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Characterization of a class of lifetime distributions

by

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Abstract: In Knopik (2005) the ageing class MTFR (Mean Time to Failure or Repair) of lifetime distribution was introduced. In this paper, we show that the family MTFR is closed under weak convergence of distribution and convolution. We prove that the dual family MTFR^{D} (in a particular case) is closed under mixtures.

Keywords: ageing classes, IFR, IFRA, NBUE, MTFR, HNBUE, convolution, limit distribution, mixture.

1. Introduction

Let T be a random variable (lifetime) having a distribution function $F(t) = P\{T \leqslant t\}$ with $F(0^-) = 0$ and a finite mean value $ET = \int_0^\infty R(t)dt$, where $R(t)$ denotes a survival (reliability) function $R(t) = 1 - F(t)$.

Let

$$ET(x) = \int_0^x R(t)dt \tag{1}$$

Definition 1.1 *The random variable* $T \in \text{MTFR}$ (MTFR^D), *if the function*

$$h(x) = \frac{F(x)}{ET(x)} \quad \textit{for} \;\; x > 0 \tag{2}$$

is non-decreasing (non-increasing).

In particular, it is well known that (see Klefsjö, 1982; Knopik, 2005; Marschall, 1972)

IFR $\subset$ MTFR $\subset$ NBUE,

where IFR denotes *increasing failure rate* function and NBUE *new better then used in expectation* classes of distributions. In Barlow (1979) it was proved for the absolutely continuous distribution that

$$\mathrm{IFRA} \subset \mathrm{MTFR}\,.$$

In this paper, we prove this inclusion for any random variable. The class MTFR (Knopik, 2005) is closed under the operation of maximum for independent and absolutely continuous random variables and is closed under the operation of minimum for independent, identically distributed and absolutely continuous random variables. In this paper, we show that the family MTFR is closed under weak convergence of distributions and convolution and we prove that the dual family $\mathrm{MTFR}^{\mathrm{D}}$ is closed under non-crossing mixtures and absolutely continuous random variables.

2. Properties of the class MTFR

2.1. Inclusion IFRA ⊂ MTFR

Let

$$g(x) = \frac{-\ln R(x)}{x} \quad \text{for} \quad x \in \{x : x > 0 \ \text{and} \ R(x) > 0\}\,. \tag{3}$$

It is known that, if the function $g(x)$ is non-decreasing if and only if $T \in \mathrm{IFRA}$. By (3), we have

$$F(x) = 1 - e^{-xg(x)} \quad \text{for} \ x > 0.$$

To show the inclusion IFRA ⊂ MTFR, we need two lemmas.

Lemma 2.1 *If $g(x)$ is non-decreasing, then*

$$\int_0^x e^{-tg(t)} dt \geqslant \frac{F(x)}{g(x)} \quad \textit{for} \ g(x) > 0.$$

Proof. By the fact that $g(x)$ is non-decreasing, we have

$$\int_0^x e^{-tg(t)} dt \geqslant \int_0^x e^{-tg(x)} dt = \frac{1}{g(x)} \{1 - e^{-xg(x)}\} \quad \text{for} \ g(x) > 0.$$

■

Lemma 2.2 *If $g(x)$ is non-decreasing, then*

$$\int_x^{x+y} e^{-tg(t)} dt \leqslant \frac{1}{g(x)} \{e^{-xg(x)} - e^{-(x+y)g(x+y)}\} \quad \textit{for} \ g(x) > 0.$$

Proof. The function $g(x)$ is non-decreasing, since

$$\int_x^{x+y} e^{-tg(t)}dt \leqslant \int_x^{x+y} e^{-tg(x)}dt = \frac{1}{g(x)}\{e^{-xg(x)} - e^{-(x+y)g(x)}\}$$

and finally

$$\int_x^{x+y} e^{-tg(t)}dt \leqslant \frac{1}{g(x)}\{e^{-xg(x)} - e^{-(x+y)g(x+y)}\}. \qquad \blacksquare$$

Proposition 2.1

IFRA $\subset$ MTFR

Proof. By definition, we have $T \in$ MTFR if and only if

$$\frac{F(x)}{ET(x)} \leqslant \frac{F(x+y)}{ET(x+y)} \quad \text{for } x > 0,\ y \geqslant 0. \tag{4}$$

The inequality (4) is equivalent to

$$\int_x^{x+y} e^{-tg(t)}dt\,\{1 - e^{-xg(x)}\} \leqslant \int_0^x e^{-tg(t)}dt\,\{e^{-xg(x)} - e^{-(x+y)g(x+y)}\}. \tag{5}$$

From Lemma 2.1 we obtain

$$\int_0^x e^{-tg(t)}dt\,\{e^{-xg(x)} - e^{-(x+y)g(x+y)}\} \geqslant \frac{1}{g(x)}\{1 - e^{-xg(x)}\}\{e^{-xg(x)} - e^{-(x+y)g(x+y)}\} \tag{6}$$

By Lemma 2.2, we have

$$\frac{1}{g(x)}\{1 - e^{-xg(x)}\}\{e^{-xg(x)} - e^{-(x+y)g(x+y)}\} \geqslant \{1 - e^{-xg(x)}\}\int_x^{x+y} e^{-tg(t)}dt \tag{7}$$

From (6) and (7), we obtain (5), this completes the proof of Proposition 2.1. $\blacksquare$

2.2. Limit distributions

Let F_n be the distribution function of X_n. The sequence $\{X_n\}$ is called convergent in distribution to X if $\lim\limits_{n\to\infty} F_n(t) = F(t)$ for all continuity points t of $F(t)$. Then we write $F_n \xrightarrow{LD} F$.

Let A be a class of distribution functions. Then A^{LD} denotes is the class obtained by taking limits in distributions of sequences of member A.

Proposition 2.2 *The class* MTFR *and dual class* $\mathrm{MTFR}^{\mathrm{D}}$ *are closed under limit in distribution. We write*

$$\mathrm{MTFR}^{\mathrm{LD}} = \mathrm{MTFR} \qquad (\mathrm{MTFR}^{\mathrm{D}})^{\mathrm{LD}} = \mathrm{MTFR}^{\mathrm{D}}\,.$$

Proof. Let $T_n \in$ MTFR, T_n has the distribution function $F_n(t)$ and the reliability function $R_n(t)$. We suppose that $F_n \xrightarrow{LD} F$, then $\lim\limits_{n\to\infty} R_n(t) = R(t)$ for all continuity points of $R(t)$.

Let

$$ET_n(t) = \int_0^t R_n(x)dx\,.$$

It is known (Deshpande, 1986) that $ET_n(t)/\,ET_n$ is a distribution function, which is called the equilibrium distribution and $g_n(t) = R_n(t)/\,ET_n$ is the density function of equilibrium distribution. It plays an important role in renewal theory. In Basu (1984) it is proved, that $F_n \xrightarrow{LD} F$ implies $\lim\limits_{n\to\infty} ET_n = ET$, where $ET = \int_0^\infty R(t)dt$ for HNBUE (HNBUE is a class called *harmonic new better than used in expectation*, see Basu, 1984).

It is known that MTFR $\subset$ NBUE $\subset$ HNBUE. Thus, we have

$$\lim_{n\to\infty} g_n(t) = \lim_{n\to\infty} \frac{R_n(t)}{ET_n} = \frac{R(t)}{ET}\,. \tag{8}$$

The limit (8) is the local limit theorem for density functions $g_n(t)$. According to the Scheffe theorem, the local limit theorem implies the integral limit theorem (Billigsley, 1968, Stoyanow, 1989). Thus, we have

$$\lim_{n\to\infty} \frac{ET_n(t)}{ET_n} = \frac{ET(t)}{ET}\,, \tag{9}$$

where $ET(t) = \int_0^t R(x)dx$.

Let

$$h_n(t) = \frac{F_n(t)}{ET_n(t)}\,.$$

By the assumptions and (8), (9) we obtain

$$\lim_{n\to\infty} h_n(t) = \frac{F(t)}{ET(t)}\,.$$

If functions $h_n(t)$ are non-decreasing, then $h(t) = F(t)/\,ET(t)$ is also non-decreasing. Hence $T \in \mathrm{MTFR}$. For dual class $\mathrm{MTFR}^{\mathrm{D}}$ the proof is analogous. ■

2.3. Mixture of distributions

Let F_α be a set of distributions functions, where the index α is governed by the distribution function $G(x)$. Mixture $F(x)$ of $F_\alpha(x)$ according to $G(x)$ is given by Barlow (1981),

$$F(x) = \int_{-\infty}^{\infty} F_\alpha(x) dG(\alpha). \tag{10}$$

To see that class MTFR is not closed under mixtures, we consider a mixture of non-identical exponential distributions. In this case the failure rate function is strictly decreasing, so F can not be MTFR.

If T is an absolutely continuous random variable, then Definition 1.1 is equivalent to

$$f(t)\,ET(t) - F(t)R(t) \geqslant 0 \quad \text{for } \ t \geqslant 0, \tag{11}$$

where $f(t)$ is the density function of T.

PROPOSITION 2.3 *We suppose that $F(x)$ is the mixture of $F_\alpha(t)$, $\alpha \in A$, with each $F_\alpha \in \mathrm{MTFR}^D$ and no two distinct $F_\alpha(t)$ and $F_{\alpha'}(t)$ crossing on $(0, \infty)$. Then $T \in \mathrm{MTFR}^D$.*

Proof. If $T \in \mathrm{MTFR}^D$ then by (11), we obtain

$$F^2(t) \leqslant F(t) - f(t)\,ET(t)\,. \tag{12}$$

By the Chebyschew inequality for similarity ordered function (Barlow, 1981), we obtain

$$F^2(t) = \int_{-\infty}^{\infty} F_\alpha(t)\,dG(\alpha) \int_{-\infty}^{\infty} F_\alpha(t)\,dG(\alpha) \leqslant \int_{-\infty}^{\infty} F_\alpha^2(t)\,dG(\alpha).$$

According to (12), we write

$$F^2(t) \leqslant \int_{-\infty}^{\infty} [F_\alpha(t) - f_\alpha(t)\,ET_\alpha(t)]\,dG(\alpha) = F(t) - f(t)\,ET(t),$$

where $f_\alpha(t)$ is the density function corresponding to distribution function $F_\alpha(t)$, and

$$ET_\alpha(t) = \int_0^t R_\alpha(x)dx.$$

Thus, we have $T \in \mathrm{MTFR}^D$. ■

2.4. Convolution

Proposition 2.4 *If T_1 and T_2 are independent, $T_1, T_2 \in$ MTFR, then $T = T_1 + T_2 \in$ MTFR.*

Proof. Let the random variables T_1, T_2, T have distribution functions $F_1(t)$, $F_2(t)$, $F(t)$ and reliability function $R_1(t)$, $R_2(t)$, $R(t)$ respectively.

Let

$$ET_i(t) = \int_0^t R_i(x)\,dx \quad \text{for } \ i = 1, 2.$$

It is known that

$$F(t) = \int_0^t F_1(t - z)\,dF_2(z), \tag{13}$$

$$R(t) = \int_0^t R_1(t - z)\,dF_2(z), \tag{14}$$

$$ET(t) = \int_0^t ET_1(t - z)\,dF_2(z). \tag{15}$$

Next, we show that

$$ET(u + v)F(u) \leqslant ET(u)F(u + v) \quad \text{for } \ u, v \geqslant 0. \tag{16}$$

From (13), (14), (15) for the left-hand sides of (16), we have

$$ET(u + v)F(u) = \int_0^{u+v}\int_0^u ET_1(u + v - z_1)F_1(u - z)\,dF_2(z)\,dF_2(z_1) \tag{17}$$

For the right-hand sides of (16), we obtain

$$ET(u)F(u + v) = \int_0^{u+v}\int_0^u F_1(u + v - z_1)\,ET_1(u - z)\,dF_2(z)\,dF_2(z_1). \tag{18}$$

If $z_1 \leqslant z + v$, then

$$\begin{aligned} &ET_1(u+v-z_1)F_1(u-z) - F_1(u+v-z_1)\, ET_1(u-z) \leqslant 0 \\ \text{and} \\ &ET(u+v)F(u) - ET(u)F(u+v) \leqslant 0\,. \end{aligned} \tag{19}$$

We divide the set $S = \{z, z_1) : 0 \leqslant z \leqslant u,\ 0 \leqslant z_1 \leqslant u+v\}$ into three parts:

$$\begin{aligned} S_1 &= \{(z, z_1) : 0 \leqslant z \leqslant u,\ z+v \leqslant z_1 \leqslant u+v\}, \\ S_2 &= \{(z, z_1) : 0 \leqslant z \leqslant u,\ v \leqslant z_1 < z+v\}, \\ S_3 &= \{(z, z_1) : 0 \leqslant z \leqslant u,\ 0 \leqslant z_1 < v\}. \end{aligned}$$

Now, we consider the one to one transformation of the triangles S_1 on the triangle S_2. This transformation has the form

$$(z, z_1) \xrightarrow{\alpha} (z_1 - v,\ z+v).$$

Let $g(z, z_1) = ET_1(u+v-z_1)F_1(u-z) - F_1(u+v-z_1)\, ET_1(u-z)$. It can be seen that

$$g(z, z_1) = -g(\alpha(z, z_1)$$

and

$$\iint\limits_{S_1} g(z, z_1)\, dF_2(z)\, dF_2(z_1) + \iint\limits_{S_2} g(z, z_1)\, dF_2(z)\, dF_2(z_1) = 0\,.$$

Thus

$$\iint\limits_{S} g(z, z_1)\, dF_2(z)\, dF_2(z_1) = \iint\limits_{S_3} g(z, z_1) dF_2(z) dF_2(z_1) \geqslant 0\,.$$

This completes the proof of Proposition 2.4. ■

3. Conclusion

The survey of the results proved in Section 2 shows that the ageing class MTFR have many important properties. We are interested now whether Proposition 2.3 is true without the assumption on absolute continuity. And, we conjecture that class $\mathrm{MTFR}^{\mathrm{D}}$ is not preserved under arbitrary mixtures (a crossing may occur).

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