) \, : \, v(P^\pm) = 0 \,, \quad v = 0 \text{ on } \Sigma, \, \partial_n v = 0 \text{ on } \partial\Omega \setminus \overline{\Sigma}\} \,, \tag{42}$$

$$\mathcal{D}(\overline{\mathcal{A}}^0_\Upsilon) = \mathring{H}^2(\Upsilon) = \{w \in H^2(\Upsilon) \ : \ w(\pm l) = \partial_\tau w(\pm l) = 0\} \,. \tag{43}$$

Besides that, the adjoint operators $\mathcal{A}^*_\Omega$ and $\mathcal{A}^*_\Upsilon$ are defined by the same differential expressions as before, $-\Delta_x$ and $-\partial_\tau H(\tau)\partial_\tau$, but with the following domains of definition

$$\begin{aligned} &\mathcal{D}(\mathcal{A}^*_\Omega) = \{v \ : \ v(x) = \sum_\pm \chi_\Omega(r_\pm) \left[-\frac{1}{\pi} a_\pm \ln r_\pm + b_\pm \right] + \widetilde{v}(x) \,, \\ &\widetilde{v} \in H^2(\Omega), \quad \widetilde{v}(P^\pm) = 0 \,, \quad a_\pm, b_\pm \in \mathbb{R}\} \,, \end{aligned} \tag{44}$$

$$\mathcal{D}(\mathcal{A}^*_\Upsilon) = H^2(\Upsilon) \,. \tag{45}$$

Remark 5.1 *The following embeddings take place for any* $\varepsilon > 0$

$$\mathcal{D}(\overline{\mathcal{A}}^0_\Omega) \subset V^{2,2}_\varepsilon(\Omega) \,, \quad \mathcal{D}(\mathcal{A}^*_\Omega) \subset V^{2,2}_{1+\varepsilon}(\Omega) \,.$$

Since the dimensions of the quotient spaces $\mathcal{D}(\mathcal{A}^*_\Omega)/\mathcal{D}(\overline{\mathcal{A}}^0_\Omega)$ and $\mathcal{D}(\mathcal{A}^*_\Upsilon)/\mathcal{D}(\overline{\mathcal{A}}^0_\Upsilon)$ are equal to 4, the operators $\overline{\mathcal{A}}^0_\Omega$ and $\overline{\mathcal{A}}^0_\Upsilon$ are not self-adjoint. However, by the general scheme (see Rofe-Beketov, 1969; Gorbachuk, Gorbachuk, 1984; Pavlov, 1987, and others) applied here, it follows that the operators admit self-adjoint extensions. Friedrichs' extension of the operator $\mathcal{A}^0_\Omega$ is given by the restriction of $\mathcal{A}^*_\Omega$ to the linear space

$$\{v \in \mathcal{D}(\mathcal{A}^*_\Omega) \ : \ a_\pm = 0\} = \{v \in H^2(\Omega) \ : \ v = 0 \text{ on } \Sigma, \ \partial_n v = 0 \text{ on } \partial\Omega \setminus \overline{\Sigma}\} \,.$$

In this particular case, the boundary value problem associated with the extension, is just the classical formulation of problem (9) in the space $H^2(\Omega)$. Similarly, the restriction of operator $\mathcal{A}^*_\Upsilon$ to the linear space

$$\{u \in \mathcal{D}(\mathcal{A}^*_\Upsilon) \ : \ w(\pm l) = 0\} \quad \text{or} \quad \{u \in \mathcal{D}(\mathcal{A}^*_\Upsilon) \ : \ \partial_\tau w(\pm l) = 0\}$$

corresponds to the Dirichlet, or Neumann problem for the ordinary differential equations (19), respectively.

For modelling of asymptotic solutions to problem (3) by means of the self-adjoint extensions we have to impose some conditions. First, we should consider the unbounded operator $\mathcal{A}^0 = \operatorname{diag}\{\mathcal{A}^0_\Omega, \mathcal{A}^0_\Upsilon\}$ defined on the cartesian product $\mathbb{L} = L_2(\Omega) \times L_2(\Upsilon)$, with the domain of definition $\mathcal{D}(\mathcal{A}^0_\Omega) \times \mathcal{D}(\mathcal{A}^0_\Upsilon)$. Second, the parameters of the extension $\mathfrak{A}$ should be determined, i.e. the relations between the coefficients $a_\pm, b_\pm$ and the values $w(\pm l), \partial_\tau w(\pm l)$ in the domain of definition $\mathcal{D}(\mathcal{A}^\star_\Omega) \times \mathcal{D}(\mathcal{A}^\star_\Upsilon)$ of the restricted operator $\mathcal{A}^\star$. The parameters should be selected in such a way that the components $\mathbf{v}(x)$ and $\mathbf{w}(\tau)$ of solutions to the abstract equation (8) turn out to be close, in a proper sense, to the parts $v_0(x)+hv_1(x)$ and $w_0(\tau)$ of asymptotic ansätze (24) and (17). Finally, we should verify that the constructed extension is self-adjoint.

PROPOSITION 5.1 *Assume that* $\mathfrak{A}$ *is an unbounded operator in the space* $\mathbb{L} = L_2(\Omega) \times L_2(\Upsilon)$*, defined by the differential expressions* $\operatorname{diag}\{-\Delta_x, -\partial_\tau H(\tau)\partial_\tau\}$ *with the domain of definition*

$$\mathcal{D}(\mathfrak{A}) = \Big\{ \ \{\mathbf{v}, \mathbf{w}\} \in \mathcal{D}(\mathcal{A}^\star_\Omega) \times \mathcal{D}(\mathcal{A}^\star_\Upsilon)\,, \quad a = TB, \quad b = T^{-1}A + T^{-\frac{1}{2}}ST^{\frac{1}{2}}B \Big\}\,, \tag{46}$$

where

$$a = \begin{bmatrix} a_+ \\ a_- \end{bmatrix},\ b = \begin{bmatrix} b_+ \\ b_- \end{bmatrix},\ A = \begin{bmatrix} w(+l) \\ w(-l) \end{bmatrix},\ B = \begin{bmatrix} -H(l)\partial_\tau w(+l) \\ H(l)\partial_\tau w(-l) \end{bmatrix}. \tag{47}$$

If T *is a positive definite matrix, and* S *is a symmetric* 2×2*-matrix, then the operator* $\mathfrak{A}$ *turns out to be the self-adjoint extension of the operator* $\mathcal{A}^0 = \operatorname{diag}\{\mathcal{A}^0_\Omega, \mathcal{A}^0_\Upsilon\}$*. By* $T^{\frac{1}{2}}$ *we denote the positive square root of the matrix* T*.*

Proof. We only need to verify the assertion: If for some $\{\mathbf{V}, \mathbf{W}\}, \{\mathbf{F}_\Omega, \mathbf{F}_\Upsilon\} \in \mathbb{L}$ the identity holds true

$$(\mathfrak{A}\{\mathbf{v}, \mathbf{w}\}, \{\mathbf{V}, \mathbf{W}\})_{\mathbb{L}} - (\{\mathbf{v}, \mathbf{w}\}, \{\mathbf{F}_\Omega, \mathbf{F}_\Upsilon\})_{\mathbb{L}} = 0 \quad \forall \{\mathbf{v}, \mathbf{w}\} \in \mathcal{D}(\mathfrak{A}) \tag{48}$$

then

$$\{\mathbf{V}, \mathbf{W}\} \in \mathcal{D}(\mathfrak{A}) \quad \text{and} \quad \mathfrak{A}\{\mathbf{V}, \mathbf{W}\} = \{\mathbf{F}_\Omega, \mathbf{F}_\Upsilon\}\,. \tag{49}$$

Let us take $\mathbf{w} \in C_0^\infty(-l, l)$ and $\mathbf{v} \in C_0^\infty(\overline{\Omega} \setminus P^\pm)$, while in addition $\mathbf{v}$ satisfies the boundary conditions $(9)_2$. Integration by parts in equation (48) leads to

$$\begin{aligned} 0 &= (-\Delta_x \mathbf{v}, \mathbf{V})_\Omega - (\mathbf{v}, \mathbf{F}_\Omega)_\Omega + (-\partial_\tau H \partial_\tau \mathbf{w}, \mathbf{W})_\Upsilon - (\mathbf{w}, \mathbf{F}_\Upsilon)_\Upsilon = \\ &= (\mathbf{v}, -\Delta_x \mathbf{V} - \mathbf{F}_\Omega)_\Omega + (\mathbf{v}, \partial_n \mathbf{V})_{\partial\Omega \setminus \Sigma} - (\partial_n \mathbf{v}, \mathbf{V})_\Sigma + (\mathbf{w}, -\partial_\tau H \partial_\tau \mathbf{W} - \mathbf{F}_\Upsilon)_\Upsilon\,. \end{aligned} \tag{50}$$

Since the linear spaces $C_0^\infty(\Omega) \subset C_0^\infty(\overline{\Omega} \setminus P^\pm)$ and $C_0^\infty(-l, l)$ are dense in $L_2(\Omega)$ and $L_2(-l, l)$, respectively, from (50) we deduce that $\mathbf{V}$ and $\mathbf{W}$ solve equation $(9)_1$ with the right-hand side $\mathbf{F}_\Upsilon$ and equation (19) with the right-hand side $\mathbf{F}_\Omega$, respectively. Since for $\mathbf{v} \in C_0^\infty(\overline{\Omega} \setminus P^\pm)$ the traces $\mathbf{v} \in L_2(\partial\Omega \setminus \Sigma)$ and $\partial_n \mathbf{v} \in L_2(\Sigma)$ are arbitrarily smooth, from (50) the boundary conditions $(9)_2$ for $\mathbf{V}$ are obtained. Being a solution to problem (9), with the right-hand side $\mathbf{F}_\Omega \in L_2(\Omega)$, the function $\mathbf{V} \in L_2(\Omega)$ belongs to the space (44) (the detailed proof of the property can be found in Nazarov, Sokołowski, 2005) for a similar boundary value problem). It can be shown using the same argument that $\mathbf{W}$ belongs to the space $(45)_2$. It remains to verify that the coefficients $\mathbf{a}_\pm, \mathbf{b}_\pm$ from the decomposition of $\mathbf{v}$ in (44) and the quantities $\mathbf{W}(\pm l), \partial_\tau \mathbf{W}(\pm l)$ are related by the formula given in (46).

For functions $\mathbf{v}, \mathbf{V} \in \mathcal{D}(\mathcal{A}^*_\Omega)$ generalized Green's formula holds

$$(-\Delta_x \mathbf{v}, \mathbf{V})_\Omega - (\mathbf{v}, -\Delta_x \mathbf{V})_\Omega = \sum_\pm (b_\pm \mathbf{a}_\pm - a_\pm \mathbf{b}_\pm)\,. \tag{51}$$

For general self-adjoint problems similar formulae were constructed in Nazarov, Plamenevskii (1992, 1994), a simple argument for the Neumann case is given in Nazarov, Sokołowski (2005). We add to (51) Green's formula on the interval $\Upsilon = (-l, l)$

$$\begin{aligned}&(-\partial_\tau H \partial_\tau \mathbf{w}, \mathbf{W})_\Upsilon - (\mathbf{w}, -\partial_\tau H \partial_\tau \mathbf{W})_\Upsilon = \\ &= \sum_\pm \pm H(\pm l)(\mathbf{w}(\pm l)\partial_\tau \mathbf{W}(\pm l) - \mathbf{W}(\pm l)\partial_\tau \mathbf{w}(\pm l))\end{aligned}$$

and use relations (46) for the coefficients of $\{\mathbf{v}, \mathbf{w}\}$

$$\begin{aligned}&(-\Delta_x \mathbf{v}, \mathbf{V})_\Omega + (-\partial_\tau H \partial_\tau \mathbf{w}, \mathbf{W})_\Upsilon - (\mathbf{v}, -\Delta_x \mathbf{V})_\Omega - (\mathbf{w}, -\partial_\tau H \partial_\tau \mathbf{W})_\Upsilon = \\ &= \sum_\pm (b_\pm \mathbf{a}_\pm - a_\pm \mathbf{b}_\pm \pm H(\pm l)\mathbf{w}(\pm l)\partial_\tau \mathbf{W}(\pm l) \mp H(\pm l)\mathbf{W}(\pm l)\partial_\tau \mathbf{w}(\pm l))) = \\ &= \langle b, \mathbf{a}\rangle - \langle a, \mathbf{b}\rangle + \langle B, \mathbf{A}\rangle - \langle A, \mathbf{B}\rangle = \\ &= \langle T^{-1}A + T^{-\frac{1}{2}} S T^{\frac{1}{2}} B, \mathbf{a}\rangle - \langle TB, \mathbf{b}\rangle + \langle B, \mathbf{A}\rangle - \langle A, \mathbf{B}\rangle = \\ &= \langle A, T^{-1}\mathbf{a} - \mathbf{B}\rangle + \langle B, T^{\frac{1}{2}} S T^{-\frac{1}{2}} \mathbf{a} - T\mathbf{b} + \mathbf{A}\rangle\,.\end{aligned} \tag{52}$$

Here $\langle\,,\rangle$ is the scalar product in $\mathbb{R}^2$.

The left-hand side of (52) coincides with the left-hand side of (48), and thus vanishes by our assumption for any pair $\{\mathbf{v}, \mathbf{w}\}$. Since the columns A and B in (47) can be made arbitrary by means of the choice of the functions $\mathbf{w}$ and $\mathbf{v}$, identity (52) implies the following relations

$$T^{-1}\mathbf{a} = \mathbf{B}\,, \quad T\mathbf{b} = \mathbf{A} + T^{\frac{1}{2}} S T^{-\frac{1}{2}} \mathbf{a}\,,$$

which readily turn into the relations

$$\mathbf{a} = T\mathbf{B}\,, \quad \mathbf{b} = T^{-1}\mathbf{A} + T^{-\frac{1}{2}} S T^{\frac{1}{2}} \mathbf{B}\,.$$

These relations define the domain of the operator $\mathfrak{A}$. Therefore, $\{\mathbf{V}, \mathbf{W}\} \in \mathcal{D}(\mathfrak{A})$, which completes the proof. ■

Remark 5.2 *Proposition* 5.1 *does not cover all possible self-adjoint extensions of the operator* $\mathcal{A}^0$*, but only of the type required in the paper. All self-adjoint extensions can be described on the basis of general results in Gorbachuk, Gorbachuk (1984), Pavlov (1987), Rofe-Beketov (1969) .*

Proposition 5.2 *There exists a number $t > 0$ such that for $||T; \mathbb{R}^2 \to \mathbb{R}^2|| < t$ equation*

$$\mathfrak{A}\{\mathbf{v}, \mathbf{w}\} = \{\mathbf{f}_\Omega, \mathbf{f}_\Lambda\} \in L_2(\Omega) \times L_2(\Upsilon)$$

with the operator $\mathfrak{A}$ from Proposition 5.1 admits the unique solution $\{\mathbf{v}, \mathbf{w}\} \in \mathcal{D}(\mathfrak{A})$ for any right-hand side $\{\mathbf{f}_\Omega, \mathbf{f}_\Lambda\} \in \mathbb{L}$.

Proof. We introduce two functions $v^\bullet \in H^2(\Omega)$ and $w^\bullet \in H^2(\Upsilon)$ in the domain of Friedrichs' extension of $\mathcal{A}_0$. Let $v^\bullet \in H^2(\Omega)$ denote the solution of problem (9) with the right-hand side $\mathbf{f}_\Omega$, and $w^\bullet \in H^2(\Upsilon)$ be the solution to problem (19) with the right-hand side $\mathbf{f}_\Upsilon$ and the boundary conditions $w^\bullet(\pm l) = 0$. We set

$$\begin{aligned} \mathbf{v}(x) &= v^\bullet(x) + a^0_+ G_+(x) + a^0_- G_-(x)\,, \\ \mathbf{w}(\tau) &= w^\bullet(\tau) + A^0_+ W_+(\tau) + A^0_- W_-(\tau)\,, \end{aligned} \tag{53}$$

where $W_\pm$ is the solution of the homogeneous equation (19) with the boundary conditions $W_+(-l) = 0, W_+(l) = 1$ and $W_-(-l) = 1, W_-(l) = 0$. We show that the columns $a^0 = \begin{bmatrix} a^0_+ \\ a^0_- \end{bmatrix}$ and $A^0 = \begin{bmatrix} A^0_+ \\ A^0_- \end{bmatrix}$ can be selected in such a way that the pair (53) belongs to the linear space (46).

Let $a^\bullet = 0, b^\bullet, A^\bullet = 0, B^\bullet$ be attributes (47) of the pair $\{v^\bullet, w^\bullet\}$. We specify the algebraic conditions, from (46), prescribed for the functions in (53)

$$\begin{aligned} a^0 &= T(B^\bullet - \mathbf{M}A^0)\,, \\ b^\bullet + \mathbf{G}a^0 &= T^{-1}A^0 + T^{-\frac{1}{2}}ST^{\frac{1}{2}}(B^0 - \mathbf{M}A^0)\,. \end{aligned} \tag{54}$$

Here $\mathbf{G}$ is matrix (31), and $\mathbf{M}$ denotes the matrix with the elements $\mathbf{M}_{\pm,\alpha} = \pm H(\pm l)\partial_\tau W_\alpha(\pm l)$, where $\alpha = \pm$. Observe that integration by parts leads to the equality

$$\mathbf{M}_{\beta,\alpha} = \int_{-l}^{l} H(\tau)\partial_\tau W_\sigma(\tau)\partial_\tau W_\alpha(\tau)d\tau\,,$$

which shows that $\mathbf{M}$ is Gram's matrix, symmetric and positive definite (these properties are not used in the sequel). We insert $(54)_1$ in $(54)_2$ and multiply by the nonsingular matrix T, which leads to the relation

$$\{\mathbb{I}_2 + T\mathbf{G}T\mathbf{M} - T^{\frac{1}{2}}ST^{\frac{1}{2}}\mathbf{M}\}A^0 = Tb^\bullet + T\mathbf{G}TB^0 - T^{\frac{1}{2}}ST^{\frac{1}{2}}B^0\,. \tag{55}$$

It is clear now that for a small t in our assumption, the matrix $\{\mathbb{I}_2 + T\mathbf{G}T\mathbf{M} - T^{\frac{1}{2}}ST^{\frac{1}{2}}\mathbf{M}\}$ is invertible. Therefore, the column A^0 can be determined from (55), the column a^0 from (54), and finally the solution of equation (8) is given by formula (53). ■

We construct the self-adjoint extension $\mathfrak{A}$, suitable for modelling of problem (3). In accordance with the asymptotic ansatz (24) we set

$$\mathbf{v}(x) = v_0(x) + hv_1(x)\,. \tag{56}$$

By formula (26) and (24) the coefficients of decomposition in (44) of the function from (56) take the form

$$\begin{aligned} &a^{\pm} = \mp hH(\pm l)\partial_\tau w_0(\pm l)\,,\\ &b^{\pm} = v_0(P^{\pm}) + h\{H(-l)\partial_\tau w_0(-l)G^0_-(P^{\pm}) - H(l)\partial_\tau w_0(l)G^0_-(P^{\pm})\}\,. \end{aligned} \tag{57}$$

Therefore, in order to define the matrix T from (56) in the proper way we put

$$\mathbf{w}(\tau) = h^{\frac{1}{2}} w_0(\tau)\,. \tag{58}$$

Now relations (57), taking into account condition (20), become

$$\begin{aligned} &a^{\pm} = \mp hH(\pm l)\partial_\tau \mathbf{w}(\pm l)\,,\\ &b^{\pm} = h^{-\frac{1}{2}}\mathbf{w}(\pm l) + h^{\frac{1}{2}}\{H(-l)\partial_\tau \mathbf{w}(-l)G^0_-(P^{\pm}) - H(l)\partial_\tau \mathbf{w}(l)G^0_-(P^{\pm})\} \end{aligned} \tag{59}$$

which means that they can be rewritten in the form (46), namely,

$$a = TB, \quad b = -T^{-1}A + h^{\frac{1}{2}}\mathbf{G}B\,,$$

where $T = h^{\frac{1}{2}}\mathbb{I}_2$, $\mathbb{I}_2$ is the unit 2×2-matrix, and $\mathbf{G}$ is matrix (31). We point out that $h^{\frac{1}{2}}\mathbf{G} = S = T^{-\frac{1}{2}}ST^{\frac{1}{2}}$, because the matrix T is proportional to the identity matrix.

THEOREM 5.1 *If in Proposition* 5.1 *we set* $T = h^{\frac{1}{2}}\mathbb{I}_2$ *and* $S = \mathbf{G}$*, then the self-adjoint extension* $\mathfrak{A}$ *of the operator* $\mathcal{A}^0$*, with the domain of definition* (46)*, can be considered as a model for the singularly perturbed problem* (3) *in the following sense: The solution* $\{\mathbf{v},\mathbf{w}\}$ *of abstract equation* (8) *with the right-hand side*

$$\{\mathbf{f}_\Omega, \mathbf{f}_\Lambda\} = \{f_\Omega, f_\Lambda\} \in \mathbb{L}$$

has the components $\mathbf{v} = v_0 + hv_1$ *and* $\mathbf{w} = h^{\frac{1}{2}}w_0$*, which contain the functions* v_0, v_1 *and* w_0 *from asymptotic ansätze* (24) *and* (17*) justified in Theorem* 4.1 *and Corollary* 4.1.

We emphasize that, by virtue of Proposition 5.2, the abstract equation (8) is uniquely solvable for a small $h > 0$.

6. Analysis of functional $\mathcal{J}$

In this section we derive a simple asymptotic formula for functional (5), in terms of solutions $\{\mathbf{v},\mathbf{w}\}$ of the abstract equation (8), which can be used as simplified model for the singularly perturbed problem (3). The task is achieved in few steps. In the first step, an error estimate is obtained for the replacement of the solution $u(h,x)$ by its global asymptotic approximation.

LEMMA 6.1 *Let $\mathcal{U}$ be the global asymptotic approximation* (32) *of solutions to problem* (3) *in the domain with ligament. Then the following inequality is valid*

$$|\mathcal{J}(u;h) - \mathcal{J}(\mathcal{U};h)| \leqslant c\sigma(h)^q h^{\frac{3}{2}}(1+|\ln h|)^3\mathcal{N}^q\,, \tag{60}$$

where the notation from Theorem 4.1 *is used and*

$$\sigma(h) = \max\{1, h^{-\frac{1}{2}-\frac{1}{q}}(1+|\ln h|)\}\,. \tag{61}$$

Proof. First, we need to establish the estimate

$$||\mathcal{U}; L_q(\Lambda_h|| \leqslant ch^{-\frac{1}{2}-\frac{1}{q}}||\mathcal{U}; H^1(\Lambda_h)||\,. \tag{62}$$

Let us straighten the ligament by the change of variables

$$x \mapsto (\nu,\tau) \mapsto \xi = (\xi_1,\xi_2) = (H(\tau)^{-1}[\nu + hH_-(\tau)],\tau)\,, \tag{63}$$

which transforms the curvilinear strip (1) onto the rectangle $\widehat{\Lambda}_h = [0,h]\times[-l_-,l_+]$. Here the lengths $l_\pm$ are chosen in such a way that $\Gamma = \{x \;:\; \tau\in[-l_-,l_+]\}$. The function $\mathcal{U}$ written in variables (63) is denoted by $\widehat{\mathcal{U}}$. Since $H(\tau)>0$ for $\tau\in[-l_-,l_+]$, the relations hold

$$\begin{aligned} &c||\mathcal{U}; H^1(\Lambda_h)|| \leqslant ||\widehat{\mathcal{U}}; H^1(\widehat{\Lambda}_h)|| \leqslant C||\mathcal{U}; H^1(\Lambda_h)||\,,\\ &c_q||\mathcal{U}; L_q(\Lambda_h)|| \leqslant ||\widehat{\mathcal{U}}; L_q(\widehat{\Lambda}_h)|| \leqslant C_q||\mathcal{U}; L_q(\Lambda_h)||\,, \end{aligned} \tag{64}$$

with positive constants c, c_q and C, C_q. Let us consider the union of rectangle $\widehat{\Lambda}_h$ with its shifts $\widehat{\Lambda}_h(j) = \{\xi\in\mathbb{R}^2 \;:\; (\xi_1 - h_j,\xi_2)\subset\widehat{\Lambda}_h\}$ along the axis ξ_1. Here $j=0,...,N$, and $N=[h^{-1}]$ is the integer part of the number h^{-1}. In this way we obtain the rectangle $\Lambda^\sharp(h)$ with the side of order 1, yet its width remains dependent on the parameter h. The function $\widehat{\mathcal{U}}$ is extended from the rectangle $\widehat{\Lambda}_h(0)$ onto the consecutive rectangle $\widehat{\Lambda}_h(1)$ as an even function over the right-hand side of $\widehat{\Lambda}_h(0)$, i.e., we set

$$\widehat{\mathcal{U}}(\xi) = \mathcal{U}(h-\xi_1,\xi_2) \quad \text{for} \quad \xi\in\widehat{\Lambda}_h(1)\,.$$

The same procedure is repeated for all pairs of neighbours $\widehat{\Lambda}_h(j)$ and $\widehat{\Lambda}_h(j+1), j=1,...,N-1$, so the function $\mathcal{U}^\sharp$ is defined on $\Lambda^\sharp_h$, furthermore, by construction it belongs to the Sobolev space $H^1(\Lambda^\sharp_h)$. The embedding theorem $H^1\subset L_q$ applied on the overlapping domains $\{\xi\in\Lambda^\sharp_h \;:\; \xi_1\in(0,\frac{3}{4})\}$ and $\{\xi\in\Lambda^\sharp_h \;:\; \xi_1-(N+1)h\in(-\frac{3}{4},0)\}$ gives the inequality

$$||\mathcal{U}^\sharp; L_q(\Lambda^\sharp_h)|| \leqslant c||\mathcal{U}^\sharp; H^1(\Lambda^\sharp_h)||\,, \tag{65}$$

with the constant c independent of the parameter h and of the function $\mathcal{U}^\sharp$. Taking into account that in both sides of (65) there are $N+1$ copies of the function $\widehat{\mathcal{U}}$, we rewrite (65) as follows

$$\Big((N+1)||\widehat{\mathcal{U}}; L_q(\widehat{\Lambda}_h)||^q\Big)^{\frac{1}{q}} \leqslant c\Big((N+1)||\widehat{\mathcal{U}}; H^1(\widehat{\Lambda}_h)||^2\Big)^{\frac{1}{2}}\,. \tag{66}$$

In view of the relation $N \sim h^{-1}$ and the norm equivalence (64), inequality (66) becomes (62). ■

Weighted Poincaré-Friedrichs' inequality, proved in Proposition 1 of Nazarov, Sokołowski (2004) combined with formula (38) for the weight $d(h,x)$ results in the inequality

$$||u; L_2(\Omega)|| + (1+|\ln h|)^{-1}||u; L_2(\Lambda(h))|| \leqslant c||du; L_2(\Omega(h))|| \leqslant \\ \leqslant c||\nabla_x u; L_2(\Omega(h))||\,,$$

which means, according to (62) and (61), that

$$||u; L_q(\Omega(h))|| \leqslant c\sigma(h)||\nabla_x u; L_2(\Omega(h))||\,.$$

From the above, and the evident relation

$$||\nabla_x u; L_2(\Omega(h))||^2 \leqslant c||d^{-1}f; L_2(\Omega(h))||\,||du; L_2(\Omega(h))||\,,$$

for the solution u of problem (3), we get

$$||u; L_q(\Omega(h))|| \leqslant c\sigma(h)||d^{-1}f; L_2(\Omega(h))|| \leqslant c\sigma(h)\mathcal{N}\,. \tag{67}$$

The last evaluation is obtained taking into account the specific representation (4) of the right-hand side f with the introduced notation (39) for norms of its components.

Now, we make use of property (6) of the functional $\mathcal{J}$ and apply estimates (40) and (66). As a result we obtain that

$$|\mathcal{J}(u;h) - \mathcal{J}(\mathcal{U};h)| \leqslant c||u-\mathcal{U}; L_q(\Omega(h))||\left(||u; L_q(\Omega(h))||^{q-1} + \right. \\ \left. (||u; L_q(\Omega(h))|| + ||u-\mathcal{U}; L_q(\Omega(h))||)^{q-1}\right) \leqslant c\mathcal{N}^q h^{\frac{3}{2}}(1+|\ln h|)^3([\sigma(h)]^{q-1} + \\ +[\sigma(h) + \sigma(h)h^{\frac{3}{2}}(1+|\ln h|)^3]^{q-1}) \leqslant \mathcal{N}^q h^{\frac{3}{2}}(1+|\ln h|)^3\sigma(h)^q$$

and the proof of lemma is completed. ■

To some extent, the second step consists in removing the nonnecessary terms from the global asymptotic approximation.

LEMMA 6.2 *The inequality holds*

$$||\mathcal{U} - v_0 - hv_1; L_q(\Omega)|| + ||\mathcal{U} - w_0; L_q(\Lambda(h))|| \leqslant ch^{1+\frac{1}{q}}\mathcal{N}\,. \tag{68}$$

Proof. In view of estimate (35) the norm in $L_q(\Omega(h))$ of the components $h\chi_\Omega Z^\pm$ and $h\chi_\Lambda Z^\pm$ of the boundary layer is bounded by

$$ch\left(h^q\int_0^{r_0}(h+r_\pm)^{-q}r_\pm dr_\pm + \int_0^{\frac{l}{2}}\exp\left(-q\delta_H h^{-1}(l-|\tau|)\right)d\tau\right)^{\frac{1}{q}} \leqslant \\ \leqslant ch(h^q I_q(h) + h)^{\frac{1}{q}} \leqslant ch^{1+\frac{1}{q}}$$

where $I_q(h) = h^{2-q}$ for $q > 2$, $I_q(h) = 1 + |\ln h|$ for $q = 2$ and $I_q(h) = 1$ for $q \in [1, 2)$.

Simple calculations, based on the representation (12), (26) and $(33)_1$ lead to

$$\mathcal{V}(h,x) - v_0(x) - hv_1(x) =$$
$$= -\sum_{\pm} \chi_\Omega(h^{-1}r_\pm)\Big\{v_0(x) - v_0(P^\pm) + h\Big(v_1(x) - c^\pm - h\pi^{-1}H(\pm l)\partial_\tau w_0(\pm l)\ln h\Big)\Big\}.$$

Using the estimate (13) for v_0 and the estimate for v_1 given after the formula (27) we find out that

$$||\mathcal{V}(h,x) - v_0(x) - hv_1(x); L_q(\Omega)|| \leqslant c\mathcal{N}\left(\int_0^h [r^q + h^q(|\ln r| + |\ln h|)^q] r dr\right)^{\frac{1}{q}} \leqslant$$
$$\leqslant c\mathcal{N}(h^{q+2} + h^q(1 + |\ln h|)^q h^2)^{\frac{1}{q}} \leqslant c\mathcal{N} h^{1+\frac{2}{q}}(1 + |\ln h|)\,.$$

Finally, taking into account $(33)_2$ and (23), (20), we obtain

$$\mathcal{W}(h,x) - w_0(\tau) = -\sum_{\pm} \chi(h^{-1}(\tau \mp l))\Big(w_0(\tau) - w_0(\pm l)\Big) +$$
$$+ hw_1(\tau) - h\sum_{\pm} \chi(h^{-1}(\tau \mp l))\widetilde{w}_1(\tau)\,.$$

Hence

$$||\mathcal{W} - w_0; L_q(\Lambda(h))|| \leqslant c\mathcal{N}\left(\int_{-ch}^{ch}\int_{l-hl/2}^{l} |l - |\tau||^q d\tau d\nu + \int_{-ch}^{ch}\int_{l-hl/2}^{l} h^q d\tau d\nu\right)^{\frac{1}{q}} \leqslant$$
$$\leqslant c(hh^{q+1} + hh^q)^{\frac{1}{q}} \leqslant ch^{1+\frac{1}{q}}\,.$$

The estimate completes the proof of the lemma. ■

The obvious formulae

$$||v_0 + hv_1; L_q(\Omega)|| \leqslant c(||v_0; H^1(\Omega)|| + h||v_1; H^1(\Omega)||) \leqslant c\mathcal{N}\,,$$
$$||w_0; L_q(\Lambda(h))|| \leqslant ch^{\frac{1}{q}}||w_0; L_q(\Upsilon)|| \leqslant ch^{\frac{1}{q}}\mathcal{N}$$

together with inequality (68) and property (6) of the functional $\mathcal{J}$ lead to the estimate

$$|\mathcal{J}(\mathcal{U}; h) - \mathcal{J}(\mathbf{u}; h)| \leqslant c\mathcal{N}^q h^{1+\frac{1}{q}}(1^{q-1} + (1 + h^{1+\frac{1}{q}})^{q-1}) \leqslant ch^{1+\frac{1}{q}}\mathcal{N}^q\,, \qquad (69)$$

where the function $\mathbf{u}$ is defined as follows

$$\mathbf{u} = v_0 + hv_1 \quad \text{on } \Omega \quad \text{and} \quad \mathbf{u} = w_0 \quad \text{on } \Lambda(h)\,.$$

Now the error estimate can be given for the replacement of the exact solution by its approximation

$$\{v_0 + hv_1, w_0\} = \{\mathbf{v}, h^{-\frac{1}{2}}\mathbf{w}\}\,,$$

obtained either by consecutive solution of problems (9), (19), (20) and (25), or by solution of abstract equation (8) with the self-adjoint operator defined in Theorem 5.1.

THEOREM 6.1 *Under the assumptions and with the notation of Theorem* 4.1 *there holds*

$$|\mathcal{J}(\mathcal{U};h)-\mathcal{J}(\{v_0+hv_1,w_0\};h)| \leqslant c\max\{h^{1+\frac{1}{q}},\sigma(h)^q h^{\frac{3}{2}}(1+|\ln h|)^3\}\mathcal{N}^q\,, \quad (70)$$

where the constant c is independent of the parameter h, and of the components in the decomposition (4) *of the right-hand side of problem* (3).

In view of the formula (61) for $\sigma(h)$, we can see that for $q \in [1,2]$ the upper bound in (70) equals

$$ch^{\frac{3}{2}}(1+|\ln h|)^3\mathcal{N}^q\,,$$

and in the case of $q > 2$ it equals

$$ch^{\frac{5-q}{2}}(1+|\ln h|)^3\mathcal{N}^q\,,$$

with the exponent of the small parameter h which is strictly greater than one only for $q < 3$.

The loss of the precision is due to the apriori estimate (67) of solutions to (3), which contains in the case of $q > 2$ the large factor $h^{-\frac{1}{2}+\frac{1}{q}}(1+|\ln h|)$. The factor comes from the estimate (62), which is asymptotically exact on the ligament (for $U = 1$ the left-hand side equals $(\mathrm{mes}_2\Lambda_h)^{\frac{1}{q}} = O(h^{\frac{1}{q}})$, and the right-hand side equals $(\mathrm{mes}_2\Lambda_h)^{\frac{1}{2}} = O(h^{\frac{1}{2}})$). At the same time, on the body Ω, the Sobolev embedding theorem gives the inequality

$$||u;L_q(\Omega)|| \leqslant c||u;H^1(\Omega)||$$

with a constant which obviously is independent of h. In other words, the worsening of estimate (70) with the growth of q is caused just by the presence of the thin ligament. The upper bound in (70) can be in principle improved only due to the small norm of f in $L_2(\Lambda(h))$ (in view of (4) the norm is of the order $O(h^{\frac{1}{2}})$), however to this end an apriori estimate of solutions is required, such that the norm on the ligament includes an additional large weight factor. Unfortunately, such a weighted estimate is not known to the authors.

COROLLARY 6.1 *Let $q \in [2,3)$, and assume that the integrand $J(x;u)$ in* (5) *verifies for any $s,t \in \mathbb{R}$ the relation*

$$|J(x;s+t)-J(x;t)-J'_u(x;t)s| \leqslant c(s^{q-2}+t^{q-2})s^2\,, \quad (71)$$

and the relation

$$|J(x(\nu,\tau);t)-J(x(0,\tau);t)| \leqslant ch|t|^q\,,$$

where $t \in \mathbb{R}$, $\tau \in \Gamma$ and $x(\nu,\tau) \in \Lambda_h$ is a point in the strip (1) *with the coordinates* (ν,τ). *Then the following asymptotic formula is valid*

$$\Big| \mathcal{J}(u;h) - \int_\Omega J(x;v_0(x))dx - h\int_\Omega J'_u(x;v_0(x))v_1(x)dx + \\ - h\int_\Upsilon H(\tau)J(x(0,\tau);w_0(\tau))d\tau \Big| \leqslant ch^{\frac{5-q}{2}}(1+|\ln h|)^3\mathcal{N}^q\,.$$

From Corollary 6.1, the case of a linear functional $u \mapsto \mathcal{J}(u;h)$ is excluded. The same analysis can be performed for the latter functional, as in the case of the energy functional (see Nazarov, Sokołowski, 2005).

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